# 2021 AMC 10A Spring Solutions

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https://live.poshenloh.com/past-contests/amc10/2021A/solutions



**1.** What is the value of

$$(2^2-2)-(3^2-3)+(4^2-4)?$$

A $1$

B $2$

C $5$

D $8$

E $12$

### Solution(s):

$$\begin{aligned}(2^2-2)-&(3^2-3)+(4^2-4)\\ &=2-6+12\\ &=8.\end{aligned}$$

Thus, **D** is the correct answer.

**2.** Portia's high school has $3$ times as many students as Lara's high school. The two high schools have a total of $2600$ students. How many students does Portia's high school have?

A $600$

B $650$

C $1950$

D $2000$

E $2050$

## Solution(s):

Let $x$ be the number of students in Lara's high school. Then Portia's high school has $3x$ students.

Therefore,

$$\begin{aligned} 3x + x &= 2600 \\ x &= 650. \end{aligned}$$

Then $3x = 1950$.

Thus, **C** is the correct answer.

**3.** The sum of two natural numbers is $17,402$. One of the two numbers is divisible by $10$. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

A $10,272$

B $11,700$

C $13,362$

D $14,238$

E $15,426$

## Solution(s):

Let $x$ and $y$ be the two numbers. WLOG, let $x$ be divisible by $10$. Then the units digit of $x$ is $0$.

If we erase the units digit, then we are essentially dividing $x$ by $10$. The problem statement also gives us that $\frac{x}{10} = y$.

Therefore,

$$\begin{aligned}\frac{x}{10} + x &= 17,402 \\ \frac{11x}{10} &= 17,402 \\ x &= 15,820.\end{aligned}$$

Then

$$x - \frac{x}{10} = 14,238.$$

Thus, **D** is the correct answer.

**4.** A cart rolls down a hill, travelling $5$ inches the first second and accelerating so that during each successive $1$-second time interval, it travels $7$ inches more than during the previous $1$-second interval. The cart takes $30$ seconds to reach the bottom of the hill. How far, in inches, does it travel?

A $215$

B $360$

C $2992$

D $3195$

E $3242$

## Solution(s):

The distance travelled every second forms an arithmetic sequence:

$$5, 5+7, 5+2\cdot 7, \ldots$$

The sum of an arithmetic sequence is \text{# of terms} \cdot \dfrac{\text{first + last}}{2}. We know the number of terms is $30$ and the first term is $5$. The last term is

$$5 + 29 \cdot 7 = 208.$$

Plugging these values into the expression yields

$$30 \cdot \frac{5+208}{2} = 15 \cdot 213 = 3195.$$

Thus, **D** is the correct answer.

**5.** The quiz scores of a class with $k > 12$ students have a mean of $8$. The mean of a collection of $12$ of these quiz scores is $14$. What is the mean of the remaining quiz scores in terms of $k$?

A $\dfrac{14-8}{k-12}$

B $\dfrac{8k-168}{k-12}$

C $\dfrac{14}{12}-\dfrac{8}{k}$

D $\dfrac{14(k-12)}{k^2}$

E $\dfrac{14(k-12)}{8k}$

## Solution(s):

The sum of the scores of everyone in the class is $8k$. The sum of the scores in the collection of $12$ is $12 \cdot 14 = 168$.

This means that the sum of the scores of everyone not in the collection is $8k - 168$. There are also $k - 12$ people not in the collection. Therefore, the average is

$$\frac{8k-168}{k-12}.$$

Thus, **B** is the correct answer.

**6.** Chantal and Jean start hiking from a trailhead toward a fire tower. Jean is wearing a heavy backpack and walks slower. Chantal starts walking at $4$ miles per hour. Halfway to the tower, the trail becomes really steep, and Chantal slows down to $2$ miles per hour. After reaching the tower, she immediately turns around and descends the steep part of the trail at $3$ miles per hour. She meets Jean at the halfway point. What was Jean's average speed, in miles per hour, until they meet?

