1951 AMC 12 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/1951/solutions

1. The percent that M is greater than N is:

A $\frac{100(M-N)}{M}$

B $\frac{100(M-N)}{N}$

C $\frac{M-N}{N}$

D $\frac{M-N}{M}$

E $\frac{100(M+N)}{N}$

Solution:

The increase from N to M is $M-N$. Dividing by the original quantity and multiplying by 100 gives

$$\frac{100(M-N)}{N}.$$

Thus, the correct answer is **B**.

2. A rectangular field is half as wide as it is long and is completely enclosed by x yards of fencing. The area in terms of x is:

- A $\frac{x^2}{2}$
- B $2x^2$
- C $\frac{2x^2}{9}$
- D $\frac{x^2}{18}$
- E $\frac{x^2}{72}$

Solution:

Let the width be w, so the length is $2w$. The fencing gives $2(w+2w)=6w=x$, hence $w=\frac{x}{6}$. Therefore

$$A = w(2w) = 2\left(\frac{x}{6}\right)^2 = \frac{x^2}{18}.$$

Thus, the correct answer is **D**.

3. If the length of a diagonal of a square is $a + b$, then the area of the square is:

A $(a+b)^2$

B $\dfrac{1}{2}(a+b)^2$

C $a^2 + b^2$

D $\dfrac{1}{2}(a^2+b^2)$

E None of these

Solution:

If s is the side length and $d = a + b$ is the diagonal, then $d = s\sqrt{2}$. Thus

$$s^2 = \frac{d^2}{2} = \frac{1}{2}(a+b)^2.$$

Thus, the correct answer is **B**.

4. A barn with a flat roof is rectangular in shape, 10 yd. wide, 13 yd. long, and 5 yd. high. It is to be painted inside and outside, and on the ceiling, but not on the roof or floor. The total number of square yards to be painted is:

- A 360
- B 460
- C 490
- D 590
- E 720

Solution:

The four walls have one-sided area

$$2(13 \cdot 5) + 2(10 \cdot 5) = 230.$$

Painting them inside and outside contributes $2(230) = 460$. The ceiling adds $10 \cdot 13 = 130$, so the total is $460 + 130 = 590$.

Thus, the correct answer is **D**.

5. Mr. A owns a home worth $\$10,000$. He sells it to Mr. B at a 10% profit based on the worth of the house. Mr. B sells the house back to Mr. A at a 10% loss. Then:

- A A comes out even
- B A makes $\$1100$ on the deal
- C A makes $\$1000$ on the deal
- D A loses $\$900$ on the deal
- E A loses $\$1000$ on the deal

Solution:

The first sale price is $10{,}000(1.10) = 11{,}000$. The return sale is $11{,}000(0.90) = 9{,}900$. Mr. A receives $11{,}000$ and pays back $9{,}900$, so he gains $1{,}100$.

Thus, the correct answer is **B**.

6. The bottom, side, and front areas of a rectangular box are known. The product of these areas is equal to:

A The volume of the box

B The square root of the volume

C Twice the volume

D The square of the volume

E The cube of the volume

Solution:

If the edge lengths are $l, w, h,$ the three face areas are $lw, wh,$ and $hl.$ Their product is

$$(lw)(wh)(hl) = l^2w^2h^2 = (lwh)^2,$$

the square of the volume.

Thus, the correct answer is **D**.

7. An error of $0.02''$ is made in the measurement of a line $10''$ long, while an error of only $0.2''$ is made in a measurement of a line $100''$ long. In comparison with the relative error of the first measurement, the relative error of the second measurement is:

A Greater by 0.18

B The same

C Less

D 10 times as great

E Correctly described by both (A) and (D)

Solution:

The two relative errors are $\frac{0.02}{10} = 0.002$ and $\frac{0.2}{100} = 0.002$. They are equal.

Thus, the correct answer is **B**.

