2024 AMC 10B Solutions

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1. In a long line of people, the 1013th person from the left is also the 1010th person from the right. How many people are in the line?

A 2021

B 2022

C 2023

D 2024

E 2025

Solution:

There are 1012 people to the left of this spot and 1009 to the right. Add those two groups plus the person themselves: $1012 + 1009 + 1 = 2022.$ Or, just as fast, the two positions overlap on one person, so $1013 + 1010 - 1 = 2022.$ Thus, **B** is the correct answer.

2. What is $10! - 7! \cdot 6!$?

A -120

B 0

C 120

D 600

E 720

Solution:

Write $10! = 10 \cdot 9 \cdot 8 \cdot 7! = 720 \cdot 7!$. But $720 = 6!$ too, so $10! = 6! \cdot 7! = 7! \cdot 6!$. The two terms are the same. That makes $10! - 7! \cdot 6! = 0$. Therefore, the answer is **B**.

3. For how many integer values of x is $|2x| \leq 7\pi$?

A 16

B 17

C 19

D 20

E 21

Solution:

Divide by 2 to get $|x| \leq 3.5\pi \approx 10.996$. The integers that fit run from -10 up to 10, and there are 21 of them. Thus, **E** is the correct answer.

4. Balls numbered $1, 2, 3, \ldots$ are placed in bins $A, B, C, D,$ and E so that the first ball is placed in A, the next two are placed in B, the next three are placed in C, the next four are placed in D, the next five are placed in E, and then the next six go in A, etc. For example, $22, 23, \ldots, 28$ are placed in B. Which bin contains ball 2024?

- A A
- B B
- C C
- D D
- E E

Solution:

Group g holds g balls, so the first g groups swallow $\frac{g(g+1)}{2}$ of them. Now $\frac{63 \cdot 64}{2} = 2016$ and $\frac{64 \cdot 65}{2} = 2080$, which puts ball 2024 in group 64 (balls 2017 through 2080). The bins cycle A, B, C, D, E, so group g lands in bin number $(g-1) \bmod 5$. For $g = 64$ that's $63 \bmod 5 = 3$, bin D. Therefore, the answer is **D**.

5. In the following expression, Melanie changed some of the plus signs to minus signs:

$$1 + 3 + 5 + 7 + \cdots + 97 + 99$$

When the new expression was evaluated, it was negative. What is the least number of plus signs that Melanie could have changed to minus signs?

A 14

B 15

C 16

D 17

E 18

Solution:

The full sum is $1 + 3 + \cdots + 99 = 50^2 = 2500$. Flipping a term t drops the total by $2t$, so to go negative the flipped terms have to add up to more than 1250. The greedy move is to flip the biggest odd numbers: flipping the top k gives $99 + 97 + \cdots = k(100 - k)$. We want $k(100 - k) > 1250$. At $k = 14$ it's only 1204, but at $k = 15$ it jumps to 1275. So 15 flips do it. Thus, **B** is the correct answer.

6. A rectangle has integer side lengths and an area of 2024. What is the least possible perimeter of the rectangle?

- A 160
- B 180
- C 222
- D 228
- E 390

Solution:

The perimeter $2(\ell + w)$ with $\ell w = 2024$ is smallest when ℓ and w are as close together as possible. Factor $2024 = 2^3 \cdot 11 \cdot 23$. The divisor pair nearest $\sqrt{2024} \approx 45$ is 44×46, which gives perimeter $2(44 + 46) = 180$. Therefore, the answer is **B**.

7. What is the remainder when $7^{2024} + 7^{2025} + 7^{2026}$ is divided by 19?

- A 0
- B 1
- C 7
- D 11
- E 18

Solution:

Pull out the common power: $7^{2024} + 7^{2025} + 7^{2026} = 7^{2024}(1 + 7 + 49) = 7^{2024} \cdot 57$. And $57 = 3 \cdot 19$, so the product is a multiple of 19. The remainder is 0. Thus, **A** is the correct answer.