A $\frac{12}{13}$

B $1$

C $\frac{13}{12}$

D $\frac{24}{13}$

E $2$

## Solution(s):

Let $2d$ be the distance from the fire tower, where $d > 0$.

Then Chantal hiked for

$$\frac{d}{4} + \frac{d}{2} + \frac{d}{3} = \frac{13d}{12}$$

hours.

If Jean travelled $d$ miles in $\frac{13d}{12}$ hours, then his speed was

$$d \div \frac{13d}{12} = \frac{12}{13}$$

miles per hour.

Thus, **A** is the correct answer.

**7.** Tom has a collection of $13$ snakes, $4$ of which are purple and $5$ of which are happy. He observes that

- all of his happy snakes can add,
- none of his purple snakes can subtract, and
- all of his snakes that can't subtract also can't add.

Which of these conclusions can be drawn about Tom's snakes?

A Purple snakes can add.

B Purple snakes are happy.

C Snakes that can add are purple.

D Happy snakes are not purple.

E Happy snakes can't subtract.

## Solution(s):

Note that the third condition ensures that purple snakes can't add.

We also know that all happy snakes can add, which means that happy snakes can't be purple as well.

Thus, **D** is the correct answer.

**8.** When a student multiplied the number $66$ by the repeating decimal,

$$\underline{1}.\underline{a}\ \underline{b}\ \underline{a}\ \underline{b}\ldots = \underline{1}.\overline{\underline{a}\ \underline{b}},$$

where $a$ and $b$ are digits, he did not notice the notation and just multiplied $66$ times $\underline{1}.\underline{a}\ \underline{b}$. Later he found that his answer is $0.5$ less than the correct answer. What is the $2$-digit number $\underline{a}\ \underline{b}$?

A $15$

B $30$

C $45$

D $60$

E $75$

## Solution(s):

We can express $\underline{1}.\overline{\underline{a}\ \underline{b}}$ as an infinite geometric sum:

$$\underline{1}.\overline{\underline{a}\ \underline{b}} = 1 + .\underline{a}\ \underline{b} + .00\ \underline{a}\ \underline{b} + \cdots$$

We can therefore use the formula for the sum of a geometric sum:

$$S = \frac{\text{first term}}{1 - \text{ratio}} = \frac{.\underline{a}\ \underline{b}}{1 - \frac{1}{100}}$$

$$= \frac{100}{99}\left(.\underline{a}\ \underline{b}\right) = \frac{\underline{a}\ \underline{b}}{99}.$$

We also know that

$$1.\underline{a}\ \underline{b} = 1 + \frac{\underline{a}\ \underline{b}}{100}$$

Then

$$\begin{aligned}
66\left(1+\frac{\underline{a}\,\underline{b}}{100}\right)+.5 &= 66\left(1+\frac{\underline{a}\,\underline{b}}{99}\right)\\
\frac{66}{100}\underline{a}\,\underline{b}+.5 &= \frac{66}{99}\underline{a}\,\underline{b}\\
\frac{1}{150}\underline{a}\,\underline{b} &= .5\\
\underline{a}\,\underline{b} &= 75.
\end{aligned}$$

Thus, **E** is the correct answer.

**9.** What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers $x$ and $y$?

- A $0$
- B $\frac{1}{4}$
- C $\frac{1}{2}$
- D $1$
- E $2$

## Solution(s):

Expanding, we get

$$\begin{aligned}
&x^2y^2-2xy+1+x^2+2xy+y^2\\
&=x^2y^2+x^2+y^2+1.
\end{aligned}$$

Note that every square must be non-negative. Therefore, the minimum value is when all the terms except $1$ are $0$, making the sum $1$.

This is attainable when $x=y=0$.

Thus, **D** is the correct answer.