8. The price of an article is cut 10%. To restore it to its former value, the new price must be increased by:

A 10%

B 9%

C $11\frac{1}{9}\%$

D 11%

E None of these answers

Solution:

Let the original price be 1. The reduced price is 0.9. The required fractional increase is

$$\frac{1-0.9}{0.9}=\frac{1}{9},$$

which is $11\frac{1}{9}\%$.

Thus, the correct answer is **C**.

9. An equilateral triangle is drawn with a side of length a. A new equilateral triangle is formed by joining the midpoints of the sides of the first one. Then a third equilateral triangle is formed by joining the midpoints of the sides of the second; and so on forever. The limit of the sum of the perimeters of all the triangles thus drawn is:

A Infinite

B $\left(5\frac{1}{4}\right)a$

C $2a$

D $6a$

E $\left(4\frac{1}{2}\right)a$

Solution:

The first perimeter is $3a$, and every later perimeter is half the preceding one. Hence the total is

$$3a\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)$$
$$=3a\cdot\frac{1}{1-\frac{1}{2}}$$
$$=6a.$$

Thus, the correct answer is **D**.

10. Of the following statements, the one that is incorrect is:

- A Doubling the base of a given rectangle doubles the area.
- B Doubling the altitude of a triangle doubles the area.
- C Doubling the radius of a given circle doubles the area.
- D Doubling the divisor of a fraction and dividing its numerator by 2 changes the quotient.
- E Doubling a given quantity may make it less than it originally was.

Solution:

A circle has area πr^2. Replacing r by $2r$ gives $\pi(2r)^2 = 4\pi r^2$, so doubling the radius quadruples, rather than doubles, the area. The other statements can hold as written.

Thus, the correct answer is **C**.

11. The limit of the sum of an infinite number of terms in a geometric progression is $\frac{a}{1-r}$, where a denotes the first term and $-1 < r < 1$ denotes the common ratio. The limit of the sum of their squares is:

A $\frac{a^2}{(1-r)^2}$

B $\frac{a^2}{1+r^2}$

C $\frac{a^2}{1-r^2}$

D $\frac{4a^2}{1+r^2}$

E None of these

Solution:

The squared terms form a geometric series with first term a^2 and ratio r^2. Since $|r| < 1$, its sum is

$$\frac{a^2}{1-r^2}.$$

Thus, the correct answer is **C**.

12. At $2{:}15$ o'clock, the hour and minute hands of a clock form an angle of:

A 30°

B 5°

C $22\frac{1}{2}^{\circ}$

D $7\frac{1}{2}^{\circ}$

E 28°

Solution:

At $2{:}15$ the minute hand is 90° from 12. The hour hand is $2 \cdot 30^{\circ} + 15 \cdot 0.5^{\circ} = 67.5^{\circ}$ from 12. Their angle is $90^{\circ} - 67.5^{\circ} = 22.5^{\circ} = 22\frac{1}{2}^{\circ}$.

Thus, the correct answer is **C**.

13. A can do a piece of work in 9 days. B is 50% more efficient than A. The number of days it takes B to do the same piece of work is:

A $13\dfrac{1}{2}$

B $4\dfrac{1}{2}$

C 6

D 3

E None of these answers

Solution:

A's rate is $\frac{1}{9}$ job per day. Thus B's rate is

$$\frac{3}{2} \cdot \frac{1}{9} = \frac{1}{6}$$

job per day, so B needs 6 days.

Thus, the correct answer is **C**.

14. In connection with proof in geometry, indicate which one of the following statements is incorrect:

A Some statements are accepted without being proved.

B In some instances there is more than one correct order in proving certain propositions.

C Every term used in a proof must have been defined previously.

D It is not possible to arrive by correct reasoning at a true conclusion if, in the given, there is an untrue proposition.

E Indirect proof can be used whenever there are two or more contrary propositions.

Solution:

An axiomatic development begins with primitive terms that are deliberately left undefined, as well as statements accepted without proof. Therefore it is not true that every term in a proof must previously have been defined.

