2019 AMC 10B Solutions

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1. Alicia had two containers. The first was $\frac{5}{6}$ full of water and the second was empty. She poured all the water from the first container into the second container, at which point the second container was $\frac{3}{4}$ full of water. What is the ratio of the volume of the first container to the volume of the second container?

A $\frac{5}{8}$

B $\frac{4}{5}$

C $\frac{7}{8}$

D $\frac{9}{10}$

E $\frac{11}{12}$

Solution(s):

The ratio is equal to the ratio of the recipricols of how much they are filled. This makes the ratio

$$\frac{\left(\frac{6}{5}\right)}{\left(\frac{4}{3}\right)} = \frac{9}{10}.$$

Thus, the answer is **D**.

2. Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

- A 11
- B 15
- C 19
- D 21
- E 27

Solution(s):

We need n to not be prime, so n can only be $15, 21, 27$. Then, $n-2$ must be not prime, leaving just 27.

Thus, the answer is **E**.

3. In a high school with 500 students, 40% of the seniors play a musical instrument, while 30% of the non-seniors do not play a musical instrument. In all, 46.8% of the students do not play a musical instrument. How many non-seniors play a musical instrument?

A 66

B 154

C 186

D 220

E 266

Solution(s):

Let the number of seniors be s. Then, $500 - s$ people aren't seniors. We know 60% of seniors don't play an instrument. Then, the number of students who don't play an instrument can be represented as

$$0.3(500 - s) + 0.6(s) = 0.3s + 150$$

and

$$0.468 \cdot 500 = 234.$$

Thus,

$$0.3s + 150 = 234$$

$$s = 280.$$

This makes the number of non-seniors equal to 220. Since 70% of non seniors play instruments, we have the total number as

$$220 \cdot 0.7 = 154.$$

Thus, the answer is **B**.

4. All lines with equation $ax + by = c$ such that a, b, c form an arithmetic progression pass through a common point. What are the coordinates of that point?

- A $(-1, 2)$
- B $(0, 1)$
- C $(1, -2)$
- D $(1, 0)$
- E $(1, 2)$

Solution(s):

Let $d = b - a$.

Then, we have

$$(a, b, c) = (a, a + d, a + 2d).$$

Thus,

$$ax + (a + d)y = a + 2d.$$

If we match the parts of a and d, we get

$$ax + ay = a$$

and

$$dy = 2d$$

for all a, d. Therefore, we have

$$y = 2, x + y = 1$$

implying that

$$x = -1.$$

This makes the pair $(-1, 2)$.

Thus, the answer is **A**.

5. Triangle ABC lies in the first quadrant. Points A, B, and C are reflected across the line $y = x$ to points A', B', and C', respectively. Assume that none of the vertices of the triangle lie on the line $y = x$. Which of the following statements is *not* always true?

A Triangle $A'B'C'$ lies in the first quadrant.

B Triangles ABC and $A'B'C'$ have the same area.

C The slope of line AA' is -1.

D The slopes of lines AA' and CC' are the same.

E Lines AB and $A'B'$ are perpendicular to each other.

Solution(s):

Choice A must be true since the reflection of the first quadrant is itself, so anything inside stays inside after a reflection.

Choice B must be true as a reflection keeps the same area.

Choice C must be true as a reflection will have a perpendicular slope to the line its reflected about, so its slope is $-\frac{1}{1} = -1$.

Choice D must be true as they both have the slope of -1.

Choice E can be false as if

$$A = (2, 1),$$

$$B = (3, 2),$$

then AB and its reflection both have slope -1, making them parallel. Therefore, they can be not perpendicular.

Thus, the answer is **E**.

6. A positive integer n satisfies the equation

$$(n+1)! + (n+2)! = n! \cdot 440.$$

What is the sum of the digits of n?

- A 3
- B 8
- C 10
- D 11
- E 12

Solution(s):

We can rewrite the left side as

$$(n+1)n! + (n+2)(n+1)n!$$

$$= ((n+2)^2 - 1)n!,$$

so

$$((n+2)^2 - 1)n! = 440n!$$

Therefore,

$$(n+2)^2 = 441,$$

so $n = 19$. The sum of its digits is 10.

Thus, the answer is **C**.

7. Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

- A 18
- B 21
- C 24
- D 25
- E 28

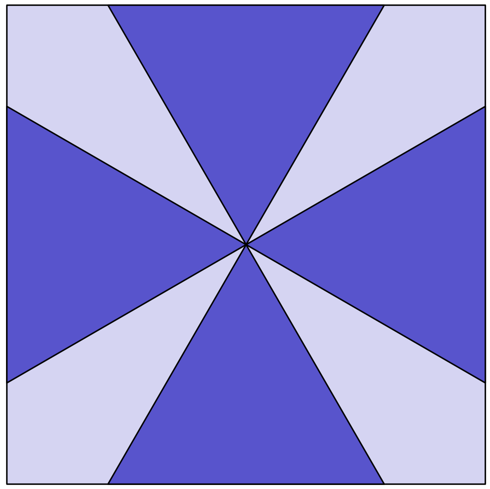
Solution(s):

Let the number of cents she has be c. Then, c is a multiple of $12, 14,$ and 15. Thus, it must be a multiple of 420.

Let $c = 420k$ for some k. Also, $c = 20n$, so $420k = 20n$, making $n = 21k$. Since k is a whole number, the minimum possible value of n is 21.

Thus, the answer is **B**.

8. The figure below shows a square and four equilateral triangles, with each triangle having a side lying on a side of the square, such that each triangle has side length 2 and the third vertices of the triangles meet at the center of the square. The region inside the square but outside the triangles is shaded. What is the area of the shaded region?



A 4

B $12 - 4\sqrt{3}$

C $3\sqrt{3}$

D $4\sqrt{3}$

E $16 - 4\sqrt{3}$

Solution(s):

The altitude of the triangle is $\sqrt{3}$ using $30 - 60 - 90$ triangles, so the total base is $2\sqrt{3}$. The total amount of the base on each side that isn't in the white region is $2\sqrt{3} - 2$, so the amount from each triangle is $\sqrt{3} - 1$.

This makes 8 total triangles with base $\sqrt{3} - 1$ and altitude $\sqrt{3}$, so the combined area is

$$8 \cdot \frac{(\sqrt{3})(\sqrt{3} - 1)}{2}$$

$$= 4 \cdot (3 - \sqrt{3})$$

$$= 12 - 4\sqrt{3}.$$

Thus, the answer is **B**.

9. The function f is defined by

$$f(x) = \lfloor |x| \rfloor - |\lfloor x \rfloor|$$

for all real numbers x, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r. What is the range of f?

A $\{-1, 0\}$

B The set of nonpositive integers

C $\{-1, 0, 1\}$

D $\{0\}$

E The set of nonnegative integers

Solution(s):

If x was an integer, then

$$\begin{aligned} \lfloor |x| \rfloor - |\lfloor x \rfloor| &= |x| - |x| \\ &= 0. \end{aligned}$$

If x was positive, then

$$\begin{aligned} \lfloor |x| \rfloor - |\lfloor x \rfloor| &= \lfloor x \rfloor - \lfloor x \rfloor \\ &= 0. \end{aligned}$$

Now, we must look at negative non-integers. If we have a negative non-integer, then $\lfloor |x| \rfloor$ would negate x and then round down, while $|\lfloor x \rfloor|$ would round down then negate it, effectively negating it and rounding up.

The first one rounds down and the second one rounds up, the second one is 1 larger than the first, making $f = -1$.

Therefore, the range is $\{-1, 0\}$.

Thus, the answer is **A**.

10. In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

- A 0
- B 2
- C 4
- D 8
- E infinitely many

Solution(s):

If the perimeter was 50, then $AC + BC = 40$. This means that their average is 20, so either both of them are 20 or at least one of them is less than 20.

If the area was 100 and the altitude from C was h, then

$$\frac{10h}{2} = 100$$
$$h = 20.$$

The altitude can't be less than a side, leaving

$$AC = 20$$
$$BC = 20.$$

However, this would also yield an altitude less than 20 as neither AC nor BC are perpendicular with AB, making the altitude less than the lengths of the legs.

This leaves no way to make such a triangle.

Thus, the answer is **A**.

11. Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is $9 : 1$, and the ratio of blue to green marbles in Jar 2 is $8 : 1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

- A 5
- B 10
- C 25
- D 45
- E 50

Solution(s):

Let x be the number of green marbles in Jar 1 and let y be the number of green marbles in Jar 2.

Then, the total number of marbles is

$$10x + 9y = 95,$$

implying

$$x \equiv 5 \mod 5.$$

The only possible x is $x = 5$, making $y = 5$. There are

$$9x = 9 \cdot 5 = 45$$

green marbles in Jar 1 and

$$8y = 8 \cdot 5 = 40$$

green marbles in Jar 2. Therefore, the difference is 5.

Thus, the answer is **A**.

12. What is the greatest possible sum of the digits in the base-seven representation of a positive integer less than 2019?

- A 11
- B 14
- C 22
- D 23
- E 27

Solution(s):

First, $2019 = 5613_7$. Therefore, the 4th digit from the left is at most 5. If that digit was 4 and every other digit was maximized, then we get 4666_7, with a digit sum of 22. If that digit was 5, we can only have a sum greater than 22 by making 5666_7 which is too large.

Therefore, the largest digit sum is 22.

Thus, the answer is **C**.

13. What is the sum of all real numbers x for which the median of the numbers $4, 6, 8, 17,$ and x is equal to the mean of those five numbers?

A -5

B 0

C 5

D $\frac{15}{4}$

E $\frac{35}{4}$

Solution(s):

Since there are 5 numbers, the median 3rd largest numbers. That would be 6 if $x < 6$, 8 if $x > 8$, and x if $6 \leq x \leq 8$.

In addition, the mean is equal to

$$\frac{4+6+8+17+x}{5}$$

$$= 7 + \frac{x}{5}.$$

If the mean is 6, then

$$7 + \frac{x}{5} = 6,$$

so $x = -5$.

If the mean is 8, then

$$7 + \frac{x}{5} = 8,$$

so $x = 5$, which isn't in the required range.

If the mean is x, then

$$\begin{aligned} 7 + \frac{x}{5} &= x \\ 7 &= \frac{4x}{5} \\ x &= 8.75, \end{aligned}$$

which isn't in the required range.

Therefore, the only possible x is -5, making the mean -5.

Thus, the answer is **A**.

14. The base-ten representation for $19!$ is $121,6T5,100,40M,832,H00$, where T, M, and H denote digits that are not given. What is $T+M+H$?

A 3

B 8

C 12

D 14

E 17

Solution(s):

We know p is a multiple of 5^3 and 2^3, so its a multiple of 1000. Therefore, $H=0$

We know it is a multiple of 9, so its digit sum must be a multiple of 9. As such,

$$1+2+1+6+T+5+1+$$
$$0+0+4+0+M+8$$
$$+3+2+0+0+0$$
$$=33+M+T$$

is a multiple of 9. WIth this in mind, we know that

$$M+T \equiv 3 \mod 9,$$

leaving just 3 and 12.

We also know its a multiple of 11, which means that when alternating between adding and subtracting digits, we get

$$1-2+1-6+T-5+1-$$
$$0+0-4+0-M+8-$$
$$3+2-0+0-0$$
$$=T-M-7$$

is a multiple of 11, so

$$T-M \equiv 7 \mod 11.$$

The only way to satisfy both is

$$T = 4, M = 8, H = 0.$$

Their sum is 12.

Thus, the answer is **C**.

15. Right triangles T_1 and T_2, have areas of 1 and 2, respectively. A side of T_1 is congruent to a side of T_2, and a different side of T_1 is congruent to a different side of T_2. What is the square of the product of the lengths of the other (third) sides of T_1 and T_2?

A $\frac{28}{3}$

B 10

C $\frac{32}{3}$

D $\frac{34}{3}$

E 12

Solution(s):

Let the congruent sides be a, b such that $a \leq b$. None of the triangles can have hypotenues a since $a \leq b$.

