2021 AMC 10B Fall Solutions

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https://live.poshenloh.com/past-contests/amc10/2021D/solutions

1. What is the value of

$$1234 + 2341 + 3412 + 4123?$$

A $10,000$

B $10,010$

C $10,110$

D $11,000$

E $11,110$

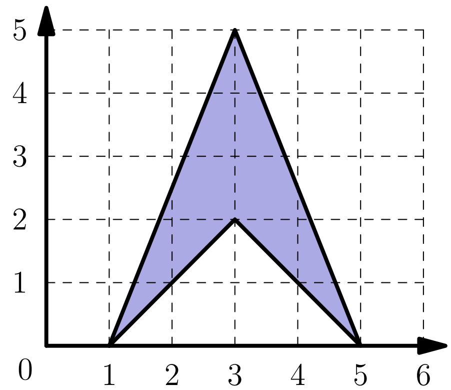
Solution(s):

We can add each individual digit, yielding $11,110$.

We can also get the sum by noticing that each digit has a sum of 10, so the sum is equal to $10 \cdot 1111 = 11,110$.

Thus, the answer is **E**.

2. What is the area of the shaded figure shown below?



A 4

B 6

C 8

D 10

E 12

Solution(s):

The area is a triangle of area

$$\frac{4 \cdot 5}{2} - \frac{2 \cdot 4}{2} = 10 - 4$$
$$= 6$$

since we subtract the area of a smaller triangle from a larger triangle.

Thus, the answer is **B**.

3. The expression

$$\frac{2021}{2020} - \frac{2020}{2021}$$

is equal to the fraction $\frac{p}{q}$ in which p and q are positive integers whose greatest common divisor is 1. What is p?

- A 1
- B 9
- C 2020
- D 2021
- E 4041

Solution(s):

We can rewrite this as

$$\frac{2021 \cdot 2021}{2020 \cdot 2021} - \frac{2020 \cdot 2020}{2020 \cdot 2021}$$

$$= \frac{2021^2 - 2020^2}{2020 \cdot 2021}.$$

This can be simplified to

$$\frac{(2021 - 2020)(2021 + 2020)}{2020 \cdot 2021}$$

$$= \frac{4041}{2020 \cdot 2021}.$$

Since 4041 is coprime with both 2020 and 2021, we know 4041 is the numerator.

Thus, the answer is **E**.

4. At noon on a certain day, Minneapolis is N degrees warmer than St. Louis. At $4{:}00$ the temperature in Minneapolis has fallen by 5 degrees while the temperature in St. Louis has risen by 3 degrees, at which time the temperatures in the two cities differ by 2 degrees. What is the product of all possible values of N?

A 10

B 30

C 60

D 100

E 120

Solution(s):

Let the temperature in Minneapolis be m and let the temperature in St. Louis be s. Then, we know that

$$|(m-5)-(s+3)|=2.$$

This means

$$|((m-s)-8|=2,$$

so

$$m-s=8\pm 2,$$

thus making the difference 6 or 10. Therefore, the product is $6\cdot 10=60$.

Thus, the answer is **C**.

5. Let $n = 8^{2022}$. Which of the following is equal to $\frac{n}{4}$?

A 4^{1010}

B 2^{2022}

C 8^{2018}

D 4^{3031}

E 4^{3032}

Solution(s):

We know

$$\begin{aligned} n &= 8^{2022} \\ &= 2^{6066} \\ &= 4^{3033}. \end{aligned}$$

Therefore,

$$\frac{n}{4} = n \cdot 4^{-1} = 4^{3032}.$$

Thus, the answer is **E**.

6. The least positive integer with exactly 2021 distinct positive divisors can be written in the form $m \cdot 6^k$, where m and k are integers and 6 is not a divisor of m. What is $m + k$?

A 47

B 58

C 59

D 88

E 90

Solution(s):

Before starting, note that if we can represent the prime factorization of an integer z as

$$z = p_1^{e_1} p_2^{e_2} \cdots,$$

then there are $(e_1 + 1)(e_2 + 1) \cdots$ distinct positive factors.

If the number in question has 2021 factors, by the previous logic,

$$2021 = (e_1 + 1)(e_2 + 1) \cdots,$$

and as the prime factorization of $2021 = 43 \cdot 47$, then our number must be $p_1^{46} p_2^{42}$ or p^{2020}.

The smallest number we can make in either of these is making $p_1 = 2, p_2 = 3$ in the first configuration, yielding

$$2^{46} 3^{42} = 16 \cdot 6^{42}.$$

Therefore,

$$m = 16$$

$$k = 42,$$

so

$$m + k = 42 + 16 = 58.$$

Thus, the answer is **B**.

7. Call a fraction $\frac{a}{b}$, not necessarily in the simplest form, "special" if a and b are positive integers whose sum is 15. How many distinct integers can be written as the sum of two, not necessarily different, special fractions?

A 9

B 10

C 11

D 12

E 13

Solution(s):

Let the denominators of both fractions be x, y. Therefore, their sums are

$$\frac{15-x}{x}+\frac{15-y}{y}$$

$$=\frac{15}{x}+\frac{15}{y}-2.$$

Thus, we need to find the number of unique integers we can get from

$$\frac{15}{x}+\frac{15}{y}.$$

If we have $x=1,3,5$, then our fraction is an integer, which would be $15,5,3$. Adding each set of pairs yields $30,20,18,10,8,6$.

If we have $x=2,6,10$, then our fractional part is a half, which would be $7.5,2,5,1.5$. Adding each set of pairs yields $15,10,9,5,4,3$.

If we have $x=4,12$, then our fractional part is a quarter or three quarters, which would be $3.75,1.25$. Adding each set of pairs yields $7.5,5,2.5$.

The unique integers from this are $30,20,15,10,9,8,6,5,4,3$, of which there are 11.