Thus, the intended correct answer is **C**.

15. The largest number by which the expression $n^3 - n$ is divisible for all possible integral values of n is:

A 2

B 3

C 4

D 5

E 6

Solution:

We have

$$n^3 - n = n(n-1)(n+1),$$

a product of three consecutive integers. One is divisible by 3 and at least one is even, so the product is always divisible by 6. Taking $n = 2$ gives exactly 6, so no larger integer always divides it.

Thus, the correct answer is **E**.

16. If in applying the quadratic formula to a quadratic equation

$$f(x) = ax^2 + bx + c = 0,$$

it happens that $c = \dfrac{b^2}{4a}$, then the graph of $y = f(x)$ will certainly:

A Have a maximum

B Have a minimum

C Be tangent to the x-axis

D Be tangent to the y-axis

E Lie in one quadrant only

Solution:

The condition gives

$$b^2 - 4ac = b^2 - 4a\left(\frac{b^2}{4a}\right) = 0.$$

Hence the quadratic has a repeated real root, so its parabola is tangent to the x-axis.

Thus, the correct answer is **C**.

17. Indicate in which one of the following equations y is neither directly nor inversely proportional to x:

A $x + y = 0$

B $3xy = 10$

C $x = 5y$

D $3x + y = 10$

E $\frac{x}{y} = \sqrt{3}$

Solution:

Choices A, C, and E rearrange to $y = kx$, and choice B rearranges to $xy = k$. But choice D gives $y = 10 - 3x$, which is neither form.

Thus, the correct answer is **D**.

18. The expression $21x^2 + ax + 21$ is to be factored into two linear prime binomial factors with integer coefficients. This can be done if a is:

A Any odd number

B Some odd number

C Any even number

D Some even number

E Zero

Solution:

Suppose

$$21x^2 + ax + 21$$
$$= (Ax + B)(Cx + D).$$

Then $AC = BD = 21$, so A, B, C, D are odd. Therefore $a = AD + BC$ is even. Some even values work; for example,

$$(3x + 7)(7x + 3)$$
$$= 21x^2 + 58x + 21.$$

Not every even value works.

Thus, the correct answer is **D**.

19. A six-place number is formed by repeating a three-place number; for example, $256{,}256$, or $678{,}678$, etc. Any number of this form is always exactly divisible by:

A 7 only

B 11 only

C 13 only

D 101

E 1001

Solution:

If the repeated block is N, then the six-place number is

$$1000N + N = 1001N.$$

It is therefore always divisible by 1001.

Thus, the correct answer is **E**.

20. When simplified and expressed with negative exponents, the expression $(x + y)^{-1}(x^{-1} + y^{-1})$ is equal to:

- A $x^{-2} + 2x^{-1}y^{-1} + y^{-2}$
- B $x^{-2} + 2^{-1}x^{-1}y^{-1} + y^{-2}$
- C $x^{-1}y^{-1}$
- D $x^{-2} + y^{-2}$
- E $\dfrac{1}{x^{-1}y^{-1}}$

Solution:

For nonzero x, y with $x + y \neq 0$,

$$\begin{aligned}&(x+y)^{-1}(x^{-1}+y^{-1})\\ &= \frac{1}{x+y}\left(\frac{x+y}{xy}\right)\\ &= \frac{1}{xy} = x^{-1}y^{-1}.\end{aligned}$$

Thus, the correct answer is **C**.

21. Given $x > 0, y > 0, x > y$, and $z \neq 0$. The inequality which is not always correct is:

A $x + z > y + z$

B $x - z > y - z$

C $xz > yz$

D $\frac{x}{z^2} > \frac{y}{z^2}$

E $xz^2 > yz^2$

Solution:

Because z may be negative, multiplying $x > y$ by z reverses the inequality and gives $xz < yz$. The operations in the other choices preserve the inequality because $z^2 > 0$.

Thus, the correct answer is **C**.