Then, in one triangle (say, T_1), we have side lengths

$$a, b, \sqrt{a^2 + b^2},$$

and in T_2, we have

$$a, \sqrt{b^2 - a^2}, b.$$

Thus, the product of the other sides would be

$$\sqrt{(b^2 - a^2)(b^2 + a^2)}$$

$$= \sqrt{b^4 - a^4},$$

making its square

$$b^4 - a^4.$$

The area of the smaller triangle T_2 is

$$\frac{a\sqrt{b^2 - a^2}}{2} = 1$$

and the area of the larger triangle T_1 is

$$\frac{ab}{2} = 2$$

Thus,

$$a^2(b^2 - a^2) = a^2b^2 - a^4$$
$$= 4,$$

and

$$a^2b^2 = 16.$$

That implies $a^4 = 12$. We can also get $a^4b^4 = 256$, so

$$b^4 = \frac{256}{12} = \frac{64}{3}.$$

Therefore,

$$b^4 - a^4 = \frac{64}{3} - 12$$
$$= \frac{28}{3}.$$

Thus, the answer is **A**.

16. In $\triangle ABC$ with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC = CD$, $DE = EB$, and the ratio $AC : DE = 4 : 3$. What is the ratio $AD : DB$?

A $2 : 3$

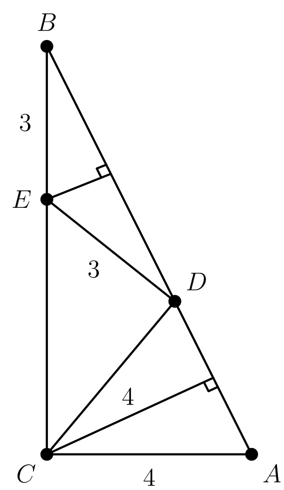
B $2 : \sqrt{5}$

C $1 : 1$

D $3 : \sqrt{5}$

E $3 : 2$

Solution(s):



Let $AC = 4$. From this, we get

$$CD = 4, ED = 3, BE = 3.$$

Notice that

$$\angle EDB = \angle EBD,$$
$$\angle CAD = \angle CDA,$$

so

$$\angle EDB + \angle CDA = 90^\circ.$$

Thus, $\angle EDC = 90^\circ$. From the Pythagorean Theorem, we get

$$EC = \sqrt{3^2 + 4^2}$$

$$= \sqrt{25}$$

$$= 5.$$

Therefore, $BC = 8$. This makes

$$\tan BAC = \frac{8}{4} = 2.$$

Now, we can get

$$BD = 2 \cdot 3 \cos B$$

$$= 6 \sin A$$

and

$$AD = 2 \cdot 4 \cos A$$

$$= 8 \cos A.$$

Thus,

$$\frac{AD}{BD} = \frac{8 \cos A}{6 \sin A}$$

$$= \frac{4}{3 \tan A}$$

$$= \frac{4}{3 \cdot 2}$$

$$= \frac{2}{3}.$$

Thus, the answer is **A**.

17. A red ball and a green ball are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1, 2, 3....$ What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

A $\frac{1}{4}$

B $\frac{2}{7}$

C $\frac{1}{3}$

D $\frac{3}{8}$

E $\frac{3}{7}$

Solution(s):

Given that at the two balls were tossed into seperate balls, the probability that ball in the higher-numbered bin is red is $\frac{1}{2}$. Thus we must find

$$\frac{P(\text{Balls in different bins})}{2}$$
$$= \frac{1 - P(\text{Balls in same bins})}{2}$$

by complementary counting.

The probability that both balls are in bin k is

$$2^{-k} \cdot 2^{-k} = 4^{-k}.$$

The probability that they are both in the same bin is therefore

$$\sum_{k=1}^{\infty} 4^{-k}.$$

Using the geometric sequence formula, we get this to be

$$\frac{1}{4} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{3}.$$

Therefore, our answer is

$$\frac{1 - \frac{1}{3}}{2} = \frac{1}{3}.$$

Thus, the answer is **C**.

18. Henry decides one morning to do a workout, and he walks $\frac{3}{4}$ of the way from his home to his gym. The gym is 2 kilometers away from Henry's home. At that point, he changes his mind and walks $\frac{3}{4}$ of the way from where he is back toward home. When he reaches that point, he changes his mind again and walks $\frac{3}{4}$ of the distance from there back toward the gym. If Henry keeps changing his mind when he has walked $\frac{3}{4}$ of the distance toward either the gym or home from the point where he last changed his mind, he will get very close to walking back and forth between a point A kilometers from home and a point B kilometers from home. What is $|A-B|$?

