2024 AMC 10A Solutions

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1. What is the value of $9901 \cdot 101 - 99 \cdot 10101$?

A 2

B 20

C 21

D 200

E 2020

Solution:

Just compute each piece. We have $9901 \cdot 101 = 990100 + 9901 = 1000001$, and $99 \cdot 10101 = 999999$. Subtracting, $1000001 - 999999 = 2$. Thus, **A** is the correct answer.

2. A model used to estimate the time it will take to hike to the top of a mountain on a trail is of the form $T = aL + bG$, where a and b are constants, T is the time in minutes, L is the length of the trail in miles, and G is the altitude gain in feet. The model estimates that it will take 69 minutes to hike to the top if a trail is 1.5 miles long and ascends 800 feet, as well as if a trail is 1.2 miles long and ascends 1100 feet. How many minutes does the model estimate it will take to hike to the top if the trail is 4.2 miles long and ascends 4000 feet?

- A 240
- B 246
- C 252
- D 258
- E 264

Solution:

Subtract the two equations $1.5a + 800b = 69$ and $1.2a + 1100b = 69$ to kill the 69. That leaves $0.3a - 300b = 0$, so $a = 1000b$. Now substitute: $1500b + 800b = 2300b = 69$, so $b = 0.03$ and $a = 30$. Then $T = 30(4.2) + 0.03(4000) = 126 + 120 = 246$. Therefore, the answer is **B**.

3. What is the sum of the digits of the smallest prime that can be written as a sum of 5 distinct primes?

- A 5
- B 7
- C 9
- D 10
- E 11

Solution:

Suppose 2 is one of the five primes. Then the total is even and bigger than $2,$ so it's composite. That means all five primes must be odd. The five smallest odd primes give $3+5+7+11+13=39=3\cdot 13,$ which isn't prime. We can't hit 41 with five distinct odd primes, but $3+5+7+11+17=43$ is prime. So the smallest such prime is $43,$ and its digit sum is $4+3=7.$ Thus, **B** is the correct answer.

4. The number 2024 is written as the sum of not necessarily distinct two-digit numbers. What is the least number of two-digit numbers needed to write this sum?

- A 20
- B 21
- C 22
- D 23
- E 24

Solution:

Each two-digit number is at most 99, so k of them sum to at most $99k$. We need $99k \geq 2024$, which forces $k \geq 20.4$, so $k \geq 21$. And 21 really works: twenty 99's plus one 44 give $1980 + 44 = 2024$. Therefore, the answer is **B**.

5. What is the least value of n such that $n!$ is a multiple of 2024?

- A 11
- B 21
- C 22
- D 23
- E 253

Solution:

Factor $2024 = 2^3 \cdot 11 \cdot 23$. The prime 23 is the bottleneck: for 23 to divide $n!$, we need $n \geq 23$. At $n = 23$, the product $23!$ already has $23, 11,$ and plenty of factors of 2, so $2024 \mid 23!$. The least value is 23. Thus, **D** is the correct answer.

6. What is the minimum number of successive swaps of adjacent letters in the string ABCDEF that are needed to change the string to FEDCBA?

(For example, 3 swaps are required to change ABC to CBA; one such sequence of swaps is ABC $\to$ BAC $\to$ BCA $\to$ CBA.)

A 6

B 10

C 12

D 15

E 24

Solution:

Reversing all six letters flips the relative order of every pair, so all $\binom{6}{2} = 15$ pairs end up inverted. Each adjacent swap fixes exactly one inversion. So we need at least 15 swaps, and bubbling each letter into place hits 15 exactly. Therefore, the answer is **D**.

7. The product of three integers is $60.$ What is the least possible positive sum of the three integers?

A 2

B 3

C 5

D 6

E 13

Solution:

To keep the sum small but positive, pair two negatives with one big positive. Take $(-1)(-6)(10) = 60,$ which sums to $3.$ Run through the other factorizations of 60 and none does better. So the least positive sum is $3.$ Thus, **B** is the correct answer.

8. Amy, Bomani, Charlie, and Daria work in a chocolate factory. On Monday Amy, Bomani, and Charlie started working at 1:00 PM and were able to pack $4, 3,$ and 3 packages, respectively, every 3 minutes. At some later time, Daria joined the group, and Daria was able to pack 5 packages every 4 minutes. Together, they finished packing 450 packages at exactly 2:45 PM. At what time did Daria join the group?