Thus, the answer is **C**.

8. The greatest prime number that is a divisor of $16,384$ is 2 because $16,384 = 2^{14}$. What is the sum of the digits of the greatest prime number that is a divisor of $16,383$?

- A 3
- B 7
- C 10
- D 16
- E 22

Solution(s):

We know

$$\begin{aligned} 16,383 &= 16384 - 1 \\ &= 2^{14} - 1 \\ &= (2^7 - 1)(2^7 + 1) \\ &= 127 \cdot 129. \end{aligned}$$

Since $129 = 3 \cdot 43$, we get

$$16383 = 3 \cdot 43 \cdot 127.$$

Therefore, 127 is the largest prime factor, and the sum of its digits is 10.

Thus, the answer is **C**.

9. The knights in a certain kingdom come in two colors. $\frac{2}{7}$ of them are red, and the rest are blue. Furthermore, $\frac{1}{6}$ of the knights are magical, and the fraction of red knights who are magical is 2 times the fraction of blue knights who are magical. What fraction of red knights are magical?

A $\frac{2}{9}$

B $\frac{3}{13}$

C $\frac{7}{27}$

D $\frac{2}{7}$

E $\frac{1}{3}$

Solution(s):

Let r, b be the number of red and blue knights, and let r_m, b_m be the number of magical knights of each color. We know

$$r_m + b_m = \frac{1}{6}.$$

We also know

$$\frac{r_m}{r} = 2\frac{b_m}{b}.$$

Note that

$$\begin{aligned} b &= 1 - r \\ &= 1 - \frac{2}{7} \\ &= \frac{5}{7}. \end{aligned}$$

Therefore,

$$\frac{r_m}{\frac{2}{7}} = 2\frac{b_m}{\frac{5}{7}},$$

so $b_m = 1.25r_m$.

As such, $2.25r_m = \frac{1}{6}$, so $r_m = \frac{2}{27}$. This makes the fraction that are magical equal to

$$\frac{\frac{2}{27}}{\frac{2}{7}} = \frac{7}{27}.$$

Thus, the answer is **C**.

10. Forty slips of paper numbered 1 to 40 are placed in a hat. Alice and Bob each draw one number from the hat without replacement, keeping their numbers hidden from each other. Alice says, "I can't tell who has the larger number." Then Bob says, "I know who has the larger number." Alice says, "You do? Is your number prime?" Bob replies, "Yes." Alice says, "In that case, if I multiply your number by 100 and add my number, the result is a perfect square. " What is the sum of the two numbers drawn from the hat?

A 27

B 37

C 47

D 57

E 67

Solution(s):

Alice saying that she doesn't know the number means that she doesn't have the largest or smallest possible numbers, which are $1, 40$. Bob, now knows who has the largest number. Thus, he may have 1 or 40. He may also have 2 or 39 as he would know Alice doesn't have $1, 40$. This means 2 would mean he has the lesser number, and 39 means he has the greater number. Since his number is prime, his number must be 2.

Since his number is 2, we know 200 plus Alice's number is a sqaure. Since Alice's number is between 1 and 40, we must find a square somewhere from 201 to 240, which must be 225. This makes Alice's number 25. Therefore the sum is

$$2 + 25 = 27.$$

Thus, the answer is **A**.

11. A regular hexagon of side length 1 is inscribed in a circle. Each minor arc of the circle determined by a side of the hexagon is reflected over that side. What is the area of the region bounded by these 6 reflected arcs?

A $\frac{5\sqrt{3}}{2} - \pi$

B $3\sqrt{3} - \pi$

C $4\sqrt{3} - \frac{3\pi}{2}$

D $\pi - \frac{\sqrt{3}}{2}$

E $\frac{\pi+\sqrt{3}}{2}$

Solution(s):

The average of the area of the circle and the new figure is equal to the area of the hexagon. The hexagon's area can be taken as the area of 6 equilateral triangles with length 1, making them each have area $\frac{\sqrt{3}}{4}$. Thus, the total area of the hexagon is

$$6 \cdot \frac{\sqrt{3}}{4} = 3\frac{\sqrt{3}}{2}.$$

The radius of the circle is 1 since thats the length of the equilateral triangle, so the area of the circle is $\pi \cdot 1^2 = \pi$.

Let the area of the shape be a. Then, by the first statement, we know

$$\frac{a+\pi}{2} = \frac{3\sqrt{3}}{2},$$

so

$$a + \pi = 3\sqrt{3},$$

making

$$a = 3\sqrt{3} - \pi.$$

Thus, the answer is **B**.

12. Which of the following conditions is sufficient to guarantee that integers x, y, and z satisfy the equation

$$x(x-y)+y(y-z)+z(z-x) = 1?$$

A $x > y$ and $y = z$

B $x = y - 1$ and $y = z - 1$

C $x = z + 1$ and $y = x + 1$

D $x = z$ and $y - 1 = x$

E $x + y + z = 1$

Solution(s):

Notice

$$\begin{aligned} & x(x-y)+y(y-z)+z(z-x) \\ & = x^2+y^2+z^2-xy-xz-yz \\ & = \frac{1}{2}\big((x-y)^2+(x-z)^2 \\ & +(y-z)^2\big). \end{aligned}$$

Thus,

$$(x-y)^2+(x-z)^2+(y-z)^2 = 2.$$

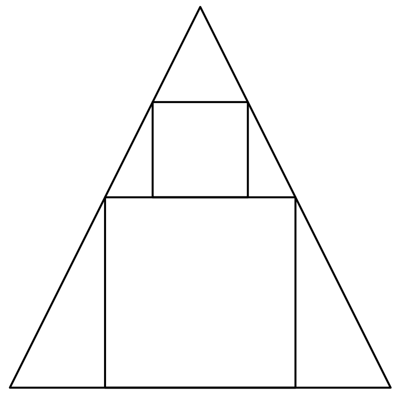
Since every term is a positive integer, we know that two of them are 1 and the other is 0, or two of them are 0 and the other is 2.