A $\frac{2}{3}$

B 1

C $1\frac{1}{5}$

D $1\frac{1}{4}$

E $1\frac{1}{2}$

Solution(s):

Suppose he starts x miles from home and then goes in the direction of the gym before coming back. The current distance between him and the gym is $2-x$, so he would have a distance of $\frac{2-x}{4}$ miles from the gym after walking. This would put him

$$2-\frac{2-x}{4}=1.5+\frac{x}{4}$$

miles from home.

Then, when he walks home, he becomes

$$\frac{1.5+\frac{x}{4}}{4}=\frac{3}{8}+\frac{x}{16}$$

miles from home.

If Henry is at one of the points, then his position would remain the same, so

$$x=\frac{3}{8}+\frac{x}{16}.$$

As such, we know that

$$16 = 6x + x$$

$$x = 0.4.$$

Thus, one of the steady state points is 0.4 miles from the house.

Using similar logic, but starting x miles from the gym and then going home, and then to the gym, we get that another point is 1.6 miles from the house.

Therefore, the points are 0.4 and 1.6 miles, so their diffence is 1.2 miles.

Thus, correct answer is **C**.

19. Let S be the set of all positive integer divisors of $100,000$. How many numbers are the product of two distinct elements of S?

A 98

B 100

C 117

D 119

E 121

Solution(s):

First, note that $100,000 = 2^5 5^5$.

Therefore, any element of S must be of the form $2^a 5^b$ with $0 \leq a$ and $b \leq 5$.

Suppose I have distinct $x, y \in S$ with

$$x = 2^a 5^b,$$
$$y = 2^c 5^d.$$

Then,

$$xy = 2^{a+c} 5^{b+d}.$$

Thus,

$$0 \leq a+c, b+d \leq 10.$$

This means that there are

$$(10+1)(10+1) = 121$$

divisors. However, there are some divisors that can only exist if $x = y$, where

$$(a, b) = (c, d).$$

If $a+c=0$, then $a=0, c=0$ must be true.

If $a+c=10$, then $a=5, c=5$ must be true.

With any other value of $a+c$, we can have $a \neq c$.

Similar stucture holds for $b + d$. Thus, if

$$a + c, b + d \in \{0, 10\},$$

then

$$(a, b) = (c, d),$$

thus making $x = y$.

This means we have to eliminate 4 choices, leaving

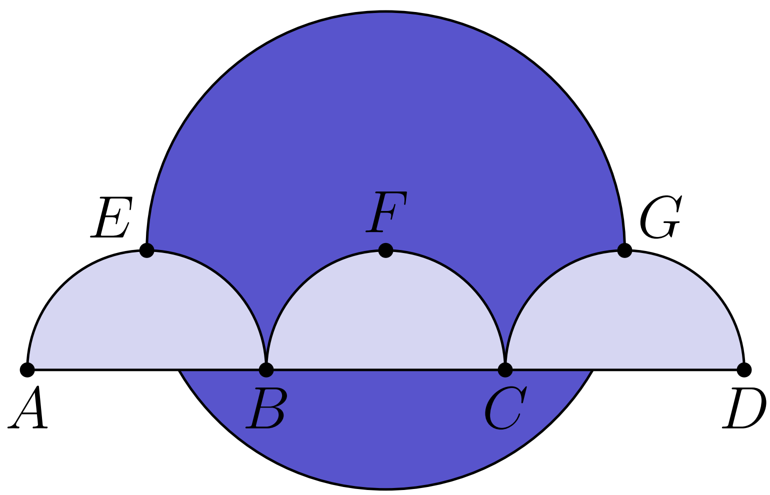
$$121 - 4 = 117.$$

Thus, the answer is **C**.

20. As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, AEB,BFC, and CGD, have their diameters on $\overline{AD}$, and are tangent to line EG at E, F, and G, respectively. A circle of radius 2 has its center on F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

$$\frac{a}{b} \cdot \pi - \sqrt{c} + d,$$

where $a, b, c,$ and d are positive integers and a and b are relatively prime. What is $a + b + c + d$?