8. Let N be the product of all the positive integer divisors of 42. What is the units digit of N?

- A 0
- B 2
- C 4
- D 6
- E 8

Solution:

Since $42 = 2 \cdot 3 \cdot 7$ has $(1+1)^3 = 8$ divisors, we can pair each divisor with its complement, so $N = 42^{8/2} = 42^4$. Only the units digit matters, and $2^4 = 16$ ends in 6, so 42^4 does too. Therefore, the answer is **D**.

9. Real numbers $a, b,$ and c have arithmetic mean $0.$ The arithmetic mean of $a^2, b^2,$ and c^2 is $10.$ What is the arithmetic mean of $ab, ac,$ and $bc?$

A -5

B $-\frac{10}{3}$

C $-\frac{10}{9}$

D 0

E $\frac{10}{9}$

Solution:

The means tell us $a+b+c=0$ and $a^2+b^2+c^2=30.$ Square the first: $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca),$ so $0=30+2(ab+bc+ca)$ and $ab+bc+ca=-15.$ Their mean is $-15/3=-5.$ Thus, **A** is the correct answer.

10. Quadrilateral $ABCD$ is a parallelogram, and E is the midpoint of the side $\overline{AD}$. Let F be the intersection of lines EB and AC. What is the ratio of the area of quadrilateral $CDEF$ to the area of triangle CFB?

A $5:4$

B $4:3$

C $3:2$

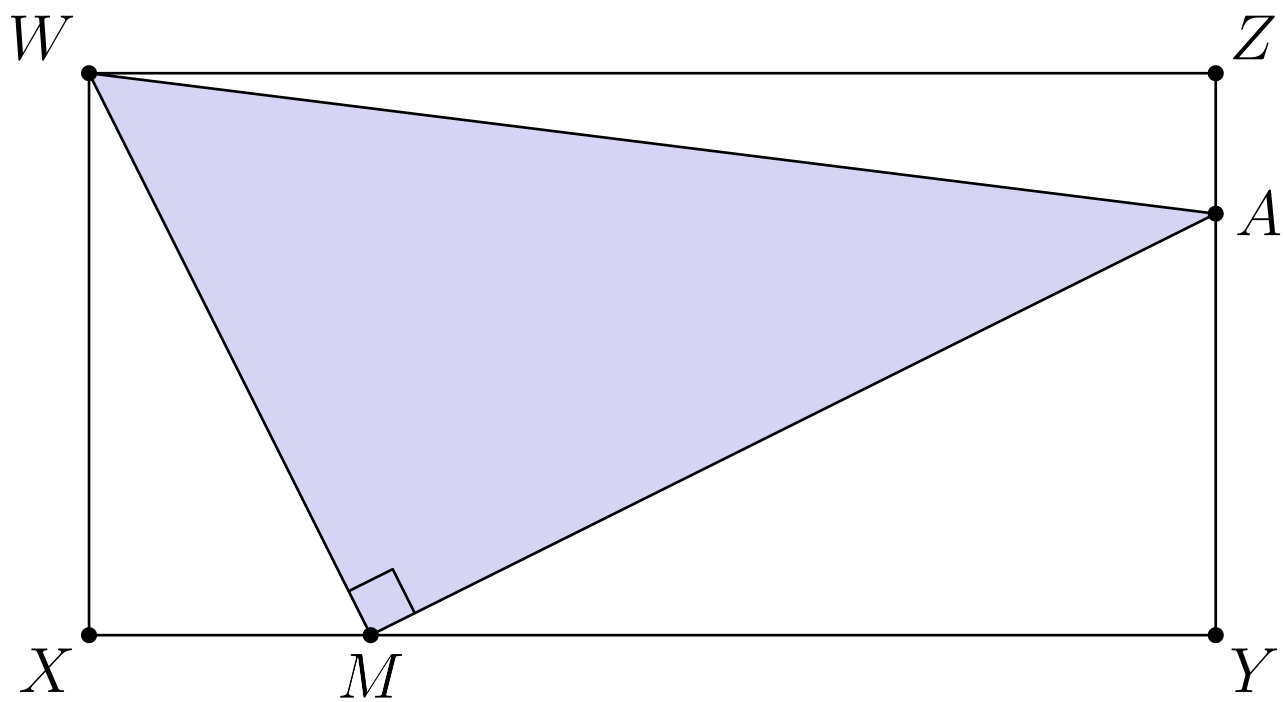
D $5:3$

E $2:1$

Solution:

Area ratios don't change under an affine map, so drop in convenient coordinates: $A=(0,0), B=(1,0), C=(1,1), D=(0,1)$, which makes $E=(0,\frac{1}{2})$. Line AC is $y=x$, and line EB runs from $(0,\frac{1}{2})$ to $(1,0)$; they cross at $F=(\frac{1}{3},\frac{1}{3})$. The shoelace formula gives quadrilateral $CDEF$ area $\frac{5}{12}$ and triangle CFB area $\frac{1}{3}$. So the ratio is $\frac{5}{12}:\frac{1}{3}=5:4$. Therefore, the answer is **A**.

11. In the figure below $WXYZ$ is a rectangle with $WX = 4$ and $WZ = 8$. Point M lies on $\overline{XY}$, point A lies on $\overline{YZ}$, and $\angle WMA$ is a right angle. The areas of $\triangle WXM$ and $\triangle WAZ$ are equal. What is the area of $\triangle WMA$?



A 13

B 14

C 15

D 16

E 17

Solution:

Set $X = (0,0), Y = (8,0), W = (0,4), Z = (8,4)$, so $M = (m,0)$ and $A = (8,a)$. The right angle means $\overrightarrow{MW} \cdot \overrightarrow{MA} = 0$, which gives $-m(8-m) + 4a = 0$, that is $m(8-m) = 4a$. Equal areas $[WXM] = 2m$ and $[WAZ] = 4(4-a)$ force $m = 8 - 2a$, so $a = \frac{8-m}{2}$. Substitute back and $(8-m)(2-m) = 0$. Taking $M \neq Y$ leaves $m = 2$ and $a = 3$. Then $[WMA] = 32 - [WXM] - [MYA] - [AZW] = 32 - 4 - 9 - 4 = 15$. Thus, **C** is the correct answer.

12. A group of 100 students from different countries meet at a mathematics competition. Each student speaks the same number of languages, and, for every pair of students A and B, student A speaks some language that student B does not speak, and student B speaks some language that student A does not speak. What is the least possible total number of languages spoken by all the students?

A 9

B 10

C 12

D 51

E 100

Solution:

Give each student the set of languages they speak. The condition says no one's set sits inside another's. Everyone speaks the same number k of languages, and two distinct k-element sets can never contain each other, so all we need is 100 different k -subsets of the n languages, i.e. $\binom{n}{k} \geq 100$. With $n = 8$ the best we can manage is $\binom{8}{4} = 70$, short of 100. But $\binom{9}{4} = 126 \geq 100$. So 9 languages are both enough and necessary. Therefore, the answer is **A**.

13. Positive integers x and y satisfy the equation $\sqrt{x}+\sqrt{y}=\sqrt{1183}$. What is the minimum possible value of $x+y$?

A 585

B 595

C 623

D 700

E 791

Solution:

Since $1183=7\cdot 169=7\cdot 13^2$, we have $\sqrt{1183}=13\sqrt{7}$. Square $\sqrt{x}+\sqrt{y}=13\sqrt{7}$ to get $x+y+2\sqrt{xy}=1183$. So $\sqrt{xy}$ is rational, which forces each of x, y to be 7 times a perfect square: $x=7a^2, y=7b^2$ with $a+b=13$. Now $x+y=7(a^2+b^2)$, smallest when a and b are as equal as we can make them. Take $a=6, b=7$ for $7(36+49)=595$. Thus, **B** is the correct answer.

14. A dartboard is the region B in the coordinate plane consisting of points (x, y) such that $|x| + |y| \leq 8$. A target T is the region where $(x^2 + y^2 - 25)^2 \leq 49$. A dart is thrown at a random point in B. The probability that the dart lands in T can be expressed as $\dfrac{m}{n}\pi$, where m and n are relatively prime positive integers. What is $m + n$?