Since they must be squares, it has to be the first condition. If we have a term with $(a-b)^2 = 0$, then $a = b$. If we have $(a-b)^2 = 1$, then, $|a-b| = 1$.

Thus, two of x, y, z must be equal and the other number must be 1 away from the equal numbers. This is guaranteed with the condition $x = z$ and $y - 1 = x$.

Thus, the answer is **D**.

13. A square with side length 3 is inscribed in an isosceles triangle with one side of the square along the base of the triangle. A square with side length 2 has two vertices on the other square and the other two on sides of the triangle, as shown. What is the area of the triangle?



A $19\frac{1}{4}$

B $20\frac{1}{4}$

C $21\frac{3}{4}$

D $22\frac{1}{2}$

E $23\frac{3}{4}$

Solution(s):

Firstly, note the area is equal to

$$\frac{bh}{2}.$$

Now, if we take the smaller triangle and scale it up to the bigger triangle, we multiply by $\frac{3}{2}$. The base of the smaller triangle is 3, so the base of the larger triangle is $\frac{3}{2} \cdot 3 = \frac{9}{2}$. This makes the area equal to $\frac{9}{4}h$.

Now, the heights of the squares go down in a scaling factor of $\frac{2}{3}$. This means if we continually put smaller and smaller squares at the top, their lengths would be multiplied by $\frac{2}{3}$, so the total height is

$$
\begin{aligned}
&3+3\cdot\frac{2}{3}+3\cdot\left(\frac{2}{3}\right)^2\cdots\\
&=3(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2\cdots\\
&=3\cdot\frac{1}{1-\frac{2}{3}}\\
&=3\cdot\frac{1}{\frac{1}{3}}\\
&=9.
\end{aligned}
$$

Therefore, the area is

$$9\cdot\frac{9}{4}=\frac{81}{4}=20\frac{1}{4}.$$

Thus, the answer is **B**.

14. Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4?

A $\frac{3}{4}$

B $\frac{57}{64}$

C $\frac{59}{64}$

D $\frac{187}{192}$

E $\frac{63}{64}$

Solution(s):

We will first count the number of ways to have the product be not divisible be 4. This can be done if the product is odd, where all numbers are odd, or the product is even but not a multiple of 4, in which 5 die are odd and the other die is 2 or 6.

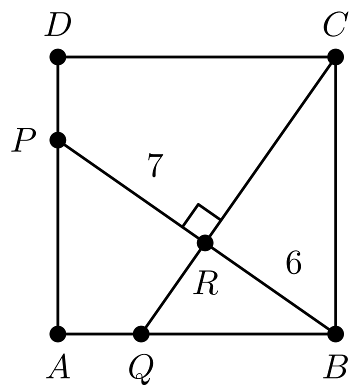
In the first case, we can do this with a probability of $\frac{1}{2}^6 = \frac{1}{64}$.

In the second case, there is a $\frac{2}{6}$ probability that a chosen dice is 2 or 6, a $\frac{1}{2}^5$ probability of the other die being even, and we have 6 ways to choose the choosen dice. This makes the probability $6 \cdot \frac{2}{6} \cdot \frac{1}{2}^5 = \frac{4}{64}$.

Therefore, the total probability that the product isn't divisible by 4 is $\frac{5}{64}$ making the probability that it is divisible equal to $\frac{59}{64}$.

Thus, the answer is **C**.

15. In square $ABCD$, points P and Q lie on $\overline{AD}$ and $\overline{AB}$, respectively. Segments $\overline{BP}$ and $\overline{CQ}$ intersect at right angles at R, with $BR = 6$ and $PR = 7$. What is the area of the square?



A 85

B 93

C 100

D 117

E 125

Solution(s):

We know

$$\begin{aligned}\angle QBR &= 90^\circ - \angle BQC \\ &= 90^\circ - (90^\circ - \angle BCQ) \\ &= \angle BCQ.\end{aligned}$$

Since

$$\angle BCQ = \angle ABP,$$

$$\angle PAB = \angle QBC,$$

and $AB = BC$, we know $PAB \cong QBC$.

Therefore, $QC = PB = 13$. As such, $QR = RC$.

Also, since $QRB \sim BRC$, we know

$$QR \cdot RC = RB^2 = 36.$$

Let $RC = x$. Then,

$$x + \frac{36}{x} = 13,$$

so

$$x^2 - 13x + 36 = 0$$

$$(x-4)(x-9) = 0.$$

Thus, $x = 4, 9$. We know $x = 9$ since its the greater number. Then,

$$\begin{aligned} CB^2 &= BR^2 + CR^2 \\ &= 9^2 + 6^2 \\ &= 81 + 36 \\ &= 117. \end{aligned}$$

Therefore, the area is

$$CB^2 = 117.$$

Thus, the answer is **D**.

16. Five balls are arranged around a circle. Chris chooses two adjacent balls at random and interchanges them. Then Silva does the same, with her choice of adjacent balls to interchange being independent of Chris's. What is the expected number of balls that occupy their original positions after these two successive transpositions?

- A 1.6
- B 1.8
- C 2.0
- D 2.2
- E 2.4

Solution(s):

Let Chris choose his two balls.

There is a $\frac{1}{5}$ probability that Silva choose the same balls, which would make all the balls the same as the original, making it such that 5 balls are the same.

There is a $\frac{2}{5}$ probability that Silva choose one of the same balls, which would make 2 the balls the same as the original.

There is a $\frac{2}{5}$ probability that Silva choose none of the same balls as Chris, so 4 balls are in different positions. This makes 1 ball the same.