A 13

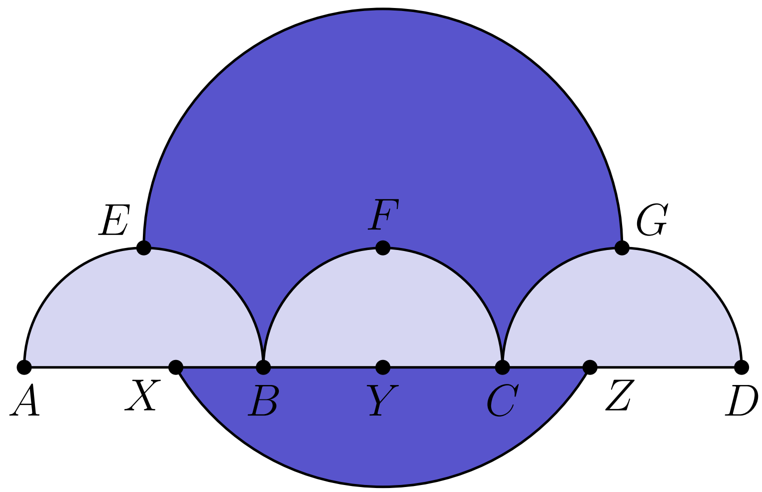
B 14

C 15

D 16

E 17

Solution(s):



Firstly, notice $FD = 1$, so the arc XZ must have length

$$2\arccos\frac{1}{2} = \frac{2\pi}{3}.$$

Since the area under semicircles is equal to the area of the arc minus the area of FXZ, that area is

$$\frac{\theta}{2}r^2 - \frac{FX \cdot FZ \sin XFZ}{2}$$

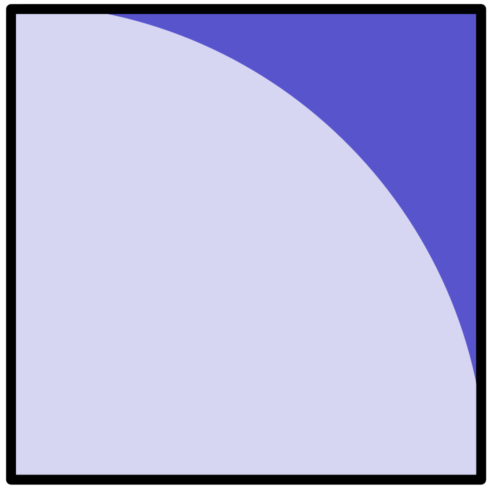
$$= \frac{2 \cdot 2^2\pi}{3 \cdot 2} - \frac{2 \cdot 2\frac{\sqrt{3}}{2}}{2}$$

$$= \frac{4}{3}\pi - \sqrt{3}.$$

Then, the gray area above EG is a semicircle

$$\frac{1}{2}r^2\pi = \frac{1}{2} \cdot 4\pi = 2\pi.$$

Finally, the gray area consists of four of the following shapes.



The squares have side length 1 so it has area 1. The quarter circle has area

$$\frac{\pi}{4}r^2 = \frac{\pi}{4}.$$

Therefore, the total amount of gray is $1 - \frac{\pi}{4}$. We multiply this by 4 since there are 4 of these shapes, yielding an area of $4 - \pi$.

The total area is

$$\frac{4}{3}\pi - \sqrt{3} + 2\pi + 4 - \pi$$

$$= \frac{7}{3}\pi - \sqrt{3} + 4.$$

This makes our answer

$$7 + 3 + 3 + 4 = 17.$$

Thus, the answer is **E**.

21. Debra flips a fair coin repeatedly, keeping track of how many heads and how many tails she has seen in total, until she gets either two heads in a row or two tails in a row, at which point she stops flipping. What is the probability that she gets two heads in a row but she sees a second tail before she sees a second head?

A $\frac{1}{36}$

B $\frac{1}{24}$

C $\frac{1}{18}$

D $\frac{1}{12}$

E $\frac{1}{6}$

Solution(s):

If we flip heads first, then the only way to see the second tails before the next heads is with HTT, which ends without two heads.

Therefore, we must start with tails. Then we need a heads to ensure that we don't end on two tails, and then another tails to ensure a second tails is seen.

This means we must start as THT.

