2023 AMC 10B Solutions

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1. Mrs. Jones is pouring orange juice into four identical glasses for her four sons. She fills the first three glasses completely full but runs out of juice when the fourth glass is only $\frac{1}{3}$ full. What fraction of a glass must Mrs. Jones pour from each of the first three glasses into the fourth glass so that all four glasses will have the same amount of juice?

A $\dfrac{1}{12}$

B $\dfrac{1}{4}$

C $\dfrac{1}{6}$

D $\dfrac{1}{8}$

E $\dfrac{2}{9}$

Solution:

There's $3 + \frac{1}{3} = \frac{10}{3}$ glasses of juice in all. Split four ways, each glass ends up with $\frac{10}{3} \div 4 = \frac{5}{6}$. So a full glass has to pour out $1 - \frac{5}{6} = \frac{1}{6}$. Thus, **C** is the correct answer.

2. Carlos went to a sports store to buy running shoes. Running shoes were on sale, with prices reduced by 20% on every pair of shoes. Carlos also knew that he had to pay a 7.5% sales tax on the discounted price. He had $43. What is the original (before discount) price of the most expensive shoes he could afford to buy?

- A $\$46$
- B $\$50$
- C $\$48$
- D $\$47$
- E $\$49$

Solution:

Let P be the original price. After the discount and tax, Carlos pays $0.8P \times 1.075 = 0.86P$. He can afford it when $0.86P \leq 43$, which means $P \leq 50$. So the priciest shoes he can swing start at $\$50$. Therefore, the answer is **B**.

3. A 3-4-5 right triangle is inscribed in circle A, and a 5-12-13 right triangle is inscribed in circle B. What is the ratio of the area of circle A to the area of circle B?

A $\frac{9}{25}$

B $\frac{1}{9}$

C $\frac{1}{5}$

D $\frac{25}{169}$

E $\frac{4}{25}$

Solution:

The hypotenuse of a right triangle is a diameter of the circle around it. So circle A has diameter 5 and circle B has diameter 13. Areas scale as the square of that, giving $\left(\frac{5}{13}\right)^2 = \frac{25}{169}$. Thus, **D** is the correct answer.

4. Jackson's paintbrush makes a narrow strip with a width of 6.5 millimeters. Jackson has enough paint to make a strip 25 meters long. How many square centimeters of paper could Jackson cover with paint?

- A $162{,}500$
- B $162{,}5$
- C $1{,}625$
- D $1{,}625{,}000$
- E $16{,}250$

Solution:

Put everything in centimeters first. The strip is 6.5 mm $= 0.65$ cm wide and 25 m $=$ 2500 cm long. Its area is $0.65 \times 2500 = 1625$ square centimeters. Therefore, the answer is **C**.

5. Maddy and Lara see a list of numbers written on a blackboard. Maddy adds 3 to each number in the list and finds that the sum of her new numbers is $45.$ Lara multiplies each number in the list by 3 and finds that the sum of her new numbers is also $45.$ How many numbers are written on the blackboard?

A 10

B 5

C 6

D 8

E 9

Solution:

Let S be the original sum and n the number of entries. Maddy adds 3 to each, so her total is $S + 3n = 45.$ Lara triples each, so hers is $3S = 45,$ giving $S = 15.$ Then $15 + 3n = 45,$ so $n = 10.$ Thus, **A** is the correct answer.