Therefore, the expected value is

$$\begin{aligned} 5 \cdot \frac{1}{5} + 2 \cdot \frac{2}{5} + 1 \cdot \frac{2}{5} &= \frac{11}{5} \\ &= 2.2. \end{aligned}$$

Thus, the answer is **D**.

17. Distinct lines ℓ and m lie in the xy-plane. They intersect at the origin. Point $P(-1, 4)$ is reflected about line ℓ to point P', and then P' is reflected about line m to point P''. The equation of line ℓ is $5x - y = 0$, and the coordinates of P'' are $(4, 1)$. What is the equation of line m?

A $5x + 2y = 0$

B $3x + 2y = 0$

C $x - 3y = 0$

D $2x - 3y = 0$

E $5x - 3y = 0$

Solution(s):

Let the line from the origin to the point P be p. Let the line from the origin to the point P'' be r.Let $\theta_\ell, \theta_m, \theta_r$ be the angle from each of the lines to the origin. If line a is reflected across line b, then the mean of the angles of a and its reflection is equal to the angle of b. If a' is the reflection, we get

$$\frac{\theta_a + \theta_{a'}}{2} = \theta_b$$

which means

$$\theta_{a'} = 2\theta_b - \theta_a.$$

Bringing this to our current problem, the angle after the first reflection is $2\theta_\ell - \theta_p$. Then, the angle after the second reflection is

$$2\theta_m - (2\theta_\ell - \theta_p)$$

$$= 2(\theta_m - \theta_\ell) + \theta_p.$$

Notice that P'' is P rotated 90° clockwise, so it lowers the angle by 90°. This means

$$2(\theta_m - \theta_\ell) + \theta_p = \theta_p - 90^\circ.$$

Therefore, $\theta_m - \theta_\ell = -45^\circ$. This further implies $\theta_m = \theta_\ell - 45^\circ$.

Since the slope of ℓ is 5, we get $\tan(\theta_\ell) = 5$. The tangent subtraction formula yields

$$\begin{aligned} \tan(\theta_m) &= \frac{\tan(\theta_\ell) - \tan(45^\circ)}{1 + \tan(\theta_\ell)\tan(45^\circ)} \\ &= \frac{5-1}{1+5} \\ &= \frac{4}{6} \\ &= \frac{2}{3}. \end{aligned}$$

Therefore, our line is

$$y = \frac{2}{3}x,$$

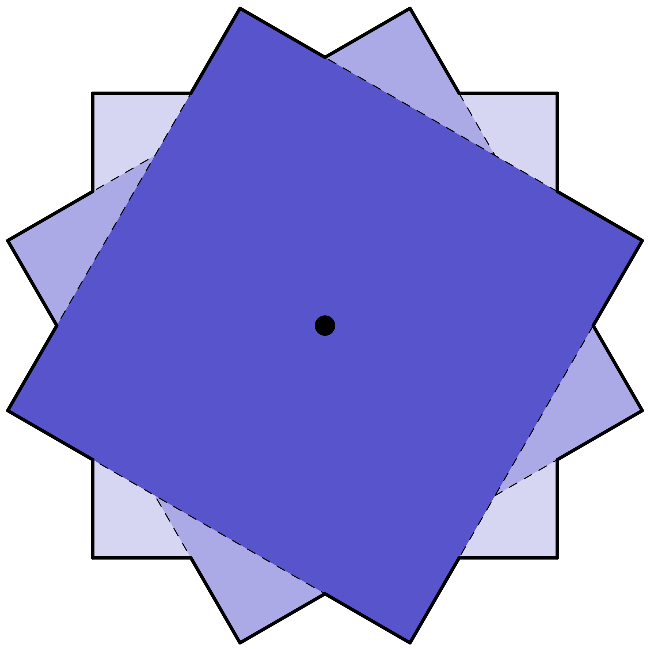
or

$$2x - 3y = 0.$$

Thus, the answer is **D**.

18. Three identical square sheets of paper each with side length 6 are stacked on top of each other. The middle sheet is rotated clockwise 30° about its center and the top sheet is rotated clockwise 60° about its center, resulting in the 24-sided polygon shown in the figure below.

The area of this polygon can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. What is $a + b + c$?



A 75

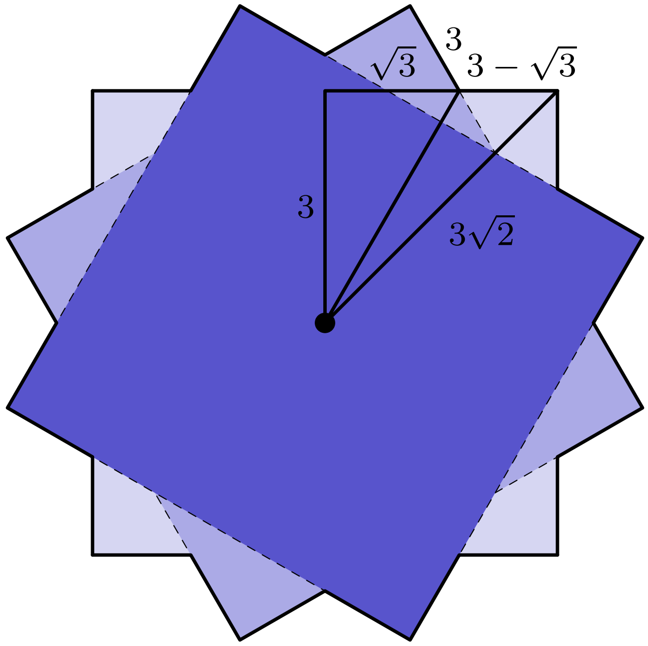
B 93

C 96

D 129

E 147

Solution(s):



This shape can be split into identical 24 triangles, as shown in the diagram. These triangles have one angle of 45° as it is an angle bisector of a right triangle.

Another angle is 15° as it is $\frac{1}{24}$ of the full way around, so the angle is $\frac{360}{24} = 15$. Thus, the last angle is 120°.