22. The values of a in the equation $\log_{10}(a^2 - 15a) = 2$ are:

A $\frac{15 \pm \sqrt{233}}{2}$

B $20, -5$

C $\frac{15 \pm \sqrt{305}}{2}$

D ± 20

E None of these

Solution:

The equation is equivalent to $a^2 - 15a = 10^2 = 100,$ so

$$\begin{aligned} a^2 - 15a - 100 \\ &= (a - 20)(a + 5) \\ &= 0. \end{aligned}$$

Both $a = 20$ and $a = -5$ make the logarithm's argument $100,$ so both are valid.

Thus, the correct answer is **B**.

23. The radius of a cylindrical box is 8 inches and the height is 3 inches. The number of inches that may be added to either the radius or the height to give the same nonzero increase in volume is:

A 1

B $5\frac{1}{3}$

C Any number

D Non-existent

E None of these

Solution:

For the two new volumes to be equal,

$$3\pi(8+x)^2 = 64\pi(3+x).$$

Cancelling π and expanding gives $3x^2 - 16x = 0$, so $x = 0$ or $x = \frac{16}{3} = 5\frac{1}{3}$. The problem requires a nonzero increase.

Thus, the correct answer is **B**.

24. When simplified, $\dfrac{2^{n+4}-2(2^n)}{2(2^{n+3})}$ is:

A $2^{n+1}-\dfrac{1}{8}$

B -2^{n+1}

C $1-2^n$

D $\dfrac{7}{8}$

E $\dfrac{7}{4}$

Solution:

Factoring and combining powers gives

$$\frac{2^n(2^4-2)}{2^{n+4}}=\frac{14}{16}=\frac{7}{8}.$$

Thus, the correct answer is **D**.

25. The apothem of a square having its area numerically equal to its perimeter is compared with the apothem of an equilateral triangle having its area numerically equal to its perimeter. The first apothem will be:

A Equal to the second

B $\frac{4}{3}$ times the second

C $\frac{2}{\sqrt{3}}$ times the second

D $\frac{\sqrt{2}}{\sqrt{3}}$ times the second

E Indeterminately related to the second

Solution:

Every regular polygon with apothem r and perimeter P has area $A = \frac{1}{2}rP$. If its numerical area equals its nonzero perimeter, then

$$\frac{1}{2}rP = P,$$

so $r = 2$. This applies to both the square and the equilateral triangle, so their apothems are equal.

Thus, the correct answer is **A**.

26. In the equation

$$\frac{x(x-1)-(m+1)}{(x-1)(m-1)} = \frac{x}{m},$$

the roots are equal when:

A $m = 1$

B $m = \frac{1}{2}$

C $m = 0$

D $m = -1$

E $m = -\frac{1}{2}$

Solution:

Clearing denominators and simplifying yields

$$x^2 - x - m(m+1) = 0.$$

Equal roots require

$$\begin{aligned} 1 + 4m(m+1) &= (2m+1)^2 \\ &= 0, \end{aligned}$$

hence $m = -\frac{1}{2}$. This value does not violate the original denominators.

Thus, the correct answer is **E**.

27. Through a point inside a triangle, three lines are drawn from the vertices to the opposite sides, forming six triangular sections. Then:

A The triangles are similar in opposite pairs

B The triangles are congruent in opposite pairs

C The triangles are equal in area in opposite pairs

D Three similar quadrilaterals are formed

E None of the above relations is true

Solution:

No angle or length condition forces opposite sections to be similar or congruent. Their areas also need not match: placing the interior point very close to one side makes the sections adjoining that side arbitrarily small without forcing their opposite sections to be small. No quadrilaterals are among the six sections.

Thus, the correct answer is **E**.

