2023 AMC 10A Solutions

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1. Cities A and B are 45 miles apart. Alice and Beth start biking from A and B at speeds of 18 mph and 12 mph, respectively. How far away from city A will they be when they meet?

A 20

B 24

C 25

D 26

E 27

Solution:

They ride toward each other, so their speeds add. That closes the 45-mile gap at $18 + 12 = 30$ mph, and they meet after $45/30 = 1.5$ hours. Alice starts at A, so by then she's gone $18 \cdot 1.5 = 27$ miles. Thus, **E** is the correct answer.

2. The weight of $\frac{1}{3}$ of a large pizza together with $3\frac{1}{2}$ cups of orange slices is the same as the weight of $\frac{3}{4}$ of a large pizza together with $\frac{1}{2}$ cup of orange slices. A cup of orange slices weighs $\frac{1}{4}$ of a pound. What is the weight, in pounds, of a large pizza?

A $1\frac{4}{5}$

B 2

C $2\frac{2}{5}$

D 3

E $3\frac{3}{5}$

Solution:

Let p be the pizza's weight. A cup of orange slices is $\frac{1}{4}$ pound, so the two sides balance as $\frac{1}{3}p + \frac{7}{2} \cdot \frac{1}{4} = \frac{3}{4}p + \frac{1}{2} \cdot \frac{1}{4}$, that is $\frac{1}{3}p + \frac{7}{8} = \frac{3}{4}p + \frac{1}{8}$. Collect the pizza terms: $\frac{6}{8} = \left(\frac{3}{4} - \frac{1}{3}\right) p = \frac{5}{12}p$. So $p = \frac{3}{4} \cdot \frac{12}{5} = \frac{9}{5} = 1\frac{4}{5}$. Therefore, the answer is **A**.

3. How many positive perfect squares less than 2023 are divisible by 5?

A 8

B 9

C 10

D 11

E 12

Solution:

If a perfect square is divisible by 5, it's divisible by 25, so it looks like $(5k)^2 = 25k^2$. We need $25k^2 < 2023$, i.e. $k^2 < 80.9$. That allows $k = 1, 2, \ldots, 8$, which is 8 squares. Thus, **A** is the correct answer.

4. A quadrilateral has all integer side lengths, a perimeter of 26, and one side of length 4. What is the greatest possible length of one side of this quadrilateral?

A 9

B 10

C 11

D 12

E 13

Solution:

In any quadrilateral each side is shorter than the sum of the other three. Call the longest side s. The rest sum to $26 - s$, so $s < 26 - s$, which gives $s < 13$ and hence $s \leq 12$. Can we hit 12? The sides $4, 12, 9, 1$ work, since $12 < 4 + 9 + 1$. So the greatest length is 12. Therefore, the answer is **D**.

5. How many digits are in the base-ten representation of $8^5 \cdot 5^{10} \cdot 15^5$?

A 14

B 15

C 16

D 17

E 18

Solution:

Factor everything into primes. $8^5 \cdot 5^{10} \cdot 15^5 = 2^{15} \cdot 5^{10} \cdot 3^5 \cdot 5^5 = 2^{15} \cdot 5^{15} \cdot 3^5 = 10^{15} \cdot 243$. That's 243 followed by 15 zeros, so it has $3 + 15 = 18$ digits. Thus, **E** is the correct answer.

6. An integer is assigned to each vertex of a cube. The value of an edge is defined to be the sum of the values of the two vertices it touches, and the value of a face is defined to be the sum of the values of the four edges surrounding it. The value of the cube is defined as the sum of the values of its six faces. Suppose the sum of the integers assigned to the vertices is 21. What is the value of the cube?

- A 42
- B 63
- C 84
- D 126
- E 252

Solution:

Count by incidences. Each edge lies on 2 faces, so the six face values together are 2 times the total of all edge values. Each vertex lies on 3 edges, so the total edge value is 3 times the vertex sum. Chaining these, the cube's value is $2 \cdot 3 \cdot 21 = 126.$ Therefore, the answer is **D**.

7. Janet rolls a standard 6-sided die 4 times and keeps a running total of the numbers she rolls. What is the probability that at some point her running total will equal 3?