A 1:25 PM

B 1:35 PM

C 1:45 PM

D 1:55 PM

E 2:05 PM

Solution:

From 1:00 to 2:45 is 105 minutes. Amy, Bomani, and Charlie pack $4+3+3=10$ packages every 3 minutes, so $\frac{10}{3}$ per minute, which is $\frac{10}{3} \cdot 105 = 350$ packages. That leaves $450 - 350 = 100$ for Daria, who packs $\frac{5}{4}$ per minute and so needs $100/\frac{5}{4} = 80$ minutes. She worked the last 80 minutes, joining $105 - 80 = 25$ minutes after 1:00. That's 1:25 PM. Therefore, the answer is **A**.

9. In how many ways can 6 juniors and 6 seniors form 3 disjoint teams of 4 people so that each team has 2 juniors and 2 seniors?

A 720

B 1350

C 2700

D 3280

E 8100

Solution:

Split the 6 juniors into three unordered pairs. There are $\frac{6!}{2!^3 3!} = 15$ ways, and the same 15 for the seniors. Each team is one junior-pair paired with one senior-pair, so we match the three junior-pairs to the three senior-pairs in $3! = 6$ ways. That's $15 \cdot 15 \cdot 6 = 1350$ sets of teams. Thus, **B** is the correct answer.

10. Consider the following operation. Given a positive integer n, if n is a multiple of 3, then you replace n by $\frac{n}{3}$. If n is not a multiple of 3, then you replace n by $n + 10$. Then continue this process. For example, beginning with $n = 4$, this procedure gives $4 \rightarrow 14 \rightarrow 24 \rightarrow 8 \rightarrow 18 \rightarrow 6 \rightarrow 2 \rightarrow 12 \rightarrow \cdots$.

Suppose you start with $n = 100$. What value results if you perform this operation exactly 100 times?

- A 10
- B 20
- C 30
- D 40
- E 50

Solution:

Just run it from $100 : 100 \rightarrow 110 \rightarrow 120 \rightarrow 40 \rightarrow 50 \rightarrow 60 \rightarrow 20 \rightarrow 30 \rightarrow 10 \rightarrow 20 \rightarrow 30 \rightarrow 10 \rightarrow \cdots$. After the 8th step we're at 10, and from there it cycles $10, 20, 30$ with period 3. So step $8 + k$ is the kth entry of the cycle. For step 100, $k = 92$, and $92 \equiv 2 \pmod 3$, which lands on 30. Therefore, the answer is **C**.