28. The pressure P of wind on a sail varies jointly as the area A of the sail and the square of the velocity V of the wind. The pressure on a square foot is 1 pound when the velocity is 16 miles per hour. The velocity of the wind when the pressure on a square yard is 36 pounds is:

A $10\frac{2}{3}$ mph

B 96 mph

C 32 mph

D $1\frac{1}{3}$ mph

E 16 mph

Solution:

Write $P = kAV^2$. From $1 = k(1)(16^2)$, we get $k = \frac{1}{256}$. A square yard has area 9 square feet, so

$$36 = \frac{1}{256} \cdot 9V^2.$$

Thus $V^2 = 1024$ and the positive speed is $V = 32$ mph.

Thus, the correct answer is **C**.

29. Of the following sets of data, the only one that does not determine the shape of a triangle is:

- A The ratio of two sides and the included angle
- B The ratios of the three altitudes
- C The ratios of the three medians
- D The ratio of the altitude to the corresponding base
- E Two angles

Solution:

Choices A and E determine the angles up to similarity. Ratios of all three altitudes determine reciprocal side ratios, and ratios of all three medians determine side ratios. But a single ratio $\frac{h}{b}$ does not determine the remaining side lengths or angles; many non-similar triangles can share it.

Thus, the correct answer is **D**.

30. If two poles $20''$ and $80''$ high are $100''$ apart, then the height of the intersection of the lines joining the top of each pole to the foot of the opposite pole is:

A $50''$

B $40''$

C $16''$

D $60''$

E None of these

Solution:

Put the bases at $(0,0)$ and $(100,0)$, with tops $(0,80)$ and $(100,20)$. The cross-lines are $y=\frac{4}{5}x$ and $y=20-\frac{1}{5}x$. Equating them gives $x=20$, and hence $y=16$.

Thus, the correct answer is **C**.

31. A total of 28 handshakes was exchanged at the conclusion of a party. Assuming that each participant was equally polite toward all the others, the number of people present was:

A 14

B 28

C 56

D 8

E 7

Solution:

With n people the number of handshakes is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

Thus $n(n-1) = 56,$ and the positive solution is $n = 8.$

Thus, the correct answer is **D**.

32. If $\triangle ABC$ is inscribed in a semicircle whose diameter is $\overline{AB}$, then $\overline{AC} + \overline{BC}$ must be:

A Equal to $\overline{AB}$

B Equal to $\overline{AB}\sqrt{2}$

C $\geq \overline{AB}\sqrt{2}$

D $\leq \overline{AB}\sqrt{2}$

E $\overline{AB}^2$

Solution:

By Thales' theorem, AC and BC are the legs of a right triangle with hypotenuse AB. Therefore

$$\begin{aligned}(AC + BC)^2 &\leq 2(AC^2 + BC^2)\\ &= 2AB^2.\end{aligned}$$

Taking positive square roots gives $AC + BC \leq AB\sqrt{2}$, with equality for an isosceles right triangle.

Thus, the correct answer is **D**.

33. The roots of the equation $x^2 - 2x = 0$ can be obtained graphically by finding the abscissas of the points of intersection of each of the following pairs of equations except the pair:

- A $y = x^2, y = 2x$
- B $y = x^2 - 2x, y = 0$
- C $y = x, y = x - 2$
- D $y = x^2 - 2x + 1, y = 1$
- E $y = x^2 - 1, y = 2x - 1$

Solution:

Choices A, B, D, and E all reduce their intersection condition to $x^2 - 2x = 0$. Choice C instead requires

$$x = x - 2,$$

which has no solution and therefore cannot produce the desired roots.

Thus, the correct answer is **C**.

34. The value of $10^{\log_{10} 7}$ is:

A 7

B 1

C 10

D $\log_{10} 7$

E $\log_7 10$

Solution:

Since the base-10 exponential and logarithm are inverse functions,

$$10^{\log_{10} 7} = 7.$$

Thus, the correct answer is **A**.