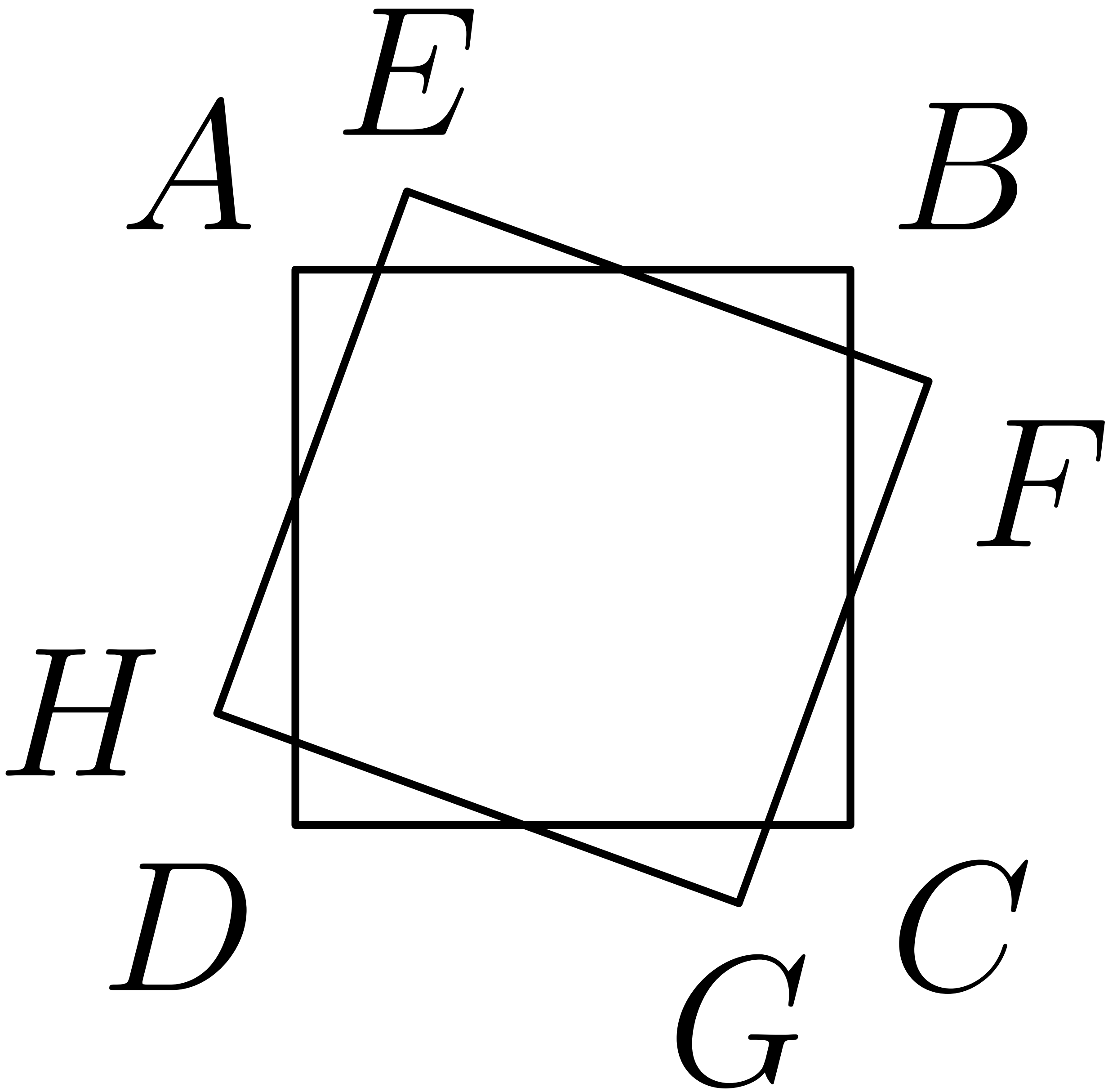
6. Let $L_1 = 1, L_2 = 3,$ and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1.$ How many terms in the sequence $L_1, L_2, L_3, \ldots, L_{2023}$ are even?

- A 673
- B 1011
- C 675
- D 1010
- E 674

Solution:

Track the parities: $1, 3, 4, 7, 11, 18, \ldots$ run odd, odd, even, then repeat with period $3.$ So L_n is even exactly when $3 \mid n.$ Among $1 \leq n \leq 2023,$ that's $\lfloor 2023/3 \rfloor = 674$ multiples of $3.$ Therefore, the answer is **E**.

7. Square $ABCD$ is rotated 20° clockwise about its center to obtain square $EFGH$, as shown below. What is the degree measure of $\angle EAB$?