- A 39
- B 71
- C 73
- D 75
- E 135

Solution:

B is the square $|x| + |y| \leq 8$, with area $2 \cdot 8^2 = 128$. The target condition $(x^2 + y^2 - 25)^2 \leq 49$ unpacks to $|x^2 + y^2 - 25| \leq 7$, that is $18 \leq x^2 + y^2 \leq 32$, an annulus of area $\pi(32 - 18) = 14\pi$. Does it fit inside B? The distance from the origin to an edge $x + y = 8$ is $\frac{8}{\sqrt{2}} = 4\sqrt{2} = \sqrt{32}$, exactly the outer radius, so yes, the annulus sits inside the square. The probability is $\frac{14\pi}{128} = \frac{7}{64}\pi$, giving $m + n = 71$. Therefore, the answer is **B**.

15. A list of 9 real numbers consists of $1, 2.2, 3.2, 5.2, 6.2, 7,$ as well as x, y, z with $x \leq y \leq z$. The range of the list is 7, and the mean and median are both positive integers. How many ordered triples (x, y, z) are possible?

A 1

B 2

C 3

D 4

E infinitely many

Solution:

The six fixed numbers total 24.8, so an integer mean needs $x + y + z = 9k - 24.8$ for some positive integer k. A range of 7 pins down the overall min and max (the fixed values already stretch from 1 to 7), and the median is the 5th smallest of the nine numbers, which has to be a positive integer. Grind through where x, y, z can sit and exactly three triples survive: $(0, 5, 6.2)$, $(0.1, 4, 7.1)$, and $(6, 6.2, 8)$. So there are 3. Thus, **C** is the correct answer.

16. Jerry likes to play with numbers. One day, he wrote all the integers from 1 to 2024 on the whiteboard. Then he repeatedly chose four numbers on the whiteboard, erased them, and replaced them with either their sum or their product. (For example, Jerry's first step might have been to erase $1, 2, 3,$ and $5,$ and then write either $11,$ their sum, or $30,$ their product, on the whiteboard.) After repeatedly performing this operation, Jerry noticed that all the remaining numbers on the board were odd. What is the maximum possible number of integers on the board at that time?

A 1010

B 1011

C 1012

D 1013

E 1014

Solution:

Among $1, \ldots, 2024$ there are 1012 even numbers and 1012 odd. Each operation eats 4 numbers and writes back $1,$ so the count falls by $3;$ to keep it high we want as few operations as possible. Every even number has to go, and a move that produces an odd result can clear at most 3 evens at once (one odd plus three evens sums to odd). Clearing all 1012 evens therefore takes at least $\lceil 1012/3 \rceil = 338$ moves. And that's achievable: 337 moves of "one odd $+$ three evens $\rightarrow$ odd sum" wipe out 1011 evens, then one move of "three odds $+$ one even $\rightarrow$ odd sum" gets the last. That leaves $2024 - 3 \cdot 338 = 1010$ numbers. Therefore, the answer is **A**.

17. In a race among 5 snails, there is at most one tie, but that tie can involve any number of snails. For example, the result of the race might be that Dazzler is first; Abby, Cyrus, and Elroy are tied for second, and Bruna is fifth. How many different results of the race are possible?

A 180

B 361

C 420

D 431

E 720

Solution:

If nobody ties, the 5 snails finish in $5! = 120$ orders. Now allow exactly one tied group of size k with $2 \leq k \leq 5.$ Choose the group in $\binom{5}{k}$ ways, then treat it as one block, leaving $6 - k$ blocks to arrange in $(6 - k)!$ ways. Summing over $k: \binom{5}{2}4! + \binom{5}{3}3! + \binom{5}{4}2! + \binom{5}{5}1! = 240 + 60 + 10 + 1 = 311.$ Add the no-tie count: $120 + 311 = 431.$ Thus, **D** is the correct answer.

18. How many different remainders can result when the 100th power of an integer is divided by 125?

A 1

B 2

C 5

D 25

E 125

Solution:

Here $125 = 5^3$ and $\varphi(125) = 100$. If $\gcd(n, 5) = 1$, Euler's theorem gives $n^{100} \equiv 1 \pmod{125}$. And if $5 \mid n$, then n^{100} carries a factor of 5^{100}, hence of 125, so $n^{100} \equiv 0 \pmod{125}$. That leaves only two possible remainders, 0 and 1. Therefore, the answer is **B**.