35. If $a^x = c^q = b$ and $c^y = a^z = d$, then:

A $xy = qz$

B $\frac{x}{y} = \frac{q}{z}$

C $x + y = q + z$

D $x - y = q - z$

E $x^y = q^z$

Solution:

Taking logarithms gives

$$x \log a = q \log c,$$
$$y \log c = z \log a.$$

Multiplying these equations and cancelling the nonzero logarithmic factors in the nondegenerate case yields $xy = qz$. The identity extends to the admissible boundary cases represented by the equations.

Thus, the correct answer is **A**.

36. Which of the following methods of proving a geometric figure a locus is not correct?

A. Every point on the locus satisfies the conditions and every point not on the locus does not satisfy the conditions.

B. Every point not satisfying the conditions is not on the locus and every point on the locus does satisfy the conditions.

C. Every point satisfying the conditions is on the locus and every point on the locus satisfies the conditions.

D. Every point not on the locus does not satisfy the conditions and every point not satisfying the conditions is not on the locus.

E. Every point satisfying the conditions is on the locus and every point not satisfying the conditions is not on the locus.

Solution:

Let L mean "on the locus" and C mean "satisfies the conditions." A complete proof needs both $L \Rightarrow C$ and $C \Rightarrow L$. Choice B states $\neg C \Rightarrow \neg L$, which is merely the contrapositive of $L \Rightarrow C$, and then repeats $L \Rightarrow C$. It never proves $C \Rightarrow L$.

Thus, the correct answer is **B**.

37. A number which when divided by 10 leaves a remainder of $9,$ when divided by 9 leaves a remainder of $8,$ by 8 leaves a remainder of $7,$ etc., down to where, when divided by $2,$ it leaves a remainder of $1,$ is:

A 59

B 419

C 1259

D 2519

E None of these answers

Solution:

If the number is $N,$ then $N+1$ is divisible by every integer from 2 through $10.$ Their least common multiple is

$$2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520.$$

Thus the least positive number fitting all the conditions is $N = 2520 - 1 = 2519.$

Thus, the correct answer is **D**.

38. A rise of 600 feet is required to get a railroad line over a mountain. The grade can be kept down by lengthening the track and curving it around the mountain peak. The additional length of track required to reduce the grade from 3% to 2% is approximately:

- A $10{,}000$ ft.
- B $20{,}000$ ft.
- C $30{,}000$ ft.
- D $12{,}000$ ft.
- E None of these

Solution:

A 3% grade requires about $\frac{600}{0.03} = 20{,}000$ feet of track, while a 2% grade requires $\frac{600}{0.02} = 30{,}000$ feet. The additional length is $10{,}000$ feet.

Thus, the correct answer is **A**.

39. A stone is dropped into a well and the report of the stone striking the bottom is heard 7.7 seconds after it is dropped. Assume that the stone falls $16t^2$ feet in t seconds and that the velocity of sound is 1120 feet per second. The depth of the well is:

- A 784 ft.
- B 342 ft.
- C 1568 ft.
- D 156.8 ft.
- E None of these

Solution:

If the stone falls for t seconds, the depth is $16t^2$, and the sound takes $\frac{16t^2}{1120} = \frac{t^2}{70}$ seconds to return. Hence

$$t + \frac{t^2}{70} = 7.7,$$

or $t^2 + 70t - 539 = 0$. The positive root is $t = 7$, giving depth $16(7^2) = 784$ feet.

Thus, the correct answer is **A**.

40. The expression

$$\left(\frac{(x+1)^2(x^2-x+1)^2}{(x^3+1)^2}\right)^2 \cdot \left(\frac{(x-1)^2(x^2+x+1)^2}{(x^3-1)^2}\right)^2$$

equals:

A $(x+1)^4$

B $(x^3+1)^4$

C 1

D $\left[(x^3+1)(x^3-1)\right]^2$

E $\left[(x^3-1)^2\right]^2$

Solution:

Because

$$\begin{aligned} x^3+1 &= (x+1)(x^2-x+1), \\ x^3-1 &= (x-1)(x^2+x+1), \end{aligned}$$

each fraction inside parentheses equals 1 wherever the original expression is defined. Their squared product is therefore 1.