Our sequence must also be alternating except the last two coins, or else there would be two consecutive flips that are the same causing it to stop. This means our sequence must be $THTH \cdots THH$. This means that there is exactly one sequence of size n for all odd n greater than 5, each with a probability of $\left(\frac{1}{2}\right)^n$. Thus, the total probability is

$$\frac{1}{2}^5 + \frac{1}{2}^7 + \cdots$$

$$= \frac{1}{2}^5 \left(1 + \frac{1}{4} + \frac{1}{4}^2 \cdots\right)$$

$$= \frac{1}{32} \cdot \frac{1}{1 - \frac{1}{4}}$$

$$= \frac{1}{32} \cdot \frac{4}{3}$$

$$= \frac{1}{24}.$$

Thus, the answer is **B**.

22. Raashan, Sylvia, and Ted play the following game. Each starts with $\$1$. A bell rings every 15 seconds, at which time each of the players who currently have money simultaneously chooses one of the other two players independently and at random and gives $\$1$ to that player. What is the probability that after the bell has rung 2019 times, each player will have $\$1$?

(For example, Raashan and Ted may each decide to give $\$1$ to Sylvia, and Sylvia may decide to give her dollar to Ted, at which point Raashan will have $\$0$, Sylvia will have $\$2$, and Ted will have $\$1$, and that is the end of the first round of play. In the second round Rashaan has no money to give, but Sylvia and Ted might choose each other to give their $\$1$ to, and the holdings will be the same at the end of the second round.)

A $\frac{1}{7}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E $\frac{2}{3}$

Solution(s):

Suppose that they are sitting in a circle, where they can move the money clockwise or counterclockwise.

Also suppose that at some point, they each have one dollar. Then, they each get a dollar afterwards if they all send the dollar in the same direction. This has a probability of

$$2 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

since there are 2 directions and a probability of one half that they chose that direction. Otherwise, two people send it to the same person, which has a probability of $\frac{3}{4}$.

Suppose that we are at a point where one person has two dollars, one person has one dollar, and the other person has no money.

If the person with one dollar sends it to the person with two dollars, then after the person with two dollars sends money to someone, he has 2 dollars still, and another person has one.

If both the person with one dollar and the person with two dollars send it to the other person, we will still have a $2-1-0$ configuration.

However, if the person with one dollar sends it to the person with no money and the person with two dollars sends it to the person with one dollar, we get a $1-1-1$ configuration. This happens with probability $\frac{1}{4}$.

Therefore, the only possible configurations are $1-1-1$ or $2-1-0$, and each has a $\frac{1}{4}$ probability that the next one is $1-1-1$.

As such, whatever the configuration is after 2018 rings, the probability of the next one being $1-1-1$ is $\frac{1}{4}$.

Thus, the answer is **B**.

23. Points $A=(6,13)$ and $B=(12,11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

A $\frac{83\pi}{8}$

B $\frac{21\pi}{2}$

C $\frac{85\pi}{8}$

D $\frac{43\pi}{4}$

E $\frac{87\pi}{8}$

Solution(s):

If the radius has radius r and the distance to some point outside the circle is d, then the distance from the point to some tangent point must be

$$\sqrt{d^2-r^2},$$

making it a constant value. Therefore, it is equidistant from A and B, so it must lie on its perpendicular bisector. The midpoint betweem A and B is $(9,12)$ and the slope between them is $-\frac{1}{3}$.

Consequently, we know that the slope of the perpendicular bisector is 3. Thus, the perpendicular bisector is on the line

$$y-12=3(x-9)$$
$$y=3x-15.$$

Thus, its intersection with the x axis is $(5,0)$.

Now, one of the tangent lines has points $(5,0)$ and $(6,13)$, so their slope is 13. Since the line perpendicular to this going through $(6,13)$ goes through the center, the center is the intersection of the lines

$$y-13=-\frac{1}{13}(x-6)$$

and

$$y = 3x - 15.$$

Using substitution, we get

$$3x - 15 - 13 = -\frac{1}{13}(x - 6)$$

$$x = \frac{37}{4}.$$

We then get

$$y = \frac{51}{4}.$$

Now, the radius is the distance from $(\frac{37}{4}, \frac{51}{4})$ and $(6, 13)$, which is

$$\sqrt{(\frac{37}{4} - 6)^2 + (\frac{51}{4} - 13)}.$$

This would simplify to

$$r = \sqrt{\frac{85}{8}}.$$

Therefore, the area is

$$\pi r^2 = \frac{85}{8}\pi.$$

Thus, the answer is **C**.