**10.** Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)$$
$$(2^8+3^8)(2^{16}+3^{16})$$
$$(2^{32}+3^{32})(2^{64}+3^{64})?$$

A $3^{127}+2^{127}$

B $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$

C $3^{128}-2^{128}$

D $3^{128}+2^{128}$

E $5^{127}$

## Solution(s):

Notice that if we multiply by $3-2=1$, we end up with a bunch of difference of squares.

$$(3-2)(2+3)=3^2-2^2$$
$$(3^2-2^2)(2^2+3^2)=3^4-2^4$$
$$\vdots$$

This ends up giving us a final value of $3^{128}-2^{128}$.

Thus, **C** is the correct answer.

**11.** For which of the following integers $b$ is the base-$b$ number $2021_b - 221_b$ not divisible by $3$?

A $3$

B $4$

C $6$

D $7$

E $8$

## Solution(s):

We can express this expression in base $10$ using the definition of bases:

$$2b^3 + 2b + 1 - 2b^2 - 2b - 1$$

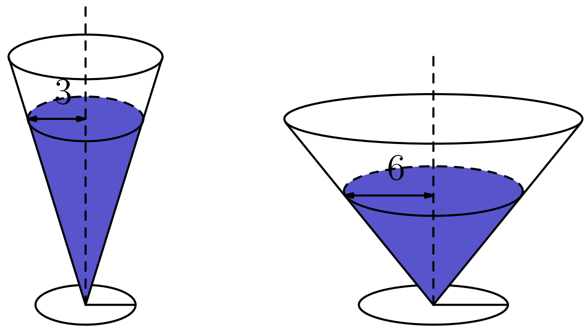
$$= 2b^3 - 2b^2 = 2b^2(b-1).$$

For this to be divisible by $3$, either $b$ or $b-1$ must be divisible by $3$.

The only answer choice that satisfies neither of these conditions is $8$.

Thus, **E** is the correct answer.

**12.** Two right circular cones with vertices facing down as shown in the figure below contains the same amount of liquid. The radii of the tops of the liquid surfaces are $3$ cm and $6$ cm. Into each cone is dropped a spherical marble of radius $1$ cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?




A $1:1$

B $47:43$

C $2:1$

D $40:13$

E $4:1$

## Solution(s):

Let the heights of the narrow wide cones be $h_1$ and $h_2$ respectively.

Then

$$\frac{1}{3}\cdot 3^2\pi h_1 = \frac{1}{3}\cdot 6^2\pi h_2,$$

which gives us

$$\frac{h_1}{h_2} = 4.$$

Also note that $\frac{3}{h_1}$ and $\frac{6}{h_2}$ must remain constant due to similar triangles.

After the marble is added to each cone, let $3x$ be the radius of the narrow cone and $6y$ be that of the wide cone.

Then the height of the narrow cone is $h_1x$ and that of the wide cone is $h_2y$ due to similar triangles.

Equating the final volumes, we get

$$\frac{1}{3}(3x)^2\pi(h_1x) = \frac{1}{3}(6y)^2\pi(h_2y),$$

which gives us

$$h_1x^3 = 4h_2y^3.$$

We know that $\frac{h_1}{h_2} = 4$, so this equation gives us that $x = y$.

The desired ratio is

$$\frac{h_1x - h_1}{h_2y - h_2} = \frac{h_1(x-1)}{h_2(y-1)} = 4.$$

Thus, **E** is the correct answer.

**13.** What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$?

A $3$

B $2\sqrt{3}$

C $4$

D $3\sqrt{3}$

E $6$

## Solution(s):

Analyzing the side lengths, we can realize that

$$\triangle ACD \text{ and } \triangle ABC$$

are both right triangles.

Therefore, we can treat $\triangle ACD$ as the base of the tetrahedron and $\overline{AB}$ as the altitude.

With this setup, we can use the formula for the volume of a pyramid, yielding

$$\frac{1}{3} \cdot \frac{3 \cdot 4}{2} \cdot 2 = 4.$$

Thus, **C** is the correct answer.