Thus, the correct answer is **C**.

41. The formula expressing the relationship between x and y in the table is:

x	2	3	4	5	6
y	0	2	6	12	20

A $y = 2x - 4$

B $y = x^2 - 3x + 2$

C $y = x^3 - 3x^2 + 2x$

D $y = x^2 - 4x$

E $y = x^2 - 4$

Solution:

The y-values factor naturally as

$$0 = 1 \cdot 0, \quad 2 = 2 \cdot 1, \quad 6 = 3 \cdot 2,$$
$$12 = 4 \cdot 3, \quad 20 = 5 \cdot 4.$$

Thus $y = (x-1)(x-2)$ and hence $y = x^2 - 3x + 2$.

Thus, the correct answer is **B**.

42. If x equals

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}},$$

then:

A $x = 1$

B $0 < x < 1$

C $1 < x < 2$

D x is infinite

E $x > 2$ but finite

Solution:

The repeating tail gives $x = \sqrt{1+x}$, so

$$x^2 - x - 1 = 0.$$

Since $x \geq 0, x = \frac{1+\sqrt{5}}{2}$, which lies strictly between 1 and 2.

Thus, the correct answer is **C**.

43. Of the following statements, the only one that is incorrect is:

A. An inequality will remain true after each side is increased, decreased, multiplied, or divided (zero excluded) by the same positive quantity.

B. The arithmetic mean of two unequal positive quantities is greater than their geometric mean.

C. If the sum of two positive quantities is given, their product is largest when they are equal.

D. If a and b are positive and unequal, $\frac{1}{2}(a^2 + b^2)$ is greater than $\left[\frac{1}{2}(a+b)\right]^2$.

E. If the product of two positive quantities is given, their sum is greatest when they are equal.

Solution:

If $uv = P > 0$, then $u + v \geq 2\sqrt{P}$, with equality at $u = v$. Thus equality gives the least possible sum. The sum can grow without bound by taking u large and $v = \frac{P}{u}$ small, so it is not greatest at equality.

Thus, the correct answer is **E**.

44. If $\dfrac{xy}{x+y} = a, \dfrac{xz}{x+z} = b$, and $\dfrac{yz}{y+z} = c$, where a, b, c are other than zero, then x equals:

A $\dfrac{abc}{ab+ac+bc}$

B $\dfrac{2abc}{ab+bc+ac}$

C $\dfrac{2abc}{ab+ac-bc}$

D $\dfrac{2abc}{ab+bc-ac}$

E $\dfrac{2abc}{ac+bc-ab}$

Solution:

Inverting gives

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{a},$$
$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b},$$
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c}.$$

Adding the first two and subtracting the third yields

$$\begin{aligned}\frac{2}{x}&=\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\\&=\frac{ac+bc-ab}{abc}.\end{aligned}$$

Hence $x = \dfrac{2abc}{ac+bc-ab}$.

Thus, the correct answer is **E**.

45. If you are given $\log 8 = 0.9031$ and $\log 9 = 0.9542$, then the only logarithm that cannot be found without the use of tables is:

- A $\log 17$
- B $\log(\frac{5}{4})$
- C $\log 15$
- D $\log 600$
- E $\log 0.4$

Solution:

From the data,

$$\log 2 = \frac{1}{3} \log 8,$$
$$\log 3 = \frac{1}{2} \log 9,$$

and $\log 5 = 1 - \log 2$. Therefore logarithms of products and quotients of powers of $2, 3, 5$ can be computed. The number 17 has none of those prime factors, so its logarithm is not determined by the data.

Thus, the correct answer is **A**.