19. In the following table, each question mark is to be replaced by "Possible" or "Not Possible" to indicate whether a nonvertical line with the given slope can contain the given number of lattice points (points both of whose coordinates are integers). How many of the 12 entries will be "Possible"?

	zero	exactly one	exactly two	more than two
zero slope	?	?	?	?
nonzero rational slope	?	?	?	?
irrational slope	?	?	?	?

A 4

B 5

C 6

D 7

E 9

Solution:

Any two lattice points give a rational slope. So a line with irrational slope holds at most one lattice point: it can have 0 (say $y = \sqrt{2}\,x + \frac{1}{2}$) or exactly 1 (say $y = \sqrt{2}\,x$), never two. A line with rational slope (zero included) through a lattice point (x_0, y_0) also passes through $(x_0 + q, y_0 + p)$ for its reduced slope $\frac{p}{q}$, so it hits infinitely many; such a line has either 0 lattice points (shift it by an irrational intercept) or more than two, never exactly one or two. So each row gives exactly two "Possible" entries. For zero and nonzero rational slope those are the "zero" and "more than two" columns; for irrational slope, the "zero" and "exactly one" columns. That's 6 in all. Thus, **C** is the correct answer.

20. Three different pairs of shoes are placed in a row so that no left shoe is next to a right shoe from a different pair. In how many ways can these six shoes be lined up?