A $\frac{2}{9}$

B $\frac{49}{216}$

C $\frac{25}{108}$

D $\frac{17}{72}$

E $\frac{13}{54}$

Solution:

The total can only reach exactly 3 through the opening rolls, and these ways are disjoint: 3 alone (probability $\frac{1}{6}$), then $1, 2$ and $2, 1$ (each $\frac{1}{36}$), and $1, 1, 1$ (probability $\frac{1}{216}$). Add them up: $\frac{36}{216} + \frac{6}{216} + \frac{6}{216} + \frac{1}{216} = \frac{49}{216}$. Thus, **B** is the correct answer.

8. Barb the baker has developed a new temperature scale for her bakery called the Breadus scale, which is a linear function of the Fahrenheit scale. Bread rises at 110 degrees Fahrenheit, which is 0 degrees on the Breadus scale. Bread is baked at 350 degrees Fahrenheit, which is 100 degrees on the Breadus scale. Bread is done when its internal temperature is 200 degrees Fahrenheit. What is this in degrees on the Breadus scale?

A 33

B 34.5

C 36

D 37.5

E 39

Solution:

The Breadus reading is linear in Fahrenheit through $(110, 0)$ and $(350, 100)$, so $B = \frac{100}{350-110}(F - 110) = \frac{5}{12}(F - 110)$. Plug in $F = 200 : B = \frac{5}{12} \cdot 90 = 37.5.$ Therefore, the answer is **D**.

9. A digital display shows the current date as an 8-digit integer, consisting of a 4-digit year, followed by a 2-digit month, followed by a 2-digit date within the month. For how many dates in 2023 will each digit appear an even number of times in the digital display for that date?

- A 5
- B 6
- C 7
- D 8
- E 9

Solution:

The year 2023 already gives two 2s (even), one 0, and one 3. So to make every digit even overall, the four digits of MM and DD have to supply one more 0, one more 3, and then two digits equal to each other, all while keeping the count of 2s even. Run through the legal months and days of 2023 under that rule and exactly 9 dates survive. Thus, **E** is the correct answer.

10. If Maureen scores an 11 on her next quiz, her mean score will go up by 1. If she gets three 11s in a row, her mean score will increase by 2. What is her current mean quiz score?