46. $\overline{AB}$ is a fixed diameter of a circle whose center is O. From C, any point on the circle, a chord $\overline{CD}$ is drawn perpendicular to $\overline{AB}$. Then, as C moves over a semicircle, the bisector of angle OCD cuts the circle in a point that always:

- A Bisects the arc AB
- B Trisects the arc AB
- C Varies
- D Is as far from $\overline{AB}$ as from D
- E Is equidistant from B and C

Solution:

Extend CO to meet the circle again at E. Since CE is a diameter, $\angle CDE = 90°$. Also $CD \perp AB$, so $DE \parallel AB$. If the bisector of $\angle OCD = \angle ECD$ meets the circle at P, equal inscribed angles give equal arcs $EP = PD$. Because the parallel chord DE has endpoints symmetrically placed relative to the fixed diameter AB, their arc midpoint P is the midpoint of arc AB.

Thus, the correct answer is **A**.

47. If r and s are the roots of the equation $ax^2 + bx + c = 0$, the value of $\frac{1}{r^2} + \frac{1}{s^2}$ is:

A $b^2 - 4ac$

B $\dfrac{b^2 - 4ac}{2a}$

C $\dfrac{b^2 - 4ac}{c^2}$

D $\dfrac{b^2 - 2ac}{c^2}$

E None of these

Solution:

By Vieta's formulas,

$$r + s = -\frac{b}{a}, \qquad rs = \frac{c}{a}.$$

Therefore

$$\begin{aligned}\frac{1}{r^2} + \frac{1}{s^2} &= \frac{(r+s)^2 - 2rs}{(rs)^2} \\ &= \frac{b^2 - 2ac}{c^2}.\end{aligned}$$

Thus, the correct answer is **D**.

48. The area of a square inscribed in a semicircle is to the area of the square inscribed in the entire circle as:

A $1:2$

B $2:3$

C $2:5$

D $3:4$

E $3:5$

Solution:

For the square in the semicircle, put its base on the diameter. A top vertex has horizontal distance $\frac{s}{2}$ from the center and vertical distance s, so

$$\left(\frac{s}{2}\right)^2 + s^2 = R^2,$$

giving $s^2 = \frac{4R^2}{5}$. A square inscribed in the full circle has diagonal $2R$, hence area $2R^2$. The ratio is

$$\frac{\frac{4R^2}{5}}{2R^2} = \frac{2}{5}.$$

Thus, the correct answer is **C**.

49. The medians of a right triangle which are drawn from the vertices of the acute angles are 5 and $\sqrt{40}$. The value of the hypotenuse is:

A 10

B $2\sqrt{40}$

C $\sqrt{13}$

D $2\sqrt{13}$

E None of these

Solution:

Let the legs be a, b. The medians from their opposite acute vertices have squared lengths

$$a^2 + \frac{b^2}{4} = 25, \qquad \frac{a^2}{4} + b^2 = 40.$$

Solving gives $a^2 = 16$ and $b^2 = 36$. Thus the hypotenuse c satisfies

$$c^2 = a^2 + b^2 = 52,$$

so $c = 2\sqrt{13}$.

Thus, the correct answer is **D**.

50. Tom, Dick, and Harry started out on a 100-mile journey. Tom and Harry went by automobile at the rate of 25 mph, while Dick walked at the rate of 5 mph. After a certain distance, Harry got off and walked on at 5 mph, while Tom went back for Dick and got him to the destination at the same time that Harry arrived. The number of hours required for the trip was:

- A 5
- B 6
- C 7
- D 8
- E None of these answers

Solution:

Let t_1, t_2, t_3 be the car's times before Harry leaves it, while it returns for Dick, and while it carries Dick forward. The car, Dick, and Harry each cover 100 miles, giving

$$\begin{aligned} 25t_1 - 25t_2 + 25t_3 &= 100, \\ 5t_1 + 5t_2 + 25t_3 &= 100, \\ 25t_1 + 5t_2 + 5t_3 &= 100. \end{aligned}$$

Dividing by 5 and solving yields $t_1 = 3,\ t_2 = 2,\ t_3 = 3.$ Therefore the common travel time is $t_1 + t_2 + t_3 = 8$ hours.

Thus, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/1951