A 60

B 72

C 90

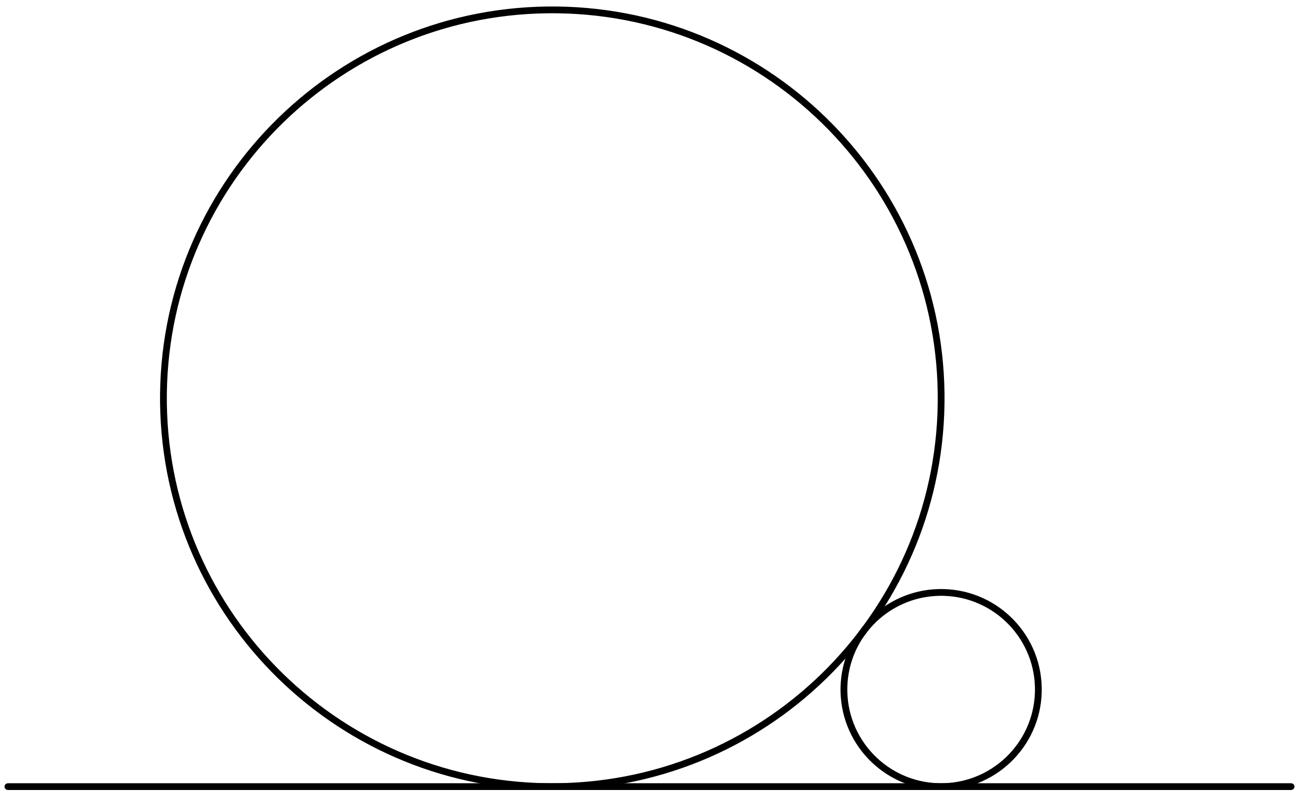
D 108

E 120

Solution:

Scan the row: wherever a left shoe touches a right shoe, they have to be mates. Look at the pattern of sides (L or R) across the six spots; every switch between L and R must sit at a matched pair. Two patterns keep all lefts together then all rights, $LLLRRR$ and $RRRLLL$. Each has a single switch, so pick the mated pair there (3 ways) and order the other two lefts (2) and two rights (2): 12 each. The other allowed patterns $LLRRRL, LRRLLR, LRRRLL, RLLLRR, RLLRRL, RRLLLR$ give 6 arrangements apiece. Altogether $2 \cdot 12 + 6 \cdot 6 = 60$. Therefore, the answer is **A**.

21. Two straight pipes (circular cylinders), with radii 1 and $\frac{1}{4}$, lie parallel and in contact on a flat floor. The figure below shows a head-on view. What is the sum of the possible radii of a third parallel pipe lying on the same floor and in contact with both?



A $\frac{1}{9}$

B 1

C $\frac{10}{9}$

D $\frac{11}{9}$

E $\frac{19}{9}$

Solution:

Two circles of radii R and r resting on the floor and touching each other have contact points a horizontal distance $2\sqrt{Rr}$ apart. So the radius-1 and radius-$\frac{1}{4}$ pipes touch the floor $2\sqrt{1 \cdot \frac{1}{4}} = 1$ apart. A third pipe of radius r sits $2\sqrt{r}$ from the big pipe's contact point and $2\sqrt{\frac{1}{4}r} = \sqrt{r}$ from the small pipe's. Nestled between them, $2\sqrt{r} + \sqrt{r} = 1$, so $\sqrt{r} = \frac{1}{3}$ and $r = \frac{1}{9}$. Sitting past the small pipe, $2\sqrt{r} - \sqrt{r} = 1$, so $r = 1$. (Past the big pipe can't happen.) The sum is $\frac{1}{9} + 1 = \frac{10}{9}$. Thus, **C** is the correct answer.

22. A group of 16 people will be partitioned into 4 indistinguishable 4-person committees. Each committee will have one chairperson and one secretary. The number of different ways to make these assignments can be written as $3^r M$, where r and M are positive integers and M is not divisible by 3. What is r?

- A 5
- B 6
- C 7
- D 8
- E 9

Solution:

Split 16 people into 4 indistinguishable groups of 4 in $\dfrac{16!}{(4!)^4\,4!}$ ways, then each committee picks a chairperson and a secretary in $4 \cdot 3 = 12$ ways, a factor of 12^4. Now count factors of 3. In $16!$ there are $\lfloor 16/3 \rfloor + \lfloor 16/9 \rfloor = 6$; the denominator $(4!)^4\,4!$ contributes $4 \cdot 1 + 1 = 5$; and 12^4 contributes 4. The exponent is $6 - 5 + 4 = 5$, so $r = 5$. Therefore, the answer is **A**.

23. The Fibonacci numbers are defined by $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. What is

$$\frac{F_2}{F_1} + \frac{F_4}{F_2} + \frac{F_6}{F_3} + \cdots + \frac{F_{20}}{F_{10}}?$$

A 318

B 319

C 320

D 321

E 322

Solution:

Use $F_{2k} = F_k L_k$, so each term $\dfrac{F_{2k}}{F_k} = L_k$, the kth Lucas number. That collapses the sum to $\sum_{k=1}^{10} L_k$. With $L_1 = 1, L_2 = 3, L_3 = 4, \ldots, L_{10} = 123$, the identity $\sum_{k=1}^{n} L_k = L_{n+2} - 3$ gives $L_{12} - 3 = 322 - 3 = 319$. Thus, **B** is the correct answer.