A 4

B 5

C 6

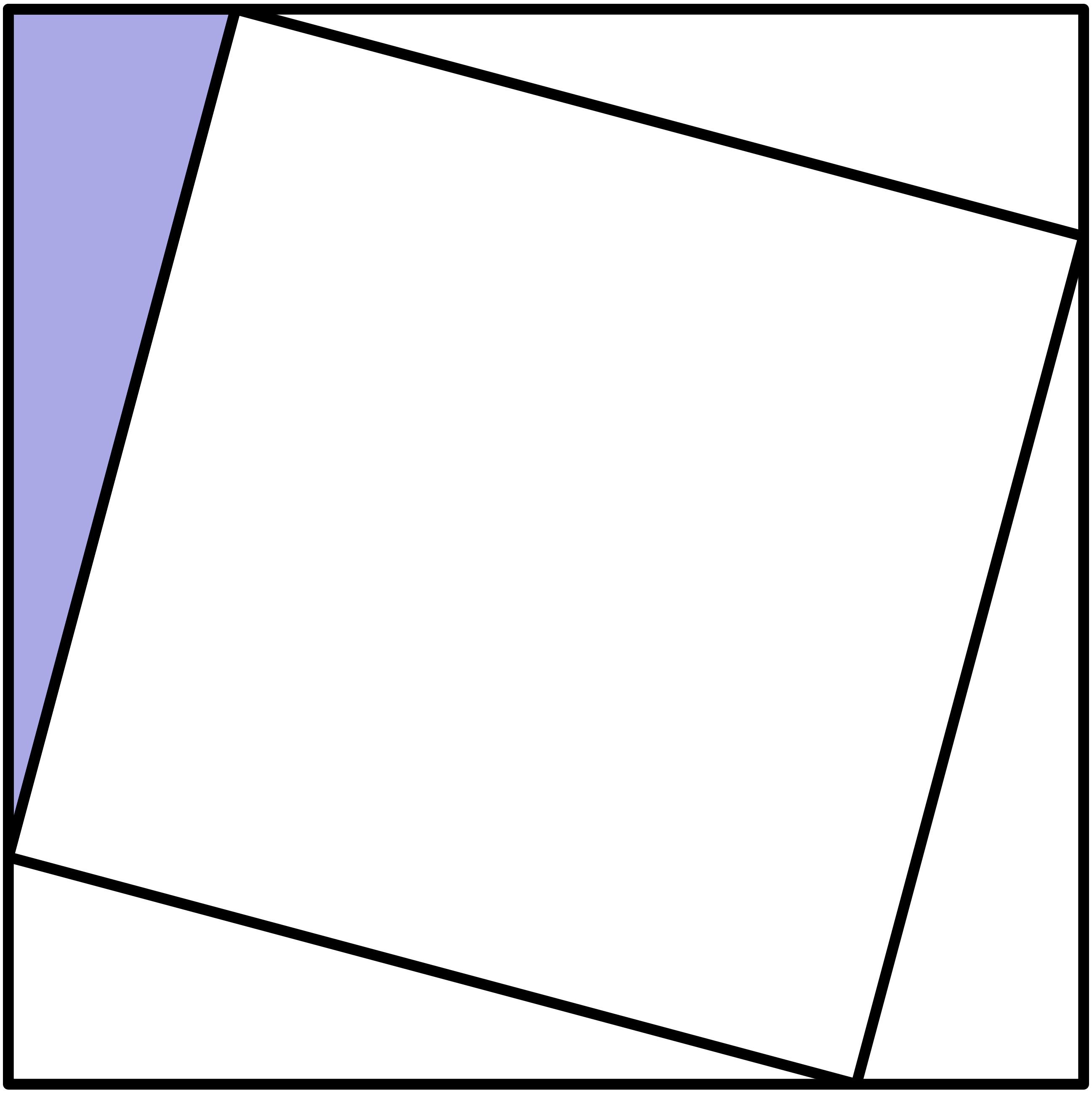
D 7

E 8

Solution:

Let m be the current mean over n quizzes. One more 11 makes the mean $m+1:$ $\frac{mn+11}{n+1}=m+1$, which tidies up to $m+n=10$. Three more 11s make it $m+2:$ $\frac{mn+33}{n+3}=m+2$, i.e. $3m+2n=27$. Solve the pair and $m=7$. Therefore, the answer is **D**.

11. A square with area 3 has a square with area 2 inscribed in it. This creates 4 smaller congruent right triangles. What is the ratio of the smaller leg to the larger leg in the shaded right triangle?



A $\frac{1}{5}$

B $\frac{1}{4}$

C $2-\sqrt{3}$

D $\sqrt{3}-\sqrt{2}$

E $\sqrt{2}-1$

Solution:

Each corner right triangle has legs a and b. A side of the outer square gives $a+b=\sqrt{3}$, and a side of the inscribed square gives $a^2+b^2=2$. Subtract to find the product: $2ab=(a+b)^2-(a^2+b^2)=3-2=1$, so $ab=\frac{1}{2}$. Then a and b are the roots of $t^2-\sqrt{3}\,t+\frac{1}{2}=0$, namely $\frac{\sqrt{3}\pm1}{2}$. The ratio of the smaller leg to the larger is $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}=\dfrac{(\sqrt{3}-1)^2}{2}=2-\sqrt{3}$. Thus, **C** is the correct answer.

12. How many three-digit positive integers N satisfy both of the following properties: N is divisible by 7, and the number formed by reversing the digits of N is divisible by 5?

A 13

B 14

C 15

D 16

E 17

Solution:

When we reverse N, its last digit is the first digit of N. For the reversal to be divisible by 5, that digit is 0 or 5. A three-digit number can't start with 0, so N starts with 5, meaning $500\le N\le 599$ (and the reversal ends in 5, always fine). Now just count multiples of 7 here: from $7\cdot 72=504$ to $7\cdot 85=595$, that's 14 numbers. Therefore, the answer is **B**.