A 24°

B 35°

C 30°

D 32°

E 20°

Solution:

Let O be the shared center. The rotation carries A to E, so $OA = OE$ and $\angle AOE = 20^\circ$. That makes triangle OAE isosceles, with base angles $\angle OAE = \frac{180^\circ - 20^\circ}{2} = 80^\circ$. The diagonal AC splits the right angle at A, so $\angle OAB = 45^\circ$. Subtracting, $\angle EAB = 80^\circ - 45^\circ = 35^\circ$. Thus, **B** is the correct answer.

8. What is the units digit of

$$2022^{2023} + 2023^{2022}?$$

A 7

B 1

C 9

D 5

E 3

Solution:

Only the units digits matter. Powers of 2 cycle $2, 4, 8, 6$ with period 4, and $2023 \equiv 3 \pmod 4$, so 2022^{2023} ends in 8. Powers of 3 cycle $3, 9, 7, 1$, and $2022 \equiv 2 \pmod 4$, so 2023^{2022} ends in 9. Add them: $8 + 9 = 17$, so the units digit is 7. Therefore, the answer is **A**.

9. The numbers 16 and 25 are a pair of consecutive positive perfect squares whose difference is 9. How many pairs of consecutive positive perfect squares have a difference of less than or equal to 2023?

A 674

B 1011

C 1010

D 2019

E 2017

Solution:

Consecutive squares k^2 and $(k+1)^2$ differ by $(k+1)^2 - k^2 = 2k+1$. We need $2k+1 \leq 2023$, which gives $k \leq 1011$. So k runs $1, 2, \ldots, 1011$, for 1011 pairs. Thus, **B** is the correct answer.

10. You are playing a game. A 2×1 rectangle covers two adjacent squares (oriented either horizontally or vertically) of a 3×3 grid of squares, but you are not told which two squares are covered. Your goal is to find at least one square that is covered by the rectangle. A "turn" consists of you guessing a square, after which you are told whether that square is covered by the hidden rectangle. What is the minimum number of turns you need to ensure that at least one of your guessed squares is covered by the rectangle?

- A 3
- B 5
- C 4
- D 8
- E 6

Solution:

Suppose every guess misses. Then the domino lies entirely on unguessed squares, so those squares include two adjacent cells. To force a hit, we need the unguessed squares to have no two adjacent. Color the grid like a checkerboard. The biggest set of pairwise non-adjacent squares has 5 cells, one color's worth: the four corners and the center. So we must guess at least $9 - 5 = 4$ squares. And 4 is enough: guess the four edge midpoints, since every domino covers one square of each color, hence one edge midpoint. Therefore, the answer is **C**.

11. Suzanne went to the bank and withdrew \$800. The teller gave her this amount using \$20 bills, \$50 bills, and \$100 bills, with at least one of each denomination. How many different collections of bills could Suzanne have received?

A 45

B 21

C 36

D 28

E 32

Solution:

Let $a, b, c \geq 1$ count the $\$20, \$50, \$100$ bills. Then $20a + 50b + 100c = 800,$ which divides down to $2a + 5b + 10c = 80.$ Both $2a$ and $10c$ are even, so $5b$ is too, forcing $b = 2t.$ Now $a = 40 - 5t - 5c \geq 1$ means $t + c \leq 7.$ With $t, c \geq 1,$ the pairs number $1 + 2 + \cdots + 6 = 21.$ Thus, **B** is the correct answer.

12. When the roots of the polynomial

$$P(x) = \prod_{i=1}^{10} (x - i)^i$$

are removed from the real number line, what remains is the union of 11 disjoint open intervals. On how many of those intervals is $P(x)$ positive?

- A 3
- B 7
- C 6
- D 4
- E 5

Solution:

For $x > 10$, every factor $(x - i)^i$ is positive, so $P(x) > 0$. Now move left. Crossing $x = i$ flips the sign only when i is odd, that is at $i = 9, 7, 5, 3, 1$. So the eleven intervals, right to left, carry signs $+, +, -, -, +, +, -, -, +, +, -$. Six are positive. Therefore, the answer is **C**.