11. How many ordered pairs of integers (m, n) satisfy

$$\sqrt{n^2 - 49} = m?$$

A 1

B 2

C 3

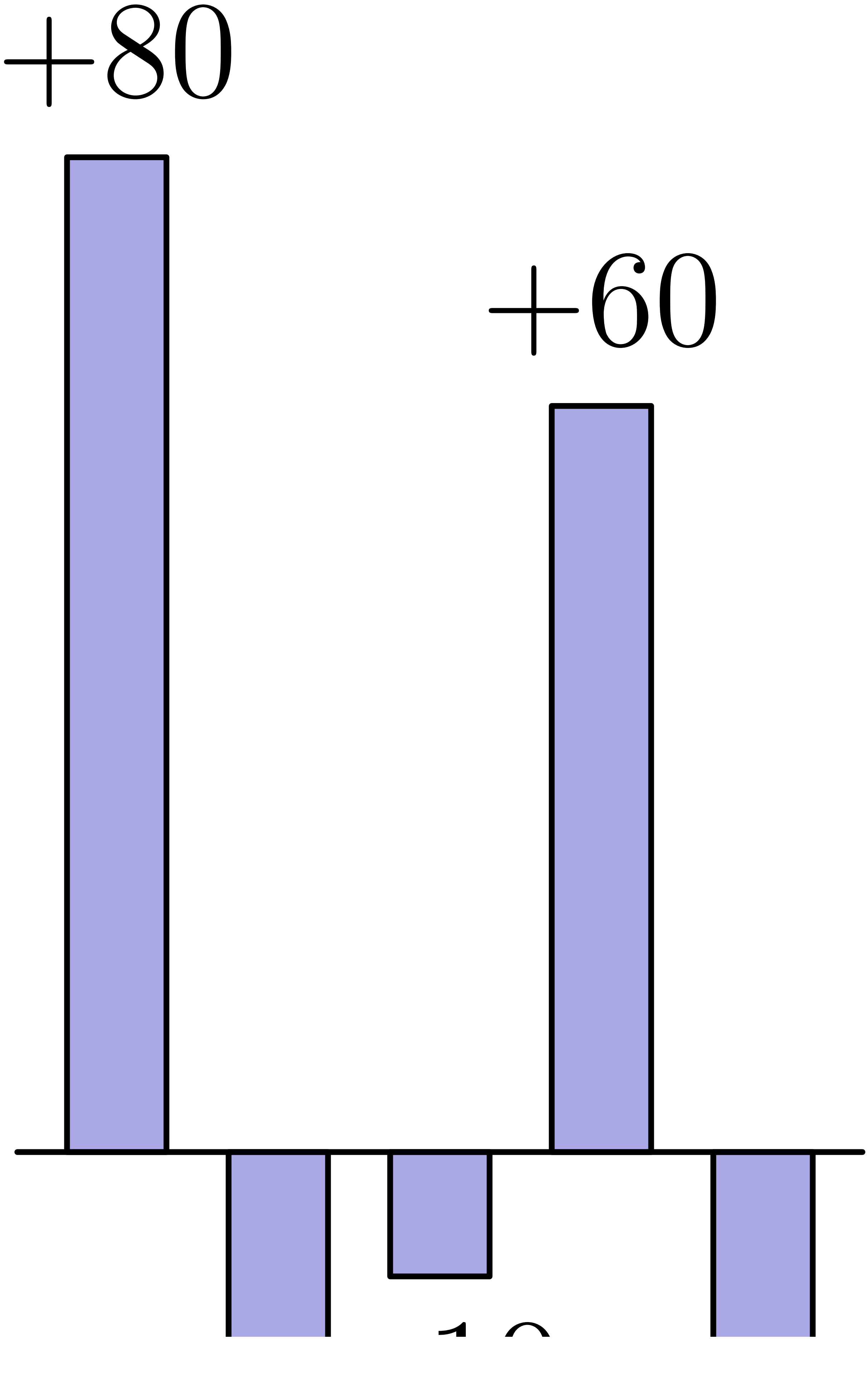
D 4

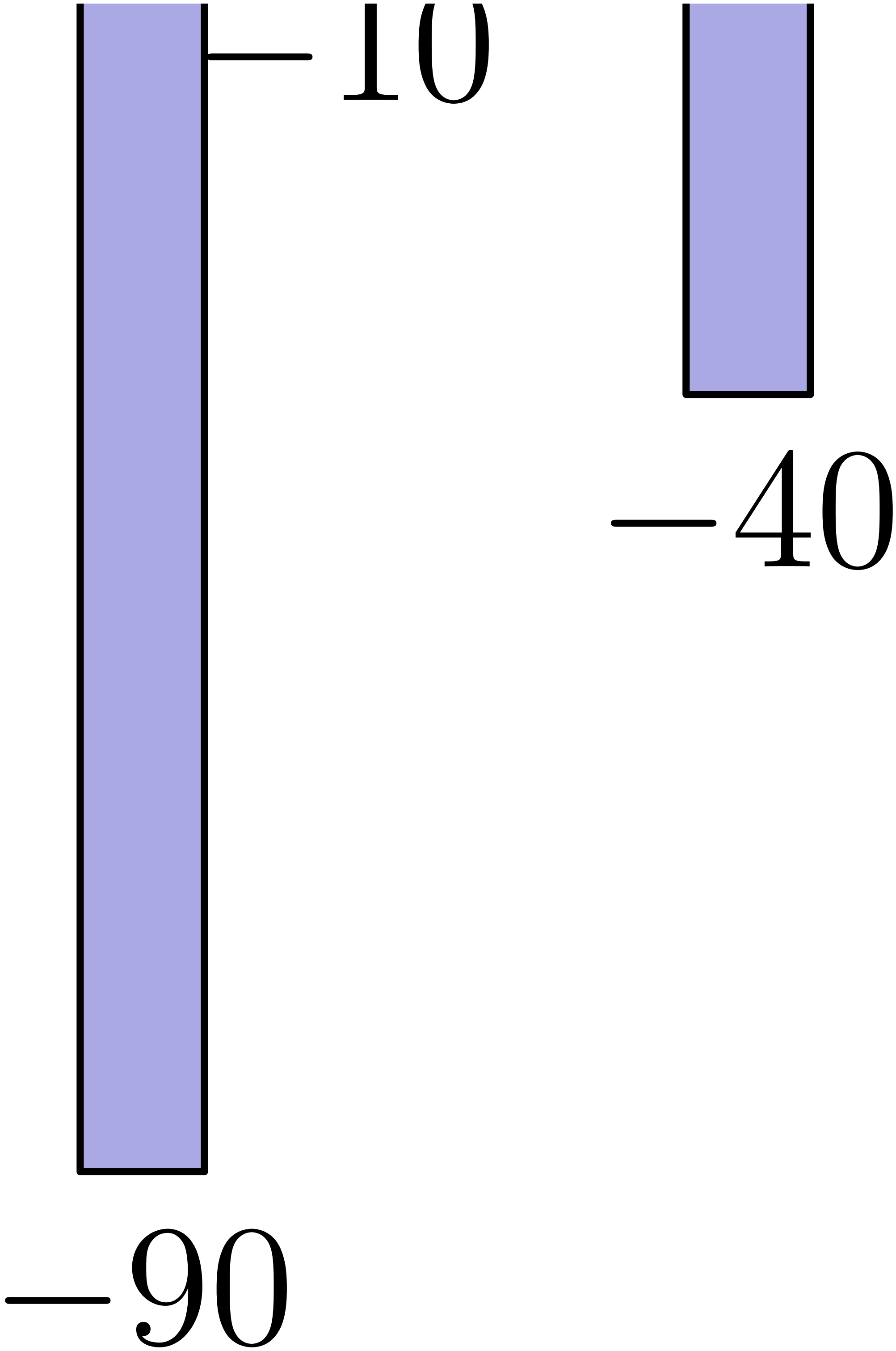
E Infinitely many

Solution:

Note $m = \sqrt{n^2 - 49} \geq 0$ has to be an integer, so $n^2 - 49 = m^2$, which means $(n - m)(n + m) = 49$. The factorizations of 49 give $|n| = 25, m = 24$ or $|n| = 7, m = 0$. So the ordered pairs (m, n) are $(24, 25), (24, -25), (0, 7), (0, -7)$. That's 4 of them. Thus, **D** is the correct answer.

12. Zelda played the *Adventures of Math* game on August 1 and scored 1700 points. She continued to play daily over the next 5 days. The bar chart below shows the daily change in her score compared to the day before. (For example, Zelda's score on August 2 was $1700 + 80 = 1780$ points.) What was Zelda's average score in points over the 6 days?


+80
+60



A 1700

B 1702

C 1703

D 1713

E 1715

Solution:

Apply the daily changes $+80, -90, -10, +60, -40$ to the starting $1700.$ The six scores are $1700, 1780, 1690, 1680, 1740, 1700.$ They add to $10290,$ so the average is $10290/6 = 1715.$ Therefore, the answer is **E**.

13. Two transformations are said to commute if applying the first followed by the second gives the same result as applying the second followed by the first. Consider these four transformations of the coordinate plane:

- a translation 2 units to the right,
- a 90° rotation counterclockwise about the origin,
- a reflection across the x-axis, and
- a dilation centered at the origin with scale factor 2.

Of the 6 pairs of distinct transformations from this list, how many commute?

A 1

B 2

C 3

D 4

E 5