13. Abdul and Chiang are standing 48 feet apart in a field. Bharat is standing in the same field as far from Abdul as possible so that the angle formed by his lines of sight to Abdul and Chiang measures 60°. What is the square of the distance (in feet) between Abdul and Bharat?

- A 1728
- B 2601
- C 3072
- D 4608
- E 6912

Solution:

Let A be Abdul, C be Chiang with $AC = 48$, and B be Bharat with $\angle B = 60^\circ$. Every point seeing AC at 60° lies on one circular arc, so all valid B sit on a circle where chord AC subtends 60°. The law of sines gives its diameter, $\frac{AC}{\sin 60^\circ} = \frac{48}{\sqrt{3}/2} = 32\sqrt{3}$. Now AB is a chord, and a chord is longest when it's a diameter. So $AB = 32\sqrt{3}$ and $AB^2 = 1024 \cdot 3 = 3072$. Thus, **C** is the correct answer.

14. A number is chosen at random from among the first 100 positive integers, and a positive integer divisor of that number is then chosen at random. What is the probability that the chosen divisor is divisible by 11?

A $\frac{4}{100}$

B $\frac{9}{200}$

C $\frac{1}{20}$

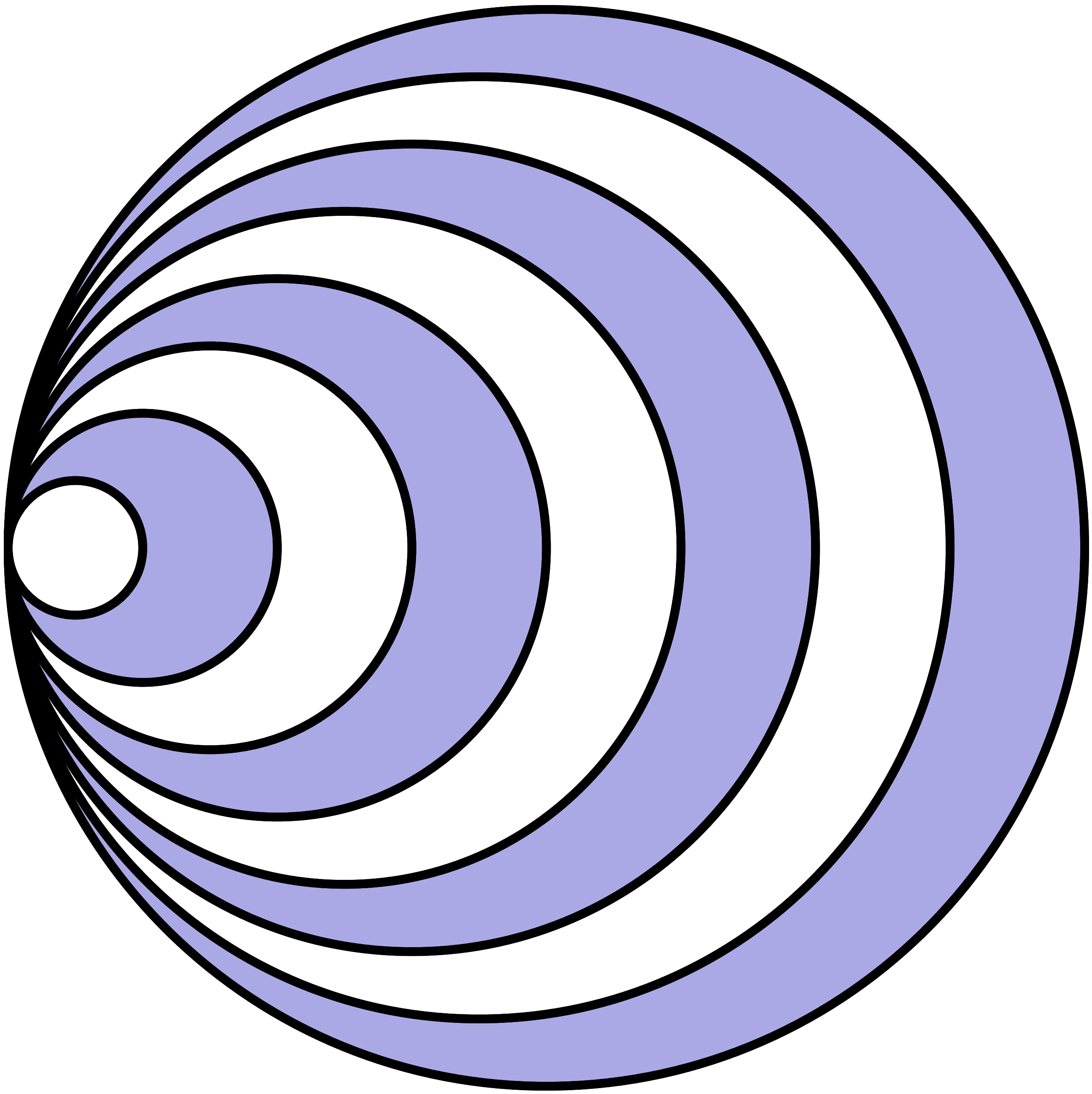
D $\frac{11}{200}$

E $\frac{3}{50}$

Solution:

A number $n \leq 100$ can only have a divisor divisible by 11 when $11 \mid n$, so $n \in \{11, 22, \ldots, 99\}$. Write $n = 11m$ with $m \leq 9$. Here $11 \nmid m$, so $d(11m) = 2\,d(m)$, and the divisors that are multiples of 11 are exactly the $d(m)$ numbers $11d$. That makes the chance $\frac{d(m)}{2\,d(m)} = \frac{1}{2}$ for each such n. Averaging over all 100 starting numbers, the probability is $\frac{1}{100}\sum_{m=1}^{9}\frac{1}{2} = \frac{9}{200}$. Therefore, the answer is **B**.

15. An even number of circles are nested, starting with a radius of 1 and increasing by 1 each time, all sharing a common point. The region between every other circle is shaded, starting with the region inside the circle of radius 2 but outside the circle of radius 1. An example showing 8 circles is displayed below. What is the least number of circles needed to make the total shaded area at least 2023π?