**14.** All the roots of the polynomial

$$z^6 - 10z^5 + Az^4 + Bz^3$$
$$+Cz^2 + Dz + 16$$

are positive integers, possibly repeated. What is the value of $B$?

A $-88$

B $-80$

C $-64$

D $-41$

E $-40$

## Solution(s):

By Vieta's formulas, we get that the product of the roots is $16$ and that their sum is $10$.

Given that all the roots are positive integers, we can see that the roots are

$$1, 1, 2, 2, 2, 2.$$

The function is therefore just

$$(z-1)^2(z-2)^4 = (z^2 - 2z + 1)$$
$$(z^4 - 8z^3 + 24z^2 - 32z + 16).$$

Calculating just the $z^3$ term, we get

$$-32z^3 - 48z^3 - 8z^3 = -88z^3.$$

Thus, **A** is the correct answer.

**15.** Values for $A, B, C$, and $D$ are to be selected from $\{1, 2, 3, 4, 5, 6\}$ without replacement (i.e. no two letters have the same value). How many ways are there to make such choices so that the two curves $y = Ax^2 + B$ and $y = Cx^2 + D$ intersect?

(The order in which the curves are listed does not matter; for example, the choices $A = 3$,$B = 2$,$C = 4$,$D = 1$ is considered the same as the choices $A = 4$, $B = 1$, $C = 3$, $D = 2$.)

- A $30$
- B $60$
- C $90$
- D $180$
- E $360$

## Solution(s):

Setting the equations equal to each other, we get

$$Ax^2 + B = Cx^2 + D$$

$$x^2(A - C) = D - B$$

$$x^2 = \frac{D - B}{A - C} \geq 0$$

since squares are non-negative.

This means $D - B$ and $A - C$ must both have the same sign.

If we choose two distinct values for $(A, C)$ and $(B, D)$, there are $2$ ways to arrange them such that the numerator and denominator both have the same sign.

We have to divide by $2$, however, since the two curves are not considered distinct.

Therefore, the total number of tuples is

$$\frac{1}{2}\binom{6}{2}\binom{4}{2} \cdot 2 = 90.$$

Thus, **C** is the correct answer.

**16.** In the following list of numbers, the integer $n$ appears $n$ times in the list for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4,$$

$$\ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

A $100.5$

B $134$

C $142$

D $150.5$

E $167$

## Solution(s):

The total number of numbers in the list is

$$1 + 2 + \cdots + 200$$

$$= \frac{200 \cdot 201}{2} = 20100.$$

We want to find the median $k$ such that $\frac{k(k+1)}{2}$ is near $\frac{20100}{2}$.

Multiplying by $2$, we want $k(k+1)$ near $20100$. Note that $\sqrt{20100} \approx 142$.

Plugging in $k = 142$ yields

$$\frac{1}{2} \cdot 142 \cdot 143 = 10153.$$

$10153 - 142 < 10050$, which shows that $142$ is our desired median ($142$ is the $10049$th and $10050$th number).

Thus, **C** is the correct answer.

**17.** Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}, BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let $O$ be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let $P$ be the midpoint of $\overline{BD}$.

Given that $OP = 11$, the length of $AD$ can be written in the form $m\sqrt{n}$, where $m$ and $n$ are positive integers and $n$ is not divisible by the square of any prime. What is $m + n$?

A $65$

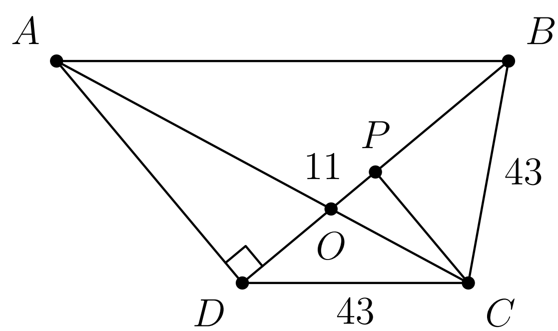
B $132$

C $157$

D $194$

E $215$

## Solution(s):




First, we can show that $\triangle BPC \sim \triangle BDA$.