Solution:

The dilation just scales about the origin, so it commutes with both the rotation and the reflection. That's 2 pairs. The translation commutes with the reflection across the x-axis too, since either order sends $(x, y) \to (x + 2, -y)$. The other three pairs fail: the translation clashes with the rotation and with the dilation, and the rotation clashes with the reflection. So 3 pairs commute. Thus, **C** is the correct answer.

14. One side of an equilateral triangle of height 24 lies on line ℓ. A circle of radius 12 is tangent to ℓ and is externally tangent to the triangle. The area of the region exterior to the triangle and the circle and bounded by the triangle, the circle, and line ℓ can be written as $a\sqrt{b} - c\pi$, where $a, b,$ and c are positive integers and b is not divisible by the square of any prime. What is $a + b + c$?

A 72

B 73

C 74

D 75

E 76

Solution:

The equilateral triangle has side $16\sqrt{3}$. Put ℓ on the x-axis with base vertex $V = (16\sqrt{3}, 0)$; the slanted side then lies on $\sqrt{3}\,x + y = 48$. The circle sits on ℓ (center height 12) and touches that side from outside, so its center is $(20\sqrt{3}, 12)$ and it meets ℓ at $T = (20\sqrt{3}, 0)$. Our region is bounded by the two tangent segments out of V (one along ℓ, one along the triangle's side) and the near arc. The tangent length is $VT = 4\sqrt{3}$, so the kite $V\text{-}T\text{-center-}P$ has area $4\sqrt{3} \cdot 12 = 48\sqrt{3}$. The angle at V is 120°, so the removed sector is 60°, with area $\frac{1}{6}\pi(12)^2 = 24\pi$. The region is $48\sqrt{3} - 24\pi$, giving $a + b + c = 48 + 3 + 24 = 75$. Therefore, the answer is **D**.