A 46

B 48

C 56

D 60

E 64

Solution:

A circle of radius r has area πr^2. So the shaded ring between radius $2k$ and $2k-1$ has area $\pi\left((2k)^2-(2k-1)^2\right)=(4k-1)\pi$. With $2n$ circles the shaded total is $\pi\sum_{k=1}^{n}(4k-1)=\pi(2n^2+n)$. We want $2n^2+n\geq 2023$. At $n=31$ it's 1953, at $n=32$ it's 2080. So $n=32$, which means $2n=64$ circles. Thus, **E** is the correct answer.

16. In a tennis tournament, each person plays every other person once. In this tournament, there are twice as many right-handed players as left-handed players, but left-handed players won 40% more games than right-handed players. How many total games were played?

A 15

B 36

C 45

D 48

E 66

Solution:

Say there are L left-handers and $2L$ right-handers, so $3L$ players and $\binom{3L}{2}$ games. Every game has one winner, and left wins are 1.4 times right wins, so the wins split $7:5$ and the total must be a multiple of 12. Try $L=3:$ that's $\binom{9}{2}=36=12\cdot 3,$ with left winning 21 and right winning $15,$ and indeed $21=1.4\cdot 15.$ It's achievable. So 36 games were played. Therefore, the answer is **B**.

17. Let $ABCD$ be a rectangle with $AB = 30$ and $BC = 28$. Points P and Q lie on BC and CD respectively so that all sides of $\triangle ABP$, $\triangle PCQ$, and $\triangle QDA$ have integer lengths. What is the perimeter of $\triangle APQ$?

A 84

B 86

C 88

D 90

E 92

Solution:

Set $A = (0, 0), B = (30, 0), C = (30, 28), D = (0, 28)$, with $P = (30, p)$ on BC and $Q = (30 - q, 28)$ on CD. The three right triangles give $AP = \sqrt{30^2 + p^2}$, $QA = \sqrt{28^2 + (30 - q)^2}$, and $PQ = \sqrt{(28 - p)^2 + q^2}$, and all must be integers. Hunt for Pythagorean triples: $p = 16$ makes $AP = 34$, and $30 - q = 21$ (so $q = 9$) makes $QA = 35$. Then $PQ = \sqrt{12^2 + 9^2} = 15$, an integer too. So the perimeter of $\triangle APQ$ is $34 + 15 + 35 = 84$. Thus, **A** is the correct answer.