Now, we extend the shortest side to make a right triangle. This has angles $45, 45, 90$. The altitude is 3, since its half of the length of the square. The base of the right triangle is 3, as its also half of the square. Now, we find the portion of the base in the original triangle by subtracting the portion outside.

The portion outside the original triangle creates a $30-60-90$ triangle when using the altitude, so its base is

$$3\tan(30^\circ) = \sqrt{3}.$$

Thus, the base of the original triangle is $3-\sqrt{3}$. Therefore, each triangle has an area of

$$\frac{3(3-\sqrt{3})}{2} = \frac{9-3\sqrt{3}}{2}.$$

Since there are 24 such angles, the total area is

$$24\left(\frac{9-3\sqrt{3}}{2}\right) = 108 - 36\sqrt{3}.$$

This makes

$$a, b, c = 108, 54, 3$$

respectively, so

$$a + b + c = 147.$$

Thus, the answer is **E**.

19. Let N be the positive integer $7777\ldots777$, a 313-digit number where each digit is a 7. Let $f(r)$ be the leading digit of the rth root of N. What is

$$f(2)+f(3)+f(4)$$
$$+f(5)+f(6)?$$

A 8

B 9

C 11

D 22

E 29

Solution(s):

We can take

$$N=\frac{7\left(10^{313}-1\right)}{9}.$$

Notice that the first digit of a number doesn't change when divided by 10. Thus, multiplying by a power of 10 would preserve the same leading digit.

Next, we can approximate N to be

$$N=\frac{7\left(10^{313}\right)}{9}$$

as this wouldn't be increase the leading digit. Now, we can case on each number functional value.

$f(2)$ means we have to find the first digit of

$$\sqrt{\frac{7\left(10^{313}\right)}{9}},$$

which has the same units digit as

$$\frac{\sqrt{\frac{7(10^{313})}{9}}}{10^{156}} = \sqrt{7.\overline{7}}.$$

This has leading digit 2, so $f(2) = 2$.

$f(3)$ means we have to find the first digit of

$$\sqrt[3]{\frac{7\left(10^{313}\right)}{9}},$$

which has the same units digit as

$$\frac{\sqrt[3]{\frac{7(10^{313})}{9}}}{10^{104}} = \sqrt[3]{7.\overline{7}}.$$

This has leading digit 1, so $f(3) = 1$.

$f(4)$ means we have to find the first digit of

$$\sqrt[4]{\frac{7\left(10^{313}\right)}{9}},$$

which has the same units digit as

$$\frac{\sqrt[4]{\frac{7(10^{313})}{9}}}{10^{78}} = \sqrt[4]{7.\overline{7}}.$$

This has leading digit 1, so $f(4) = 1$.

$f(3)$ means we have to find the first digit of

$$\sqrt[5]{\frac{7\left(10^{313}\right)}{9}},$$

which has the same units digit as

$$\frac{\sqrt[5]{\frac{7(10^{313})}{9}}}{10^{62}} = \sqrt[5]{777.\overline{7}}.$$

Note that $3^5 = 243, 4^5 = 1024$, so $\sqrt[5]{777.\overline{7}}$ has leading digit 3. Thus, $f(5) = 3$.

$f(3)$ means we have to find the first digit of

$$\sqrt[6]{\frac{7\left(10^{313}\right)}{9}},$$

which has the same units digit as

$$\frac{\sqrt[6]{\frac{7(10^{313})}{9}}}{10^{52}} = \sqrt[6]{7.\overline{7}}.$$

This has leading digit 1, so $f(6) = 1$.

We now know

$$\begin{aligned} & f(2) + f(3) + f(4) + f(5) + f(6) \\ & = 2 + 1 + 1 + 3 + 1 \\ & = 8. \end{aligned}$$

Thus, the answer is **A**.

20. In a particular game, each of 4 players rolls a standard 6-sided die. The winner is the player who rolls the highest number. If there is a tie for the highest roll, those involved in the tie will roll again and this process will continue until one player wins. Hugo is one of the players in this game. What is the probability that Hugo's first roll was a 5, given that he won the game?

A $\frac{61}{216}$

B $\frac{367}{1296}$

C $\frac{41}{144}$

D $\frac{185}{648}$

E $\frac{11}{36}$

Solution(s):

The probability that Hugo rolled a 5 given that he won is equal to the probability that Hugo rolled a 5 and won divided by the probability that he won. The probability that he wins is $\frac{1}{4}$, so dividing by this is equal to multiplying by 4. Thus, we need to just find 4 times the probability that Hugo rolled a 5 and won.

Now, to find the probability that he won given that he rolled a 5, we need to case on the number of people who tied with him.

Case 1: No one ties

This has a probability of

$$\frac{1}{6} \cdot \frac{4}{6}^3 = \frac{64}{6^4}$$

as there are 3 players who choose something from 1 to 4.

Case 2: One person ties

This has a probability of

$$\frac{1}{6}^2 \cdot \frac{4}{6}^2 \cdot \binom{3}{1} = \frac{48}{6^4}$$

as there are 3 players who choose something from 1 to 4. Now, the probability that Hugo wins here is $\frac{1}{2}$ given this event occured, so the total probability is

$$\frac{48}{6^4} \cdot \frac{1}{2} = \frac{24}{6^4}$$

Case 3: Two people tie

This has a probability of

$$\frac{1}{6}^3 \cdot \frac{4}{6}^1 \cdot \binom{3}{2} = \frac{12}{6^4}$$

as there are 3 players who choose something from 1 to 4. Now, the probability that Hugo wins here is $\frac{1}{3}$ given this event occured, so the total probability is

$$\frac{12}{6^4} \cdot \frac{1}{3} = \frac{4}{6^4}$$

Case 4: Three people tie

This has a probability of

$$\frac{1}{6}^4 = \frac{64}{6^4}$$

as there are 3 players who choose something from 1 to 4. Now, the probability that Hugo wins here is $\frac{1}{4}$ given this event occured, so the total probability is

$$\frac{1}{6^4} \cdot \frac{1}{4} = \frac{0.25}{6^4}$$

The combined probability is

$$\frac{64 + 24 + 4 + 0.25}{6^4} = \frac{92.25}{1296}.$$

This now has to be multiplyied by 4, yielding

$$\frac{369}{1296} = \frac{41}{144}.$$

Thus, the answer is **C**.