Let $\angle DBC = \alpha$. Then $\angle DCB = 180 - 2\alpha$ since $\triangle DCB$ is isosceles.

Since $\overline{AB} \parallel \overline{CD}$, we get that $\angle ABD = \alpha$. Then since $\triangle BPC$ and $\triangle BDA$ are right triangles, they are similar.

Using this fact, we get

$$2 = \frac{BD}{BP} = \frac{AB}{AC} = \frac{AB}{43}.$$

From this, we get that $AB = 86$.

We also get that $\triangle ABO \sim \triangle CDO$ from parallel sides and vertical angles.

Therefore

$$2 = \frac{AB}{CD} = \frac{BO}{OD} = \frac{BP+11}{BP-11}.$$

Solving, we get $BP = 33$ and $BD = 66$.

Using the Pythagorean Theorem on $\triangle ADB$, we get that

$$AD = \sqrt{86^2 - 66^2} = 4\sqrt{190}.$$

Thus, **D** is the correct answer.

**18.** Let $f$ be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers $a$ and $b$. Suppose that $f$ also has the property that $f(p) = p$ for every prime number $p$. For which of the following numbers $x$ is $f(x) < 0$?

A $\frac{17}{32}$

B $\frac{11}{16}$

C $\frac{7}{9}$

D $\frac{7}{6}$

E $\frac{25}{11}$

## Solution(s):

Note that for any number of the form $p^e$ where $p$ is prime,

$$f(p^e) = ef(p) = ep.$$

This can be seen by applying the function property multiple times.

Also note that

$$\begin{aligned} f(a) &= f\left(\frac{a}{b} \cdot b\right) \\ &= f\left(\frac{a}{b}\right) + f(b), \end{aligned}$$

which gives us

$$f\left(\frac{a}{b}\right) = f(a) - f(b).$$

We can know calculate each answer choice one-by-one.

$$f\left(\frac{17}{32}\right) = f(17) - f(32)$$

$$= 17 - 5 \cdot 2 = 7.$$

$$f\left(\frac{11}{16}\right) = f(11) - f(16)$$
$$= 11 - 4 \cdot 2 = 3$$

$$f\left(\frac{7}{9}\right) = f(7) - f(9)$$
$$= 7 - 2 \cdot 3 = 1$$

$$f\left(\frac{7}{6}\right) = f(7) - f(6)$$
$$= 7 - f(2) - f(3) = 2$$

$$f\left(\frac{25}{11}\right) = f(25) - f(11)$$
$$= 2 \cdot 5 - 11 = -1$$

Thus, **E** is the correct answer.

**19.** The area of the region bounded by the graph of

$$x^2 + y^2 = 3|x - y| + 3|x + y|$$

is $m + n\pi$, where $m$ and $n$ are integers. What is $m + n$?

A $18$

B $27$

C $36$

D $45$

E $54$

## Solution(s):

We can case on the signs of $x - y$ and $x + y$.

Case $1$ :

$$|x - y| = x - y,$$

$$|x + y| = x + y$$

Substituting and simplifying, we get

$$x^2 - 6x + y^2 = 0$$
$$(x - 3)^2 + y^2 = 3^2.$$

This is a circle with radius $3$ centered at $(3, 0)$.

Case $2$ :

$$|x - y| = y - x,$$

$$|x + y| = x + y$$

As above, we get

$$x^2 + y^2 - 6y = 0$$
$$x^2 + (y - 3)^2 = 3^2.$$

This is a circle with radius $3$ centered at $(0, 3)$.

Case $3$ :

$$|x-y|=x-y,$$

$$|x+y|=-x-y$$

Again, we get

$$x^2+y^2+6y=0$$
$$x^2+(y+3)^2=3^2.$$

This is a circle with radius $3$ centered at $(0,-3)$.