18. A rhombic dodecahedron is a solid with 12 congruent rhombus faces. At every vertex, 3 or 4 edges meet, depending on the vertex. How many vertices have exactly 3 edges meeting?

A 5

B 6

C 7

D 8

E 9

Solution:

Each rhombus has 4 edges, and every edge is shared by 2 faces, so $E = \frac{12 \cdot 4}{2} = 24.$ With $F = 12,$ Euler's formula gives $V = 2 - F + E = 14.$ Suppose x vertices have 3 edges and the other $14 - x$ have $4.$ The degrees sum to twice the edge count: $3x + 4(14 - x) = 2E = 48,$ so $x = 8.$ Therefore, the answer is **D**.

19. The line segment formed by $A(1, 2)$ and $B(3, 3)$ is rotated to the line segment formed by $A'(3, 1)$ and $B'(4, 3)$ about the point $P(r, s)$. What is $|r - s|$?

A $\frac{1}{4}$

B $\frac{1}{2}$

C $\frac{3}{4}$

D $\frac{2}{3}$

E 1

Solution:

A rotation keeps its center equidistant from each point and its image. So P is equidistant from A and A', and from B and B', which puts it at the intersection of two perpendicular bisectors. The bisector of BB' from $(3, 3)$ to $(4, 3)$ is $x = 3.5$. The bisector of AA' from $(1, 2)$ to $(3, 1)$ is $2x - y = 2.5$. Then $y = 2(3.5) - 2.5 = 4.5$, so $P = (3.5, 4.5)$ and $|r - s| = |3.5 - 4.5| = 1$. Thus, **E** is the correct answer.

20. Each square in a 3×3 grid of squares is colored red, white, blue, or green so that every 2×2 square contains one square of each color. One such coloring is shown below (letters denote the colors, with the center square white). How many different colorings are possible?

B	R	B
G	W	G
R	B	R

A 24

B 48

C 60

D 72

E 96

Solution:

Label the cells row by row $a, b, c/d, e, f/g, h, i$. The top-left block a, b, d, e is a permutation of the four colors, so $4! = 24$ ways. The block $\{b, c, e, f\}$ is also all four colors, and b, e are fixed, so $\{c, f\}$ is the remaining two in some order: 2 ways. Same story for $\{g, h\}$, the two colors apart from d, e, another 2 ways. That leaves i, forced to whatever color is missing from $\{e, f, h\}$, and that only works when $f \neq h$. Of the $2 \cdot 2 = 4$ order combinations, exactly one has $f = h$, so 3 survive. The total is $24 \cdot 3 = 72$. Therefore, the answer is **D**.

21. There is a unique polynomial $P(x)$ of least degree with leading coefficient 1 satisfying all of the following:

1 is a root of $P(x)-1, 2$ is a root of $P(x-2), 3$ is a root of $P(3x)$, and 4 is a root of $4P(x)$.

All the roots of $P(x)$ except one are integers. If the one non-integer root can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers, what is $m+n$?

A 41

B 43

C 45

D 47

E 49

Solution:

Translate each condition into a value: $P(1)=1, P(0)=0, P(9)=0$, and $P(4)=0$. So $0,4,9$ are roots. Could a cubic do it? A monic cubic with those roots has $P(1)=(1)(-3)(-8)=24\neq 1$, so no. The least-degree monic polynomial is degree $4:$ $P(x)=x(x-4)(x-9)(x-c)$. Now $P(1)=(1)(-3)(-8)(1-c)=24(1-c)=1$, so $1-c=\frac{1}{24}$ and $c=\frac{23}{24}$. That's the lone non-integer root, so $m+n=23+24=47$. Thus, **D** is the correct answer.

22. Circles C_1 and C_2 have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and is externally tangent to C_3. What is the radius of C_4?