21. Regular polygons with 5, 6, 7, and 8 sides are inscribed in the same circle. No two of the polygons share a vertex, and no three of their sides intersect at a common point. At how many points inside the circle do two of their sides intersect?

A 52

B 56

C 60

D 64

E 68

Solution(s):

Suppose we have two regular polygons with a and b sides such that $a \geq b$. On, the circle, there area b arcs from one point of the polygon to the other. If a line goes from 1 arc to another arc, then it intersects the polygon twice, as there are two lines. The polygon with a sides goes across each of the b arcs, so it crosses $2b$ lines.

To get the intersections of the points in our given setup, we can take the sum of the intersections with each pair of polygons. In each of the pairs, we take 2 times the lower number, and take the sum of this. This would be

$$3 \cdot 2 \cdot 5 + 2 \cdot 2 \cdot 6 + 1 \cdot 2 \cdot 7$$
$$= 68.$$

Thus, the answer is **E**.

22. For each integer $n \geq 2$, let S_n be the sum of all products jk, where j and k are integers and $1 \leq j < k \leq n$. What is the sum of the 10 least values of n such that S_n is divisible by 3?

A 196

B 197

C 198

D 199

E 200

Solution(s):

The products that are added from S_{n-1} to S_n are those where $k = n$ and $j < n$. The sum of these are the sums of all positive integers less than $n-1$ times n, making it

$$n \cdot \frac{n(n-1)}{2} = \frac{n^2(n-1)}{2}.$$

This makes

$$S_n = S_{n-1} + \frac{n^2(n-1)}{2}.$$

If $n \equiv 0, 1 \mod 3$, then

$$\frac{n^2(n-1)}{2}$$

is a multiple of 3, so

$$S_n \equiv S_{n-1} \mod 3.$$

If $n \equiv 2 \mod 3$, then

$$\frac{n^2(n-1)}{2} \equiv \frac{2^2 \cdot 1}{2}$$
$$\equiv 2 \mod 3,$$

so $S_n \equiv S_{n-1} + 2 \mod 3$.

This means that the remainder of S_n when divided by 3 is increased by 2 when $n \equiv 2 \mod 3$, and its remainder is the same otherwise.

$$S_2 = 1 \cdot 2$$
$$= 2$$

as the only pair (j, k) is $(1, 2)$.

The it must increase by $2 \mod 3$ twice to be a multiple of 3, so the first n that works is $n = 8$. Then, $n = 8, 9, 10$ have S_n being multiples of 3. After 8, the next occurence is after 3 more changes, making $n = 17, 18, 19$ work, and then $n = 26, 27, 28$ with 3 changes after that. Finally, the 10th number that works is $n = 35$.

We can now find the sum to be

$$3 \cdot 9 + 3 \cdot 18 + 3 \cdot 27 + 35$$
$$= 197.$$

(Note that I did 3 times some number as it was the average of the tuple of 3 numbers around it.)

Thus, the answer is **B**.

23. Each of the 5 sides and the 5 diagonals of a regular pentagon are randomly and independently drawn as solid or dashed with equal probability. What is the probability that there will be a triangle whose vertices are among the vertices of the pentagon such that all of its sides have the same stroke type?

A $\frac{2}{3}$

B $\frac{105}{128}$

C $\frac{125}{128}$

D $\frac{253}{256}$

E 1

Solution(s):

We will do this with complementary counting, meaning we will find the probability that no triangle has three sides of the same stroke first.

Now we will case on the stroke configurations of the outside edges. I will use the notation $a-b-c-d$ or some shortened version of this to specify the stroke, with the numbers being how many of one stroke there is, and a dash means that it switches strokes afterwards. There must be an even number of numbers in the configurations, except for the case where there is just one number. Therefore, the configurations that are possible can be found by adding an even number of whole numbers that add to 5. Therefore, these are the possible configurations:

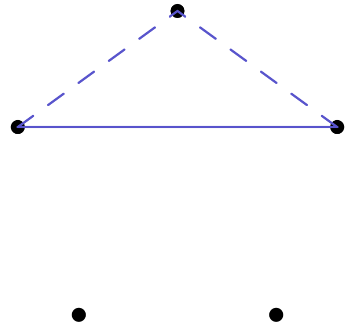
$$5,$$

$$4-1,$$

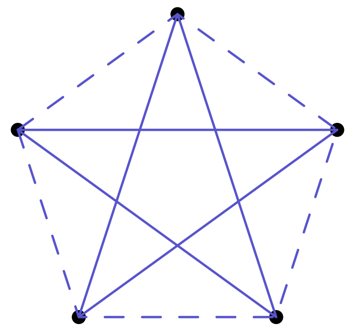
$$3-2,$$

$$2-1-1-1.$$

Notice how all the configurations require at least one adjacent pair of edges to be the same stroke type. If a pairing of adjacent sides exists like this, then the connection must be the opposite stroke as shown below



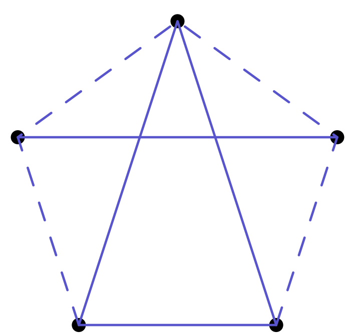
Case 1: The configuration 5



This can occur with probability of $\frac{1}{16}$ as there are two stroke types, each with a $\frac{1}{32}$ chance of happening. Due to the pairing of every adjacent side, we know the diagonals must all be the same stroke, which has a probability of $\frac{1}{32}$. Thus, the probability of this case is $\frac{1}{16 \cdot 32} = \frac{1}{512}$.