13. What is the area of the region in the coordinate plane defined by the inequality

$$\big||x|-1\big|+\big||y|-1\big|\le 1?$$

A 2

B 8

C 4

D 15

E 12

Solution:

Substitute $u=|x|, v=|y|$. Then $|u-1|+|v-1|\le 1$ is a diamond centered at $(1,1)$ with diagonals of length 2, so it has area 2, and it sits entirely in $u, v \ge 0$. The map $(x,y)\mapsto(|x|,|y|)$ is four-to-one over $u, v > 0$, so the full region has area $4\times 2=8$. Thus, **B** is the correct answer.

14. How many ordered pairs of integers (m, n) satisfy the equation

$$m^2 + mn + n^2 = m^2n^2?$$

- A 7
- B 1
- C 3
- D 6
- E 5

Solution:

If $m = 0$, the equation forces $n^2 = 0$, giving $(0, 0)$. Otherwise both are nonzero; assume $|m| \leq |n|$. Then $m^2n^2 = m^2 + mn + n^2 \leq 3n^2$, so $m^2 \leq 3$ and $m = \pm 1.$ Take $m = 1 : 1 + n + n^2 = n^2$ gives $n = -1.$ Take $m = -1 : n = 1.$ That leaves $(0, 0), (1, -1), (-1, 1),$ three in all. Therefore, the answer is **C**.

15. What is the least positive integer m such that $m \cdot 2! \cdot 3! \cdot 4! \cdot 5! \cdots 16!$ is a perfect square?

A 30

B 30030

C 70

D 1430

E 1001

Solution:

Group the product as $(2!\,3!)(4!\,5!)\cdots(14!\,15!)\cdot 16!$. Since $(2k)!(2k+1)! = \left((2k)!\right)^2(2k+1)$, each pair is a perfect square times an odd number. Those odd numbers $3, 5, 7, 9, 11, 13, 15$ multiply to $3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, with squarefree part $7 \cdot 11 \cdot 13$. And $16! = 2^{15}3^6 5^3 7^2 \cdot 11 \cdot 13$ has squarefree part $2 \cdot 5 \cdot 11 \cdot 13$. Multiply the two: the squarefree part of the whole thing is $2 \cdot 5 \cdot 7$. That's the smallest m, namely $2 \cdot 5 \cdot 7 = 70$. Thus, **C** is the correct answer.

16. Define an *upno* to be a positive integer of 2 or more digits where the digits are strictly increasing moving left to right. Similarly, define a *downno* to be a positive integer of 2 or more digits where the digits are strictly decreasing moving left to right. For instance, the number 258 is an upno and 8620 is a downno. Let U equal the total number of upnos and let D equal the total number of downnos. What is $|U - D|$?

A 512

B 10

C 0

D 9

E 511

Solution:

An upno is just a choice of at least 2 digits, written in increasing order. A 0 can never appear: it can't lead and can't follow a smaller digit. So the digits come from $\{1, \ldots, 9\}$, giving $U = 2^9 - 1 - 9 = 502$. A downno can end in 0, so its digits are any subset of $\{0, \ldots, 9\}$ of size ≥ 2, giving $D = 2^{10} - 1 - 10 = 1013$. So $|U - D| = |502 - 1013| = 511$. Therefore, the answer is **E**.

17. A rectangular box $\mathcal{P}$ has distinct edge lengths $a, b,$ and $c.$ The sum of the lengths of all 12 edges of $\mathcal{P}$ is $13,$ the sum of the areas of all 6 faces of $\mathcal{P}$ is $\frac{11}{2},$ and the volume of $\mathcal{P}$ is $\frac{1}{2}.$ What is the length of the longest interior diagonal connecting two vertices of $\mathcal{P}?$

A 2

B $\frac{3}{8}$

C $\frac{9}{8}$

D $\frac{9}{4}$

E $\frac{3}{2}$

Solution:

The 12 edges give $4(a+b+c)=13,$ so $a+b+c=\frac{13}{4}.$ The 6 faces give $2(ab+bc+ca)=\frac{11}{2},$ so $ab+bc+ca=\frac{11}{4}.$ The space diagonal is $\sqrt{a^2+b^2+c^2}=\sqrt{(a+b+c)^2-2(ab+bc+ca)}=\sqrt{\frac{169}{16}-\frac{11}{2}}=\sqrt{\frac{81}{16}}=\frac{9}{4}.$ Thus, **D** is the correct answer.