A $\frac{1}{14}$

B $\frac{1}{12}$

C $\frac{1}{10}$

D $\frac{3}{28}$

E $\frac{1}{9}$

Solution:

Put the centers of C_1, C_2 at $\left(\pm\frac{1}{4}, 0\right)$. By symmetry the largest circle inside both sits at the origin with radius r_3, where $1 - r_3 = \frac{1}{4}$, so $r_3 = \frac{3}{4}$. Let C_4 be centered at $(0, y)$ with radius r. Internal tangency to C_1 gives $\sqrt{\frac{1}{16} + y^2} = 1 - r$, and external tangency to C_3 gives $y = \frac{3}{4} + r$. Substitute the second into the first: $\frac{1}{16} + \left(\frac{3}{4} + r\right)^2 = (1 - r)^2$. This collapses to $\frac{7}{2}r = \frac{3}{8}$, so $r = \frac{3}{28}$. Therefore, the answer is **D**.

23. Positive integer divisors a and b of N are called complementary if $ab = N$. Given that N has a pair of complementary divisors that differ by 20 and a pair of complementary divisors that differ by 23, find the sum of the digits of N.

A 11

B 13

C 15

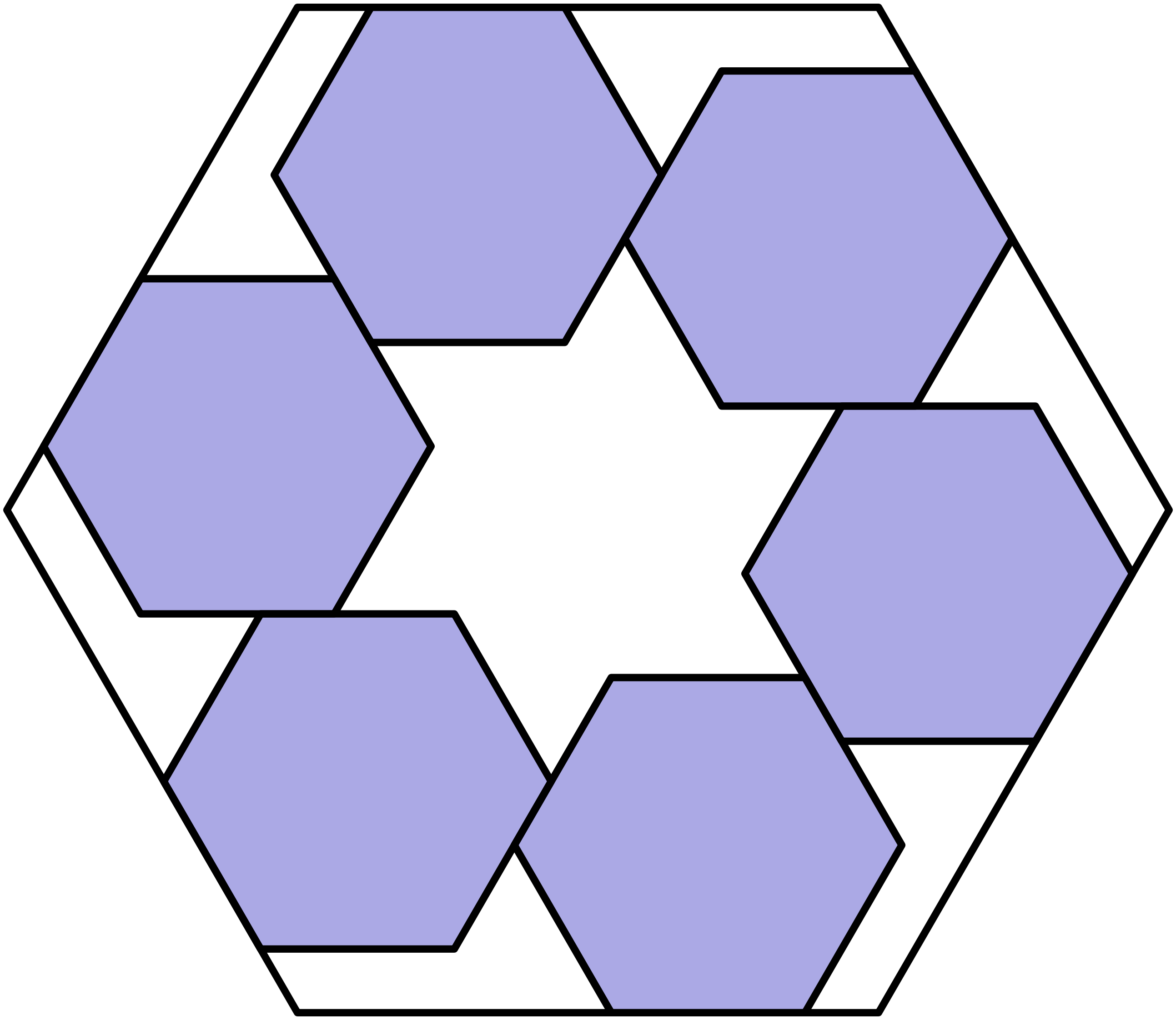
D 17

E 19

Solution:

Complementary divisors differing by 20 are b and $b + 20$ with product N, so $N = b^2 + 20b$ and $N + 100 = (b + 10)^2$. A pair differing by 23 gives $4N + 529 = (2d + 23)^2$. Set $N + 100 = k^2$. Then $4k^2 + 129 = m^2$, so $(m - 2k)(m + 2k) = 129 = 3 \cdot 43$. Take the factorization $1 \cdot 129$: it gives $m = 65, k = 32$, hence $N = 32^2 - 100 = 924$. Check it: $924 = 22 \cdot 42 = 21 \cdot 44$, and the digit sum is $9 + 2 + 4 = 15$. Thus, **C** is the correct answer.

24. Six regular hexagonal blocks of side length 1 unit are arranged inside a regular hexagonal frame. Each block lies along an inside edge of the frame and is aligned with two other blocks, as shown in the figure below. The distance from any corner of the frame to the nearest vertex of a block is $\frac{3}{7}$ unit. What is the area of the region inside the frame not occupied by the blocks?