Case 2: The configuration $4-1$

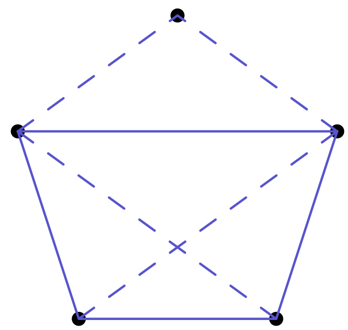
This can occur with probability of $\frac{5}{16}$ as there are two strokes that can be the 1 and 5 positions for the edge, each with a $\frac{1}{32}$ chance of happening. After filling the required diagonals, we have the configuration as shown.



This has the long triangle with a base at the bottom having all of one stroke, so there are no possible configurations.

Case 3: The configuration $3-2$

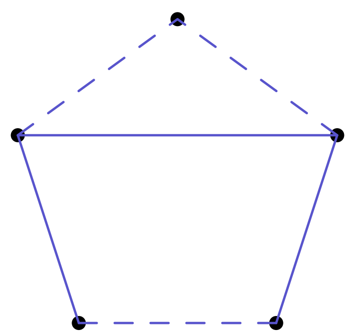
This can occur with probability of $\frac{5}{16}$ as there are two stroke types that can be the 1 and 5 positions for the edge, each with a $\frac{1}{32}$ chance of happening. After filling the required diagonals, we have the configuration as shown.



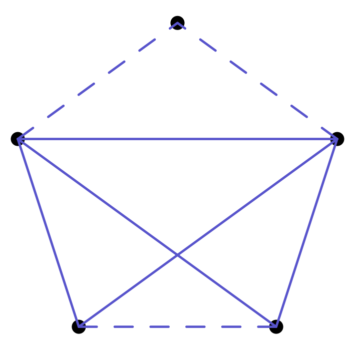
The other two diagonals have to be solid to have the longer triangles with the upper right and left edge have at least one distinct side. This makes the long triangle with the bottom side all solid, so there are no possible configurations.

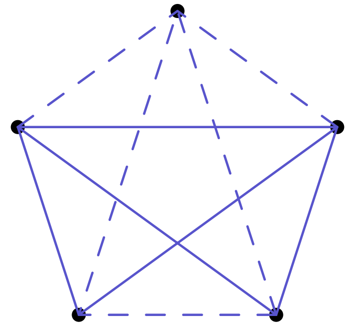
Case 3: The configuration $2-1-1-1$

This can occur with probability of $\frac{5}{16}$ as there are two strokes that can be the 2 and 5 positions for the adjacent pair, each with a $\frac{1}{32}$ chance of happening. After filling the required diagonals, we have the configuration as shown.



The other two diagonals have to be solid to have the longer triangles with the upper right and left edge have at least one distinct side. This makes the long triangle with the bottom side all solid, so there are no possible configurations. Then we fill the diagonals in the following way.





This has only one configuration, with probability $\frac{1}{32}$, so the total probability is $\frac{5}{16 \cdot 32} = \frac{5}{512}$.

Thus, the total probability of no triangles with all the same stroke is $\frac{5+1}{512} = \frac{3}{256}$.

This makes the answer to the original question $\frac{253}{256}$.

Thus, the answer is **D**.

24. A cube is constructed from 4 white unit cubes and 4 blue unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

A 7

B 8

C 9

D 10

E 11

Solution(s):

Suppose somewhere that I have an L-shape of blues, with 3 blue cubes in it. Then, there are 5 locations to put the last blue cube, each of which can't rotate onto a different configuration. We can always rotate the L-shape to be in some configuration, so each cube with some L-shape is accounted for. Now, we can count the number of configurations without any L-shapes.

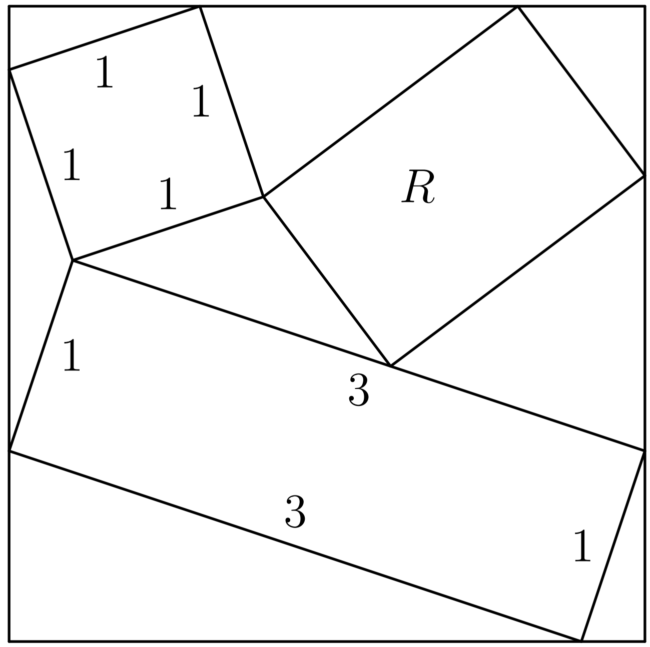
Now suppose we have two adjacent blues and no L-shapes. If we have a blue in any of the 4 positions adjacent to it, we have an L-shape, so we avoid this case. This leaves one configuration with just two adjacent blues and no L-shapes. The original pair can be rotated around, so ever configuration is accounted for.