Case $4$ :

$$|y-x|=x-y,$$

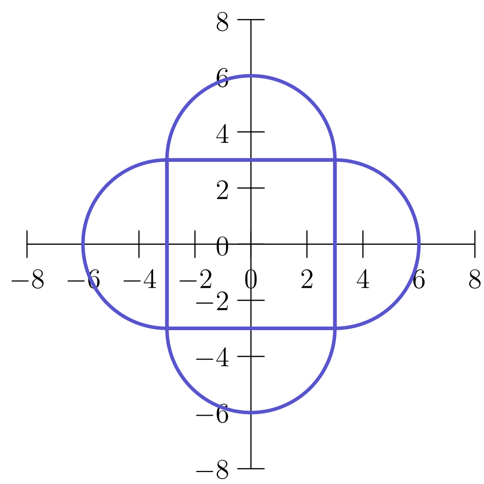
$$|x+y|=-x-y$$

Finally, we get

$$x^2+6x+y^2=0$$
$$(x+3)^2+y^2=3^2.$$

This is a circle with radius $3$ centered at $(-3,0)$.

These circles form the following graph:




The $4$ semicircles form $2$ full two circle with radius $3$ for a total area of $18\pi$.

The middle square also contributes $6^2\pi=36\pi$ to the total area. This means that the area of the region is $54\pi$.

Thus, **E** is the correct answer.

**20.** In how many ways can the sequence $1, 2, 3, 4, 5$ be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

A $10$

B $18$

C $24$

D $32$

E $44$

## Solution(s):

Note that all the sequences are symmetric about $3$.

$$x_1, x_2, x_3, x_4, x_5$$

is a valid sequence if and only if

$$6-x_1, 6-x_2, 6-x_3,$$
$$6-x_4, 6-x_5$$

is a valid sequence.

Therefore, we can just count all the sequences that begin with $1$, $2$, $31$, and $32$.

- $1, 3, 2, 5, 4$
- $1, 4, 2, 5, 3$
- $1, 4, 3, 5, 2$
- $1, 5, 2, 4, 3$
- $1, 5, 3, 4, 2$
- $2, 1, 4, 3, 5$
- $2, 1, 5, 3, 4$
- $2, 3, 1, 5, 4$
- $2, 4, 1, 5, 3$
- $2, 4, 3, 5, 1$

- $2, 5, 1, 4, 3$
- $2, 5, 3, 4, 1$
- $3, 1, 4, 2, 5$
- $3, 1, 5, 2, 4$
- $3, 2, 4, 1, 5$
- $3, 2, 5, 1, 4$

This shows that there are $16$ valid sequences starting with the above numbers. Doubling this yields the total number of sequences, $32$.

Thus, **D** is the correct answer.

**21.** Let $ABCDEF$ be an equiangular hexagon. The lines $AB, CD$, and $EF$ determine a triangle with area $192\sqrt{3}$, and the lines $BC, DE$, and $FA$ determine a triangle with area $324\sqrt{3}$. The perimeter of hexagon $ABCDEF$ can be expressed as $m + n\sqrt{p}$, where $m, n$, and $p$ are positive integers and $p$ is not divisible by the square of any prime. What is $m + n + p$?