18. Suppose a, b, and c are positive integers such that

$$\frac{a}{14} + \frac{b}{15} = \frac{c}{210}.$$

Which of the following statements are necessarily true?

I. If $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both, then $\gcd(c, 210) = 1$.

II. If $\gcd(c, 210) = 1$, then $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both.

III. $\gcd(c, 210) = 1$ if and only if $\gcd(a, 14) = \gcd(b, 15) = 1$.

A. I, II, and III

B. I only

C. I and II only

D. III only

E. II and III only

Solution:

Clear denominators to get $c = 15a + 14b$. Reduce modulo the primes of $210 = 2 \cdot 3 \cdot 5 \cdot 7 : c \equiv a \pmod 2, c \equiv 2b \pmod 3, c \equiv 4b \pmod 5$, and $c \equiv a \pmod 7$. So $\gcd(c, 210) = 1$ iff $2 \nmid a, 7 \nmid a, 3 \nmid b, 5 \nmid b$, which is exactly $\gcd(a, 14) = 1$ and $\gcd(b, 15) = 1$. That settles III, and it makes II true since the "and" implies the "or." Statement I fails, though: take $a = 1, b = 3$. Then $\gcd(a, 14) = 1$, yet $c = 57$ is divisible by 3. So only II and III hold. Therefore, the answer is **E**.

19. Sonya the frog chooses a point uniformly at random lying within the square $[0,6] \times [0,6]$ in the coordinate plane and hops to that point. She then chooses a distance uniformly at random from $[0,1]$ and a direction uniformly at random from $\{\text{north}, \text{south}, \text{east}, \text{west}\}$. All her choices are independent. She now hops the distance in the chosen direction. What is the probability that she lands outside the square?