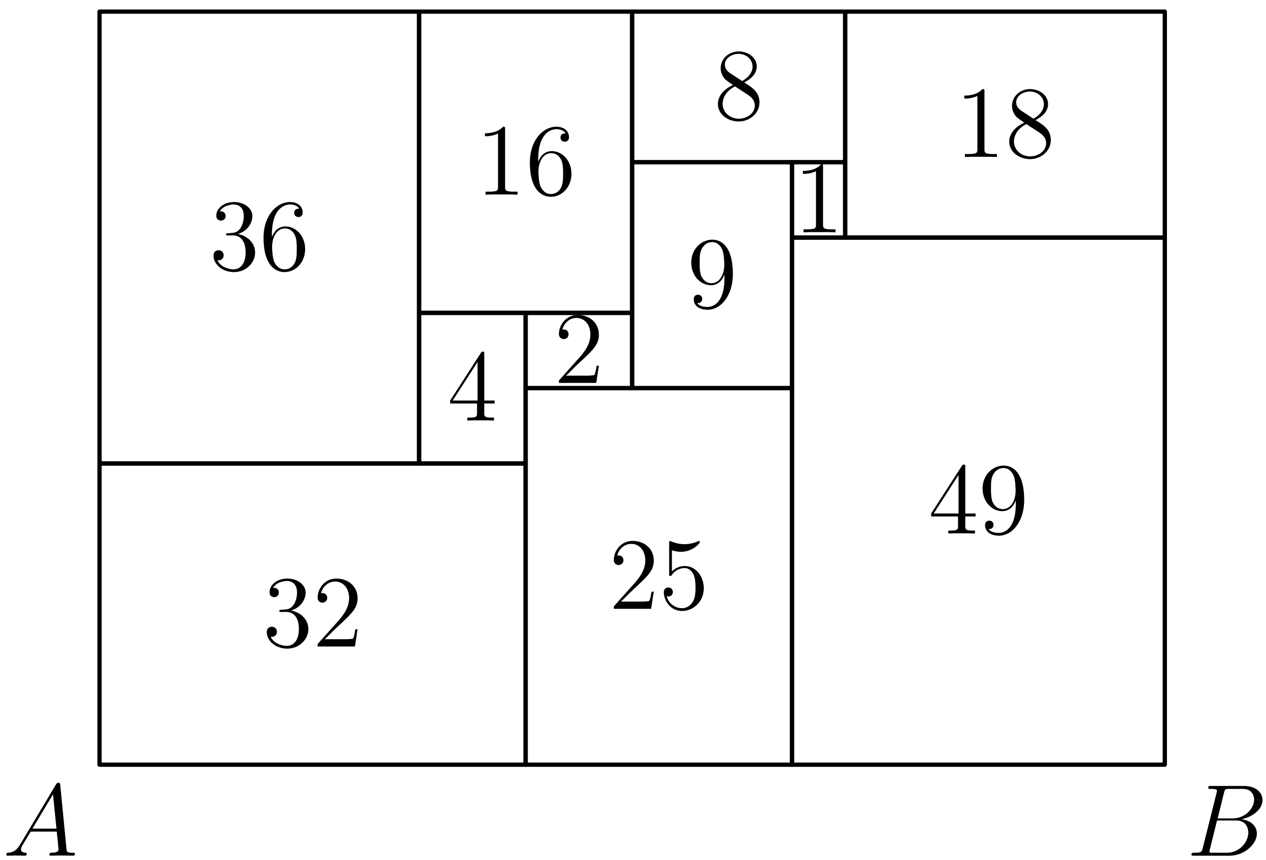
15. Let M be the greatest integer such that both $M + 1213$ and $M + 3773$ are perfect squares. What is the units digit of M?

- A 1
- B 2
- C 3
- D 6
- E 8

Solution:

Set $M + 1213 = y^2$ and $M + 3773 = x^2$. Subtracting, $x^2 - y^2 = 2560$, so $(x - y)(x + y) = 2560$. The two factors share a parity, and their product is even, so both are even: write $x - y = 2s, x + y = 2t,$ with $st = 640.$ To make M as large as possible we want $y = t - s$ as large as possible, so s as small as possible. Take $s = 1, t = 640,$ giving $y = 639.$ Then $M = 639^2 - 1213 = 407108,$ whose units digit is $8.$ Thus, **E** is the correct answer.

16. All of the rectangles in the figure below, which is drawn to scale, are similar to the enclosing rectangle. Each number represents the area of the rectangle. What is length AB?