24. Define a sequence recursively by $x_0 = 5$ and

$$x_{n+1} = \frac{x_n^2 + 5x_n + 4}{x_n + 6}$$

for all nonnegative integers n. Let m be the least positive integer such that

$$x_m \leq 4 + \frac{1}{2^{20}}.$$

In which of the following intervals does m lie?

- A $[9, 26]$
- B $[27, 80]$
- C $[81, 242]$
- D $[243, 728]$
- E $[729, \infty)$

Solution(s):

Let $a_n + 4 = x_n$. Then $a_0 = 1$ and we have to find the least m such that

$$a_m \leq \frac{1}{2^{20}}.$$

Also,

$$a_{n+1} + 4$$

$$= \frac{(a_n + 4)^2 + 5(a_n + 4) + 4}{a_n + 10},$$

so

$$a_n = \frac{a_n^2 + 13a_n + 40}{a_n + 10} - 4$$

$$= \frac{a_n(a_n + 9)}{(a_n + 10)}.$$

Thus,

$$\frac{a_{n+1}}{a_n} = \frac{a_n + 9}{a_n + 10}.$$

If $0 \leq a_n \leq 1$, then

$$\frac{a_n + 9}{a_n + 10}$$

is less than 1 and greater than 0, making

$$0 \leq a_{n+1} \leq a_n \leq 1.$$

Thus,

$$\frac{a_n + 9}{a_n + 10}$$

is between $\frac{9}{10}$ and $\frac{10}{11}$, so

$$\frac{9}{10} \leq \frac{a_{n+1}}{a_n} \leq \frac{10}{11}.$$

This means

$$\frac{9}{10}^{k} \leq a_k \leq \frac{10}{11}^{k}.$$

To find the least m, we must put $a_k = \dfrac{1}{2^{20}}$, making

$$\frac{9}{10}^{k} \leq \frac{1}{2^{20}} \leq \frac{10}{11}^{k}.$$

We can take the reciprocal, leaving

$$\frac{10}{9}^{k} \geq 2^{20} \geq \frac{11}{10}^{k}.$$

Since $(1 + \frac{1}{9})^9 \geq 2$, we know

$$9 \log_2 \left(\frac{10}{9} \right) \geq 1.$$

Thus, $\log_2(\frac{10}{9}) \geq \frac{1}{9}$. As such,

$$\frac{k}{9} \leq k \log_2\left(\frac{10}{9}\right) \leq 20$$

$$k \leq 180.$$

Since $\frac{11}{10}^5 \leq 2$ by inspection, we know

$$5 \log_2\left(\frac{11}{10}\right) \leq 1.$$

Therefore,

$$\log_2\left(\frac{11}{10}\right) \leq \frac{1}{5}.$$

Thus,

$$\frac{k}{4} \geq k \log_2(\frac{11}{10}) \geq 20$$

Which implies

$$k \geq 100.$$

This means $100 \leq m \leq 180$, so m is in the interval $[81, 242]$.

Thus, the answer is **C**.

25. How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

- A 55
- B 60
- C 65
- D 70
- E 75

Solution(s):

Our sequence starts with a 0 then has sequences of 110 and 10 in some order, where they each come after a 0.

Let the number of 110 be x and let the number of 10 be y. Then the number of terms in the sequence is

$$3x + 2y + 1 = 19,$$

making

$$3x + 2y = 18.$$

The only possible ordered pairs are

$$\begin{aligned}(x, y) = &(6, 0),\\ &(4, 3),\\ &(2, 6),\\ &(0, 9).\end{aligned}$$

Then, the number of ways to order them would be

$$\binom{x+y}{x}$$

as there are x ways to place the 110.

Therefore, the total number of ways is

$$\binom{6}{6}+\binom{7}{4}+\binom{8}{2}+\binom{9}{0}$$

$$=1+35+28+1$$

$$=65.$$

Thus, the answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc10/2019B