24. Let

$$P(m) = \frac{m}{2} + \frac{m^2}{4} + \frac{m^4}{8} + \frac{m^8}{8}.$$

How many of the values of $P(2022), P(2023), P(2024)$, and $P(2025)$ are integers?

A 0

B 1

C 2

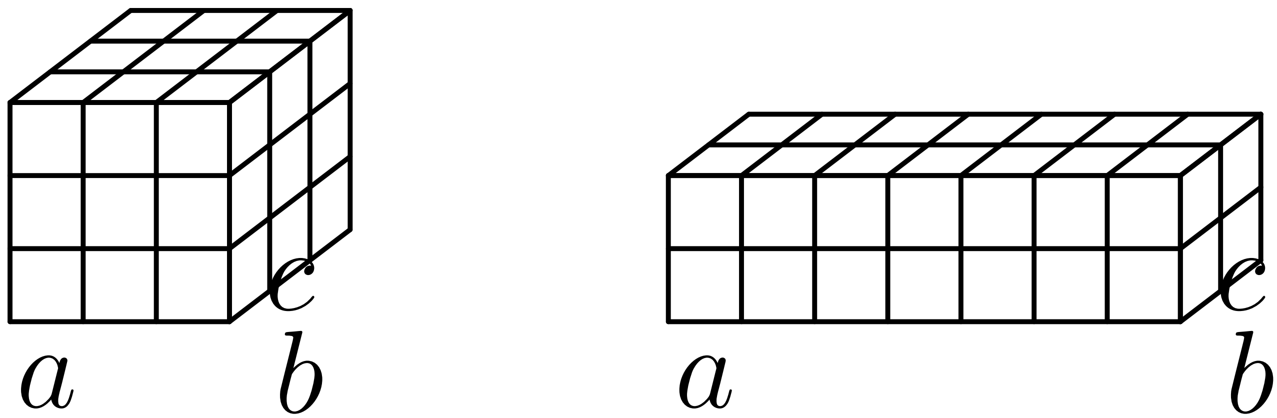
D 3

E 4

Solution:

Put everything over $8 : P(m) = \dfrac{m^8 + m^4 + 2m^2 + 4m}{8}$. If m is even, every term up top is divisible by 8. If m is odd, then $m^2 \equiv m^4 \equiv m^8 \equiv 1$ and $4m \equiv 4 \pmod{8}$, so the numerator is $1 + 1 + 2 + 4 = 8 \equiv 0 \pmod{8}$. Either way $P(m)$ is an integer, so all 4 values are integers. Therefore, the answer is **E**.

25. Each of 27 bricks (right rectangular prisms) has dimensions $a \times b \times c$, where $a, b,$ and c are pairwise relatively prime positive integers. These bricks are arranged to form a $3 \times 3 \times 3$ block, as shown on the left below. A 28th brick with the same dimensions is introduced, and these bricks are reconfigured into a $2 \times 2 \times 7$ block, shown on the right. The new block is 1 unit taller, 1 unit wider, and 1 unit deeper than the old one. What is $a + b + c$?



- A 88
- B 89
- C 90
- D 91
- E 92

Solution:

The $3 \times 3 \times 3$ block has sides $3a, 3b, 3c$. The $2 \times 2 \times 7$ block has sides $2u, 2v, 7w$, where (u, v, w) is some permutation of (a, b, c). Each new side beats its old counterpart by 1, so the multiset $\{3a + 1, 3b + 1, 3c + 1\}$ equals $\{2u, 2v, 7w\}$. Try matching the 7-side to $3(\text{one value}) + 1$. With $\{a, b, c\} = \{19, 29, 44\}$ it all lines up: $7 \cdot 19 = 133 = 3 \cdot 44 + 1, 2 \cdot 29 = 58 = 3 \cdot 19 + 1$, and $2 \cdot 44 = 88 = 3 \cdot 29 + 1$. Those are pairwise coprime, so $a + b + c = 19 + 29 + 44 = 92$. Thus, **E** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2024B