A $4 + 4\sqrt{5}$

B $10\sqrt{2}$

C $5 + 5\sqrt{5}$

D $10\sqrt[4]{8}$

E 20

Solution:

Every piece is similar to the whole rectangle, so they all share one aspect ratio. The areas $1, 2, 4, 8, 16$ (and $9, 18$) come in factor-of-2 steps, and cutting a rectangle of aspect ratio $\sqrt{2}$ across its long side gives two similar copies of half the area. That pins the ratio at $\sqrt{2}$. The total area is $36 + 16 + 8 + 18 + 1 + 9 + 4 + 2 + 32 + 25 +$

$49 = 200$. The enclosing rectangle satisfies $AB \cdot \frac{AB}{\sqrt{2}} = 200$, so $AB^2 = 200\sqrt{2}$ and $AB = \sqrt{200\sqrt{2}} = 10\sqrt[4]{8}$. Therefore, the answer is **D**.

17. Two teams are in a best-two-out-of-three playoff: the teams will play at most 3 games, and the winner of the playoff is the first team to win 2 games. The first game is played on Team A's home field, and the remaining games are played on Team B's home field. Team A has a $\frac{2}{3}$ chance of winning at home, and its probability of winning when playing away from home is p. Outcomes of the games are independent. The probability that Team A wins the playoff is $\frac{1}{2}$. Then p can be written in the form $\frac{1}{2}(m - \sqrt{n})$, where m and n are positive integers. What is $m + n$?

A 10

B 11

C 12

D 13

E 14

Solution:

Team A takes game 1 at home with probability $\frac{2}{3}$, and each away game with probability p. It can win the playoff three disjoint ways: win games $1, 2$; win 1, lose 2, win 3; lose 1, win $2, 3$. Adding those, $\frac{2}{3}p + \frac{2}{3}(1 - p)p + \frac{1}{3}p^2 = \frac{1}{2}$. This cleans up to $2p^2 - 8p + 3 = 0$, so $p = \frac{4-\sqrt{10}}{2} = \frac{1}{2}\left(4 - \sqrt{10}\right)$. Then $m = 4, n = 10$, and $m + n = 14$. Thus, **E** is the correct answer.

18. There are exactly K positive integers b with $5 \leq b \leq 2024$ such that the base-b integer 2024_b is divisible by 16 (where 16 is in base ten). What is the sum of the digits of K?

A 16

B 17

C 18

D 20

E 21

Solution:

In base b, $2024_b = 2b^3 + 2b + 4 = 2(b^3 + b + 2)$, so $16 \mid 2024_b$ exactly when $8 \mid b^3 + b + 2$. Test the residues modulo 8 : this holds precisely for $b \equiv 3, 6, 7 \pmod{8}$. Counting the b with $5 \leq b \leq 2024$ in those three classes gives $K = 758$, whose digit sum is $7 + 5 + 8 = 20$. Therefore, the answer is **D**.

19. The first three terms of a geometric sequence are the integers $a, 720,$ and b, where $a < 720 < b$. What is the sum of the digits of the least possible value of b?

A 9

B 12

C 16

D 18

E 21

Solution:

Since $720^2 = ab$, the common ratio $r = \frac{720}{a} = \frac{b}{720}$ is rational. Write $r = \frac{p}{q}$ in lowest terms with $p > q$. Then $a = \frac{720q}{p}$ and $b = \frac{720p}{q}$ are integers, which forces $p \mid 720$ and $q \mid 720$. To make b smallest, we want the smallest ratio $\frac{p}{q} > 1$ with both $p, q \mid 720$, which is $\frac{16}{15}$. That gives $b = 720 \cdot \frac{16}{15} = 768$ (and $a = 675$). The digit sum is $7 + 6 + 8 = 21$. Thus, **E** is the correct answer.

20. Let S be a subset of $\{1, 2, 3, \ldots, 2024\}$ such that the following two conditions hold:

- If x and y are distinct elements of S, then $|x - y| > 2.$
- If x and y are distinct odd elements of S, then $|x - y| > 6.$

What is the maximum possible number of elements in S?

A 436

B 506

C 608

D 654

E 675

Solution:

The two conditions say chosen numbers are at least 3 apart, and chosen odd numbers at least 7 apart. Try the pattern $1, 4, 8, 11, 14, 18, \ldots$ (residues $1, 4, 8 \pmod{10}$). Every gap is ≥ 3, and each block of 10 holds exactly one odd number, so the odds stay 10 apart. That's 3 numbers per 10. Now $\{1, \ldots, 2024\}$ is 202 full blocks plus $2021, 2024$, so the count is $202 \cdot 3 + 2 = 608$. Each block of 10 can hold at most 3 elements, so we can't do better. Therefore, the answer is **C**.