A $\frac{1}{6}$

B $\frac{1}{12}$

C $\frac{1}{4}$

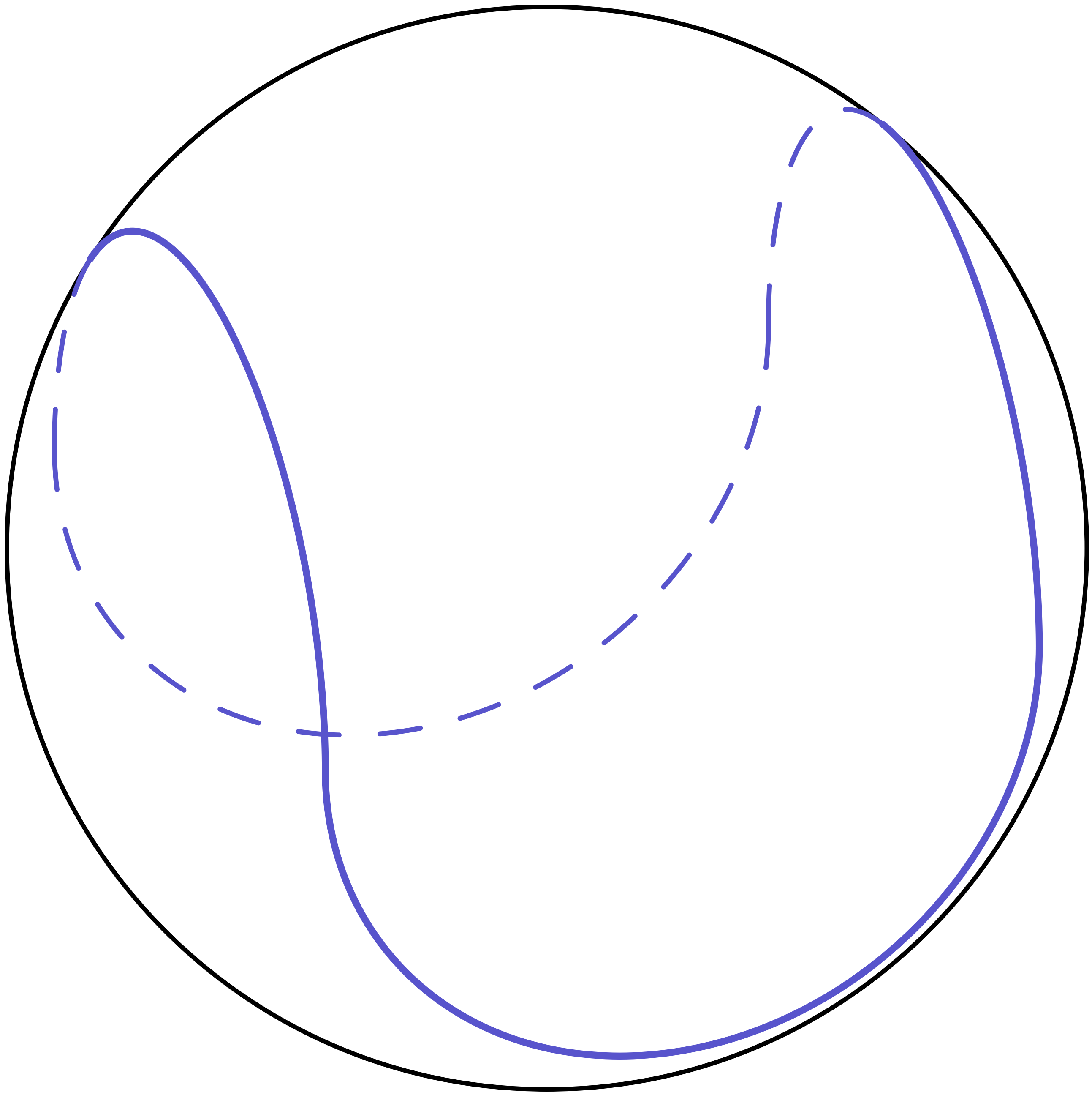
D $\frac{1}{10}$

E $\frac{1}{9}$

Solution:

The four directions behave the same by symmetry, so say she hops east. She lands outside exactly when her x-coordinate plus the hop distance d tops 6. Fix d. Her x-coordinate is uniform on $[0,6]$, so it beats $6-d$ with probability $\frac{d}{6}$. Now average over d uniform on $[0,1]: \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$. Thus, **B** is the correct answer.

Four congruent semicircles are drawn on the surface of a sphere with radius 2, as shown, creating a closed curve that divides the surface into two congruent regions. The length of the curve is $\pi\sqrt{n}$. What is n?



A 32

B 12

C 48

D 36

E 27

Solution:

The curve is four congruent semicircular arcs, so its length is 4 times one semicircle, πr, where r is the arc radius. The arcs meet at four points that form a square inscribed in a great circle of the radius-2 sphere, and each arc's diameter is a side of that square, a chord of length $2\sqrt{2}$. So $r = \sqrt{2}$. (Check it another way: the small circle sits in a plane at distance $\frac{2}{\sqrt{2}} = \sqrt{2}$ from the center, giving radius $\sqrt{2^2 - (\sqrt{2})^2} = \sqrt{2}$.) The total length is $4 \cdot \pi\sqrt{2} = \pi\sqrt{32}$, so $n = 32$. Therefore, the answer is **A**.

21. Each of 2023 balls is placed into one of 3 bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?

A $\frac{2}{3}$

B $\frac{3}{10}$

C $\frac{1}{2}$

D $\frac{1}{3}$

E $\frac{1}{4}$

Solution:

All 3^{2023} assignments are equally likely. A sign filter counts the ones with every bin odd: $\frac{1}{8}\sum_{s\in\{\pm 1\}^3}(s_1 s_2 s_3)(s_1+s_2+s_3)^{2023}$. Only $s = (1,1,1)$ and $s = (-1,-1,-1)$ carry weight, each $\pm 3^{2023}$; the other six terms total -6. So the count is $\frac{2\cdot 3^{2023}-6}{8} = \frac{3^{2023}-3}{4}$. Dividing, the probability is $\frac{3^{2023}-3}{4\cdot 3^{2023}} = \frac{1}{4} - \frac{1}{4\cdot 3^{2022}}$, a hair under $\frac{1}{4}$. Thus, **E** is the correct answer.