Suppose we have no adjacent blues. If I have some blue cube, then the 3 cubes around it must be white. Furthermore, we can't have the opposite corner being blue as that would ensure that one of the other 2 blue cubes touch the first one. This leaves just 3 locations for the other blues, so there is just one way to place it. The original blue can be rotated around, so ever configuration is accounted for.

There are therefore $5 + 1 + 1 = 7$ configurations total.

Thus, the answer is **A**.

25. A rectangle with side lengths 1 and 3, a square with side length 1, and a rectangle R are inscribed inside a larger square as shown. The sum of all possible values for the area of R can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?



A 14

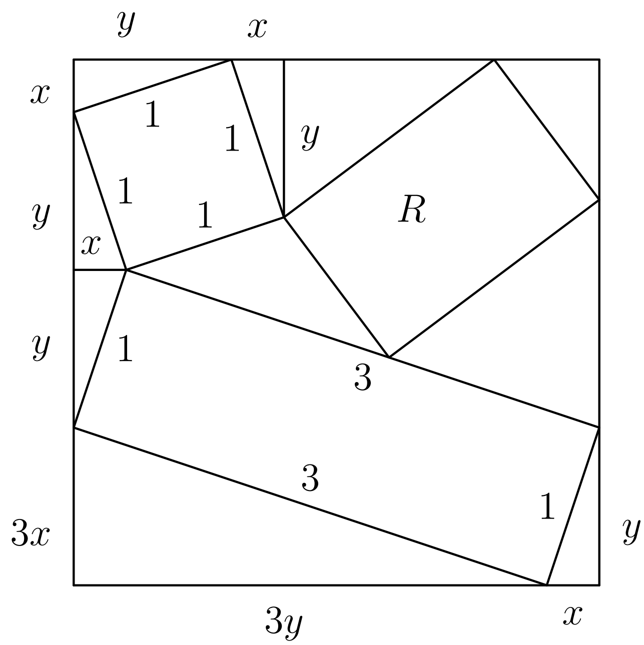
B 23

C 46

D 59

E 67

Solution(s):

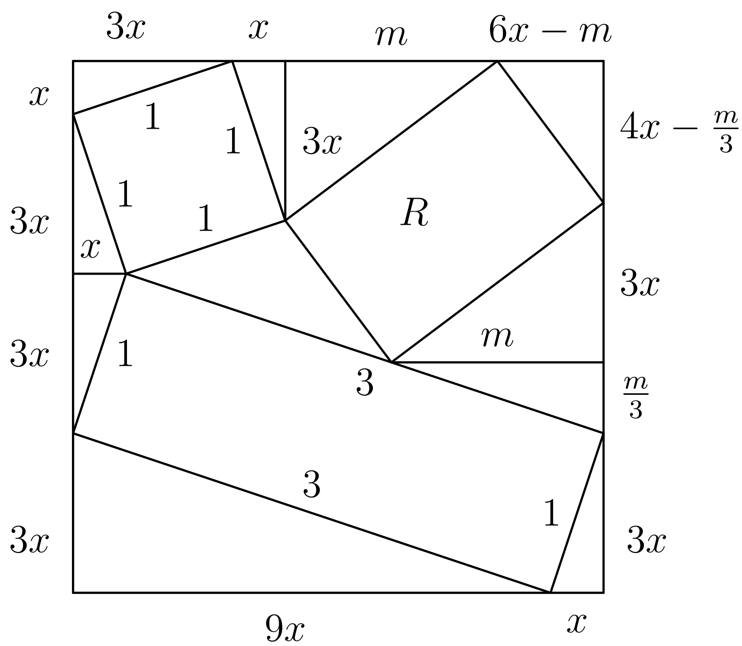


First we can fill in the diagram as such due to the similar triangles. The left side has length $4x + 2y$ and the bottom side has length $3y + x$. Thus, $3y + x = 4x + 2y$, making $3x = y$. This also means the side length is

$$4x + 2(3x) = 10x.$$

Now, the top part is partitioned by the vertical line as shown in the picture. The entire section to the right of this line has a length of $6x$. Let the part of the to the left of the the point in the rectangle be m. From there, make a horizontal line from the bottom point of the rectangle to the right side.

There now would be two right triangles with their hypoteneus being the the upper left and lower right lines in the rectangle. Since their sides are all parallel, they are similar. Since their hypoteneuses are the same, they are congruent. Therefore, we can assign them both the same side lengths. From here, we may use similar triangles to give a length of $\frac{m}{3}$ to the section below the line. We can fill the rest of the lengths out from there as shown below.



With similar triangles, we get

$$\frac{m}{3x} = \frac{4x - \frac{m}{3}}{6x - m}.$$

This means $6xm - m^2 = 12x^2 - xm$, so $m^2 - 7xm + 12x^2 = 0$. This factors to

$$(m - 3x)(m - 4x) = 0,$$

so $m = 3x$ or $m = 4x$. If $m = 3x$, then one of the side lengths is $3\sqrt{2}x$. The other right triangle has sidelengths $3x, 3x$, so it has a side length of $3\sqrt{2}x$. This makes the area $18x^2$. If $m = 4x$, then one of the side lengths is $5x$. The other right triangle has sidelengths $2x$, $\frac{8}{3}x$, so it has a side length of $\frac{10}{3}x$. This makes the area $\frac{50}{3}x^2$. The sum of the different areas is $\frac{104}{3}x^2$.

Using the Pythagorean Theorem, we get

$$(x^2) + (3x)^2 = 1,$$

so

$$10x^2 = 1.$$

Thus,

$$x^2 = \frac{1}{10}.$$

Therefore, the sum is

$$\frac{1}{10} \cdot \frac{104}{3} = \frac{52}{15}.$$

As such, the answer is is $52 + 15 = 67$.

Thus, the answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc10/2021D