21. The numbers, in order, of each row and the numbers, in order, of each column of a 5×5 array of integers form an arithmetic progression of length 5. The numbers in positions $(5,5), (2,4), (4,3)$, and $(3,1)$ are $0, 48, 16$, and 12, respectively. What number is in position $(1,2)$?

$$\begin{bmatrix} \cdot & ? & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 48 & \cdot \\ 12 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 16 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

A 19

B 24

C 29

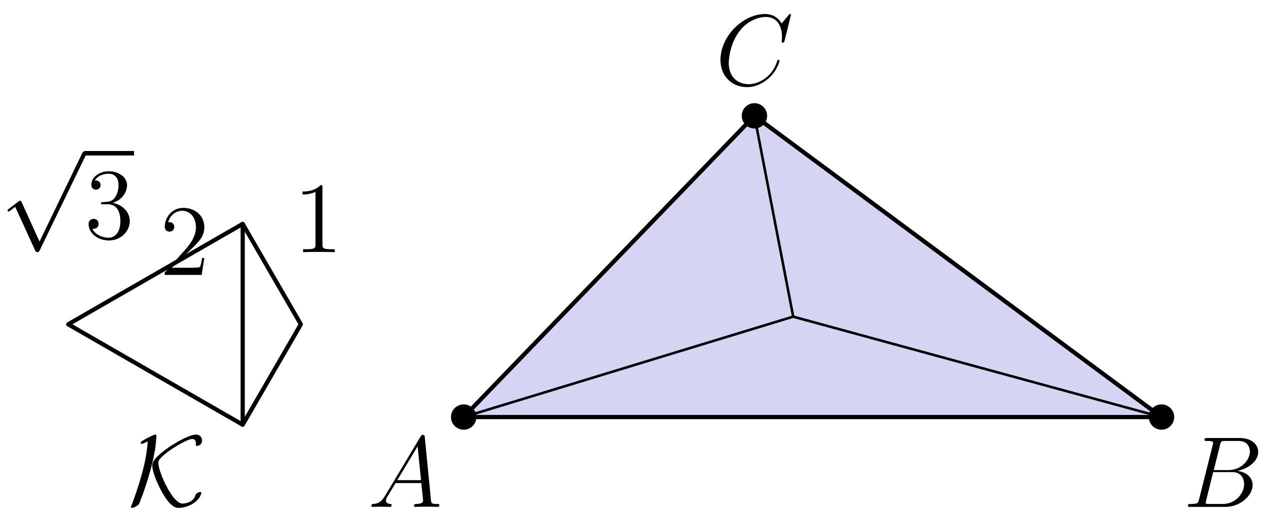
D 34

E 39

Solution:

If every row and every column is an arithmetic progression, the entry at row i, column j must take the bilinear form $f(i,j) = A + Bi + Cj + Dij$. Plug in $f(5,5) = 0$, $f(2,4) = 48$, $f(4,3) = 16$, $f(3,1) = 12$ and solve: $A = -10, B = 5, C = 22$, $D = -5$. So position $(1,2)$ is $-10 + 5 + 2 \cdot 22 - 2 \cdot 5 = 29$. Thus, **C** is the correct answer.

22. Let $\mathcal{K}$ be the kite formed by joining two right triangles with legs 1 and $\sqrt{3}$ along a common hypotenuse. Eight copies of $\mathcal{K}$ are used to form the polygon shown below. What is the area of triangle ABC?



A $2+3\sqrt{3}$

B $\frac{9}{2}\sqrt{3}$

C $\frac{10+8\sqrt{3}}{3}$

D 8

E $5\sqrt{3}$

Solution:

Each kite is two 30-60-90 triangles with legs 1 and $\sqrt{3}$ and hypotenuse 2. Trace the eight-kite figure in coordinates and the outer vertices come out to $A=(0,0)$, $B=(6,0)$, and $C=\left(\frac{5}{2},\frac{3\sqrt{3}}{2}\right)$. So triangle ABC has base $AB=6$ and height $\frac{3\sqrt{3}}{2}$, and its area is $\frac{1}{2}\cdot 6\cdot\frac{3\sqrt{3}}{2}=\frac{9\sqrt{3}}{2}$. Therefore, the answer is **B**.

23. Integers $a, b,$ and c satisfy

$$ab + c = 100, \quad bc + a = 87, \quad ca + b = 60.$$

What is $ab + bc + ca$?