A $47$

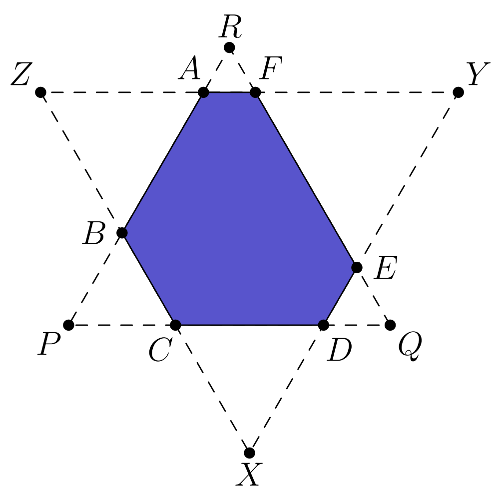
B $52$

C $55$

D $58$

E $63$

## Solution(s):




Let $P$, $Q$, $R$, $X$, $Y$, and $Z$ be the points at $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$,$\overleftrightarrow{CD} \cap \overleftrightarrow{EF}$,$\overleftrightarrow{EF} \cap \overleftrightarrow{AB}$,$\overleftrightarrow{BC} \cap \overleftrightarrow{DE}$,$\overleftrightarrow{DE} \cap \overleftrightarrow{FA}$, and $\overleftrightarrow{FA} \cap \overleftrightarrow{BC}$, respectively.

Since $ABCDEF$ is equiangular, all of its interior angles are $720^\circ \div 6 = 120^\circ$.

This means that the exterior angles are all $60^\circ$, making all the small triangles equilateral triangles. This also shows that $\triangle PQR$ and $\triangle XYZ$ are also equilateral.

We know from the problem statement that

$$[PQR] = \frac{3}{4} \cdot PQ^2 = 192\sqrt{3},$$

$$[XYZ] = \frac{3}{4} \cdot YZ^2 = 324\sqrt{3}.$$

Solving gives us that $PQ = 16\sqrt{3}$ and $YZ = 36$.

Finally, the perimeter of $ABCDEF$ is

$$AB + BC + CD + DE + EF$$

$$+FA = AZ + PC + CD + DQ$$

$$+YF + FA = (YF + FA + AZ)$$

$$+(PC + CD + DQ) = YZ$$

$$+PQ = 36 + 16\sqrt{3}.$$

With this, we have $(36 + 16 + 3 = 55)$.

Thus, **C** is the correct answer.

**22.** Hiram's algebra notes are $50$ pages long and are printed on $25$ sheets of paper; the first sheet contains pages $1$ and $2$, the second sheet contains pages $3$ and $4$, and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly $19$. How many sheets were borrowed?

A $10$

B $13$

C $15$

D $17$

E $20$

## Solution(s):

Let the sheets the roommate took be $a$ through $b$. This is the same as taking pages $2a-1$ through $2b$.

The sum of the pages taken is

$$\frac{(2a-1+2b)(2b-2a+2)}{2}.$$

The sum of the pages remaining is

$$19(50-(2b-2a+2)).$$

The sum of these equals the sum of all the pages, $\dfrac{50\cdot 51}{2}$.

Therefore

$$\frac{(2a-1+2b)(2b-2a+2)}{2}+$$
$$19(50-(2b-2a+2))=\frac{50\cdot 51}{2}.$$

Then

$$(2a+2b-39)(b-a+1)$$

$$= \frac{50 \cdot 13}{2} = 25 \cdot 13.$$

We can set

$$2a + 2b - 39 = 25$$

and

$$b - a + 1 = 13$$

to get

$$a + b = 32$$

and

$$b - a = 12.$$

This yields

$$b = 22 \text{ and } a = 12.$$

The desired answer is $22 - 10 + 1 = 13.$

Thus, **B** is the correct answer.

**23.** Frieda the frog begins a sequence of hops on a $3 \times 3$ grid of squares, moving one square on each hop and choosing at random the direction of each hop-up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square.

Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

A $\frac{9}{16}$

B $\frac{5}{8}$

C $\frac{3}{4}$

D $\frac{25}{32}$

E $\frac{13}{16}$

## Solution(s):

Let $M$ denote the center, $E$ an edge, and $C$ a corner.

The only ways Frieda can reach a corner in $4$ or less moves are

$$EC, EEC, EEEC, \text{ and } EMEC.$$

We have to find the probability of each of these cases happening.

Case $1$ : $EC$

On the first hop, Frieda necessarily moves to an $E$. From there, Frieda has a $\frac{1}{2}$ chance of going to a $C$. This gives us a probability of

$$1 \cdot \frac{1}{2} = \frac{1}{2}$$

for this case.