A $\dfrac{13\sqrt{3}}{3}$

B $\dfrac{216\sqrt{3}}{49}$

C $\dfrac{9\sqrt{3}}{2}$

D $\frac{14\sqrt{3}}{3}$

E $\frac{243\sqrt{3}}{49}$

Solution:

The uncovered region is the frame's area minus the six unit blocks. A regular hexagon of side t has area $\frac{3\sqrt{3}}{2}t^2$, so each unit block is $\frac{3\sqrt{3}}{2}$. The spacing rule, that each frame corner sits $\frac{3}{7}$ from the nearest block vertex, pins the frame's side length at 3. So the uncovered area is $\frac{3\sqrt{3}}{2} \cdot 3^2 - 6 \cdot \frac{3\sqrt{3}}{2} = \frac{27\sqrt{3}}{2} - 9\sqrt{3} = \frac{9\sqrt{3}}{2}$. Therefore, the answer is **C**.

25. If A and B are vertices of a polyhedron, define the distance $d(A, B)$ to be the minimum number of edges of the polyhedron one must traverse in order to connect A and B. For example, if AB is an edge of the polyhedron, then $d(A, B) = 1$, but if AC and CB are edges and AB is not an edge, then $d(A, B) = 2$. Let Q, R, and S be randomly chosen distinct vertices of a regular icosahedron (a regular polyhedron made up of 20 equilateral triangles). What is the probability that $d(Q, R) > d(R, S)$?

A $\frac{7}{22}$

B $\frac{1}{3}$

C $\frac{3}{8}$

D $\frac{5}{12}$

E $\frac{1}{2}$

Solution:

Fix R. Of the other 11 vertices, 5 sit at distance $1, 5$ at distance 2, and 1 (the opposite vertex) at distance 3. Pick ordered distinct Q, S from these 11 : that's $11 \cdot 10 = 110$ pairs. The ones with $d(R, Q) = d(R, S)$ number $5 \cdot 4 + 5 \cdot 4 + 1 \cdot 0 = 40$, so $P(\text{equal}) = \frac{40}{110} = \frac{4}{11}$. By the symmetry between Q and S, the $>$ and $<$ cases split the rest evenly, so $P(d(Q, R) > d(R, S)) = \frac{1-\frac{4}{11}}{2} = \frac{7}{22}$. Thus, **A** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2023A