22. How many distinct values of x satisfy

$$\lfloor x \rfloor^2 - 3x + 2 = 0,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x?

A an infinite number

B 4

C 2

D 3

E 0

Solution:

Set $n = \lfloor x \rfloor$. Then $n^2 - 3x + 2 = 0$ gives $x = \frac{n^2+2}{3}$. For this to be consistent we need $n \le \frac{n^2+2}{3} < n + 1$. The left side, $n^2 - 3n + 2 \ge 0$, holds for every integer n. The right side, $n^2 - 3n - 1 < 0$, holds only for $n \in \{0, 1, 2, 3\}$. Those give $x = \frac{2}{3}, 1, 2, \frac{11}{3}$, so there are 4 distinct values. Therefore, the answer is **B**.

23. An arithmetic sequence has $n \geq 3$ terms, initial term a, and common difference $d > 1$. Carl wrote down all the terms in this sequence correctly except for one term, which was off by 1. The sum of the terms he wrote was 222. What was $a + d + n$?

A 24

B 20

C 22

D 28

E 26

Solution:

The true sum is $S = na + \frac{n(n-1)}{2}d$. Since one term is off by 1, the written total satisfies $222 = S \pm 1$, so $S = 221$ or 223. Also $2S = n\big(2a + (n-1)d\big)$. If $S = 223$, which is prime, no factorization with $n \geq 3$ yields valid a, d. So take $S = 221 = 13 \cdot 17$. With $n = 13$, we get $2a + 12d = 34$, that is $a + 6d = 17$, so $a = 5, d = 2$, and $d > 1$ checks out. Then $a + d + n = 5 + 2 + 13 = 20$. Thus, **B** is the correct answer.

24. What is the perimeter of the boundary of the region consisting of all points which can be expressed as $(2u - 3w,\ v + 4w)$ with $0 \le u \le 1, 0 \le v \le 1$, and $0 \le w \le 1$?

A $10\sqrt{3}$

B 10

C 12

D 18

E 16

Solution:

Fix w. As u, v sweep $[0,1]^2$, the point $(2u - 3w,\ v + 4w)$ fills a 2×1 axis-aligned rectangle with lower-left corner $(-3w, 4w)$. Now let w run from 0 to 1. The rectangle slides along the vector $(-3, 4)$, which has length 5. So the region is the Minkowski sum of that rectangle and the segment, and its perimeter is the rectangle's perimeter plus twice the segment length: $2(2+1) + 2 \cdot 5 = 16$. Therefore, the answer is **E**.

25. A regular pentagon with area $\sqrt{5}+1$ is printed on paper and cut out. The five vertices of the pentagon are folded into the center of the pentagon, creating a smaller pentagon. What is the area of the new pentagon?

A $4-\sqrt{5}$

B $\sqrt{5}-1$

C $8-3\sqrt{5}$

D $\dfrac{\sqrt{5}+1}{2}$

E $\dfrac{2+\sqrt{5}}{3}$

Solution:

Let the original pentagon have circumradius R. Folding a vertex to the center creases along the perpendicular bisector of the center-to-vertex segment, a line at distance $\frac{R}{2}$ from the center. Those five creases bound the new regular pentagon, whose apothem is $\frac{R}{2}$ (the original apothem was $R\cos 36^\circ$). So the new pentagon is similar with ratio $\frac{R/2}{R\cos 36^\circ}$, and its area is the old area times $\left(\frac{1}{2\cos 36^\circ}\right)^2$. Plug in $\cos 36^\circ = \frac{1+\sqrt{5}}{4}$: the factor becomes $\frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}$. So the new area is $(\sqrt{5}+1)\cdot\frac{3-\sqrt{5}}{2} = \sqrt{5}-1$. Thus, **B** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2023B