- A 212
- B 247
- C 258
- D 276
- E 284

Solution:

Add the three equations: $(ab + bc + ca) + (a + b + c) = 247$. Now subtract them in pairs, which factors nicely as $(a - c)(b - 1) = 13$, $(b - a)(c - 1) = 27$, and $(b - c)(a - 1) = 40$. These pin down $(a, b, c) = (-9, -12, -8)$, so $a + b + c = -29$. Then $ab + bc + ca = 247 - (-29) = 276$. Thus, **D** is the correct answer.

24. A bee is moving in three-dimensional space. A fair six-sided die with faces labeled A^+, A^-, B^+, B^-, C^+, and C^- is rolled. Suppose the bee occupies the point (a, b, c). If the die shows A^+, then the bee moves to the point $(a+1, b, c)$, and if the die shows A^-, then the bee moves to the point $(a-1, b, c)$. Analogous moves are made with the other four outcomes.

Suppose the bee starts at the point $(0, 0, 0)$ and the die is rolled four times. What is the probability that the bee traverses four distinct edges of some unit cube?

A $\frac{1}{54}$

B $\frac{7}{54}$

C $\frac{1}{6}$

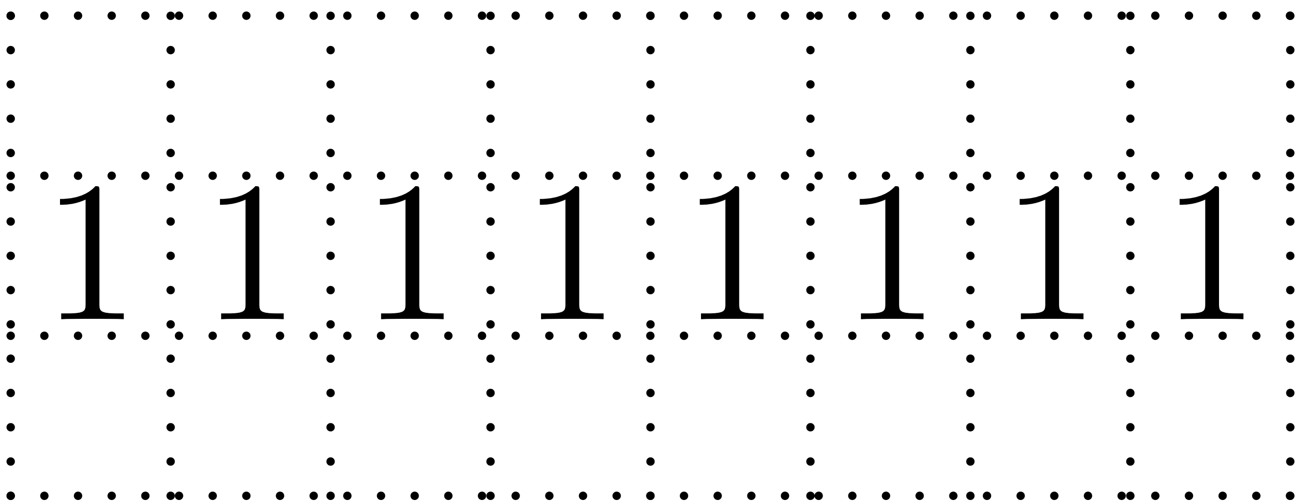
D $\frac{5}{18}$

E $\frac{2}{5}$

Solution:

Every roll moves the bee one unit along $\pm x, \pm y$, or $\pm z$, so there are $6^4 = 1296$ equally likely move sequences. A sequence works exactly when its four unit steps are four distinct edges of one unit cube, meaning the bee stays on a single cube and never repeats an edge. Enumerating these gives 168 favorable sequences, so the probability is $\frac{168}{1296} = \frac{7}{54}$. Therefore, the answer is **B**.

25. The figure below shows a dotted grid 8 cells wide and 3 cells tall consisting of $1'' \times 1''$ squares. Carl places 1-inch toothpicks along some of the sides of the squares to create a closed loop that does not intersect itself. The numbers in the cells indicate the number of sides of that square that are to be covered by toothpicks, and any number of toothpicks are allowed if no number is written. In how many ways can Carl place the toothpicks?



A 130

B 144

C 146

D 162

E 196

Solution:

Label each unit square "inside" or "outside" the loop, counting the grid's exterior as outside. The loop is then exactly the set of unit edges that separate an inside square from an outside one. A square's number counts how many of its four neighbors (left, right, up, down, with a missing neighbor being the outside exterior) are of the opposite type. So the requirement is that every middle-row square has exactly one opposite-type neighbor. Enumerate the inside/outside labelings whose boundary is a single non-self-intersecting closed loop and that meet this middle-row condition: there are 146 of them. Thus, **C** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2024A