Case $2$ : $EEC$

Again, Frieda moves to an $E$ on her first move. On the second move, however, she has a $\frac{1}{4}$ chance of wrapping around to another $E$. Then there is a $\frac{1}{2}$ chance of going to a $C$ from the $E$. This is a probability of

$$1 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}.$$

Case $3$ : $EEEC$

Using the same logic as above, this case yields a probability of

$$1 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{32}.$$

Case $ : EMEC

Finally, this case has a probability of

$$1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{1}{2} = \frac{1}{8.}$$

Adding the probabilities together yields a total probability of

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{8} = \frac{25}{32}.$$

Thus, **D** is the correct answer.

**24.** The interior of a quadrilateral is bounded by the graphs of

$$(x+ay)^2 = 4a^2$$

and

$$(ax-y)^2 = a^2,$$

where $a$ is a positive real number. What is the area of this region in terms of $a$, valid for all $a > 0$?

A $\frac{8a^2}{(a+1)^2}$

B $\frac{4a}{a+1}$

C $\frac{8a}{a+1}$

D $\frac{8a^2}{a^2+1}$

E $\frac{8a}{a^2+1}$

## Solution(s):

Note that each of the equations yields two parallel lines.

$$(x+ay)^2 = 4a^2$$

results in the two lines

$$x + ay - 2a = 0$$

and

$$x + ay + 2a = 0.$$

Both of these lines have a slope of $-\frac{1}{a}$.

Similarly,

$$(ax-y)^2 = a^2$$

results in the lines

$$ax - y - a = 0$$

and

$$ax - y + a = 0.$$

These lines have slope $a$.

Note that each pair of lines is perpendicular to the other pair of lines. This shows that the equations form a rectangle.

Recall that the formula for the distance $d$ between two parallel lines

$$\begin{cases} Ax + By + C_1 = 0 \\ Ax + By + C_2 = 0 \end{cases}$$

is

$$d = \frac{\mid C_2 - C_1 \mid}{\sqrt{A^2 + B^2}}.$$

Using this formula, we get that the distance between the first pair of lines is

$$\frac{4a}{\sqrt{a^2+1}}.$$

Similarly, the distance between the second pair of lines is

$$\frac{2a}{\sqrt{a^2+1}}.$$

These are the side lengths of the rectangle. Multiplying yields the area

$$\frac{8a^2}{a^2+1}.$$

Thus, **D** is the correct answer.

**25.** How many ways are there to place $3$ indistinguishable red chips, $3$ indistinguishable blue chips, and $3$ indistinguishable green chips in the squares of a $3 \times 3$ grid so that no two chips of the same color are directly adjacent to each other, either vertically or horizontally?

A $12$

B $18$

C $24$

D $30$

E $36$

## Solution(s):

Let the colors be $A, B$, and $C$. Note that we can assign the $3$ colors to them in $3! = 6$ ways, so we have to multiply by $6$ at the end.

Let $A$ be in the center of the grid.

$$\begin{matrix} ? & ? & ? \\ ? & A & ? \\ ? & ? & ? \end{matrix}$$

The other $A$s have to either be along the diagonal or on the same side.

$$\begin{matrix} ? & ? & A \\ ? & A & ? \\ A & ? & ? \end{matrix}$$

$$\begin{matrix} A & ? & A \\ ? & A & ? \\ ? & ? & ? \end{matrix}$$

The first scenario can happen in $2$ ways since there are $2$ diagonals. The second has $4$ ways since there are $4$ sides.

Either way, the positions of the $B$s and $C$s is fixed.

$$\begin{matrix} C & B & A \\ B & A & C \\ A & C & B \end{matrix}$$

$$\begin{array}{ccc} A & B & A \\ C & A & C \\ B & C & B \end{array}$$

This is a total of $4+2=6$ ways to arrange the $A, B$, and $C$s. This gives us a total of $6 \cdot 6 = 36$ configurations.

Thus, **E** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2021A