2019 AMC 10A Solutions

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https://live.poshenloh.com/past-contests/amc10/2019A/solutions

1. What is the value of

$$2^{\left(0^{\left(1^9\right)}\right)}+\left(\left(2^0\right)^1\right)^9?$$

A 0

B 1

C 2

D 3

E 4

Solution(s):

We can evaluate this as follows.

$$\begin{aligned}
&2^{\left(0^{\left(1^9\right)}\right)}+\left(\left(2^0\right)^1\right)^9 \\
&=2^{\left(0^1\right)}+\left(1^1\right)^9 \\
&=2^0+1^9 \\
&=1+1 \\
&=2
\end{aligned}$$

Thus, **C** is the correct answer.

2. What is the hundreds digit of $(20! - 15!)$?

A 0

B 1

C 2

D 4

E 5

Solution(s):

Note that $20!$ and $15!$ both have factors of 5^3 in them.

This means that they are both divisible by a 1000, making their difference also a multiple of 1000.

Being a multiple of 1000 makes the last three digits 0, which shows that the hundreds digit is also 0.

Thus, **A** is the correct answer.

3. Ana and Bonita were born on the same date in different years, n years apart. Last year Ana was 5 times as old as Bonita. This year Ana's age is the square of Bonita's age. What is n?

A 3

B 5

C 9

D 12

E 15

Solution(s):

Let a be Ana's age and b be Bonita's age. The statement then gives us that

$$a - 1 = 5(b - 1),$$
$$a = b^2.$$

We can substitute the second equation into the first to get

$$b^2 - 1 = 5b - 5$$
$$b^2 - 5b + 4 = 0$$
$$(b - 4)(b - 1) = 0.$$

We can see that $b \neq 1$ since that would make Ana and Bonita the same age, so we know that $b = 4$.

This gives us that $a = 4^2 = 16$ and $n = 16 - 4 = 12$.

Thus, **D** is the correct answer.

4. A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn?

- A 75
- B 76
- C 79
- D 84
- E 91

Solution(s):

Note that we can pull as many as 14 balls of each color without ensuring that 15 balls of one color are drawn.

This means that we can draw all of the black, white and blue balls, along with 14 red, green, and yellow balls.

This gives us a total of

$$9 + 11 + 13 + 3 \cdot 14$$
$$= 33 + 42$$
$$= 75.$$

We need to add one at the end, however, to ensure that we get that 15th ball of some color, $75 + 1 = 76$.

Thus, **B** is the correct answer.

5. What is the greatest number of consecutive integers whose sum is 45?

A 9

B 25

C 45

D 90

E 120

Solution(s):

Note that negative integers are allowed to be in the sequence.

This means that we could form the sequence

$$-44, -43, \cdots, 44, 45,$$

which clearly adds up to 45. There are 90 terms in this sequence.

Adding another negative term wouldn't work, since that would require having to go up to 46 in the positive numbers, which puts the sum over 45.

Thus, **D** is the correct answer.

6. For how many of the following types of quadrilaterals does there exist a point in the plane of the quadrilateral that is equidistant from all four vertices of the quadrilateral?

- a square
- a rectangle that is not a square
- a rhombus that is not a square
- a parallelogram that is not a rectangle or a rhombus
- an isosceles trapezoid that is not a parallelogram

A 1

B 2

C 3

D 4

E 5

Solution(s):

Note that if a point is equidistant from all the vertices, than that point is the center of the shape's circumcircle.

The question then becomes which of these shapes is cyclic (has a circumcircle). One condition that we can use is that opposite angles are supplementary.

Clearly, a square and rectangle that is not a square work (opposite angles are right, adding up to 180°).

A rhombus that is not a square does not work, since opposite angles are equal, but they are not 90°.

A parallelogram that is not a rectangle or a rhombus faces the same problem as above, making it not cyclic as well.

An isosceles trapezoid that is not a parallelogram by definition has supplementary opposite angles, making it cyclic.

Thus, **C** is the correct answer.

7. Two lines with slopes $\frac{1}{2}$ and 2 intersect at $(2, 2)$. What is the area of the triangle enclosed by these two lines and the line $x + y = 10$?

A 4

B $4\sqrt{2}$

C 6

D 8

E $6\sqrt{2}$

Solution(s):

Let us first find the equations of the two lines. Using slope-intercept form, we know they are the form $y = ax + b$.

For the first line, we know that $a = \dfrac{1}{2}$, so we get

$$2 = \frac{1}{2} \cdot 2 + b$$

$$b = 1.$$

Similarly, for the second line, we get that $a = 2$, which gives us

$$2 = 2 \cdot 2 + b$$

$$b = -2.$$

Our two lines are now $y = \dfrac{1}{2}x + 1$ and $y = 2x - 2$.

We can rewrite $x + y = 10$ as $y = 10 - x$. Substituting this into the first line yields

$$10 - x = \frac{1}{2}x + 1$$

$$x = 6, y = 4.$$

Similarly, for the second line,

$$10 - x = 2x - 2$$

$$x = 4, y = 6.$$

The three vertices of the triangle are therefore $(2, 2), (6, 4)$, and $(4, 6)$.

Note that these vertices form an isosceles triangle (distance formula yields the three sides as $2\sqrt{5}, 2\sqrt{5}$, and $2\sqrt{2}$.

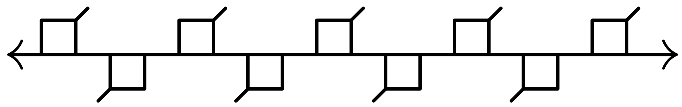
The midpoint of the base is $(5, 5)$, and applying the distance formula again tells us that the height is $3\sqrt{2}$.

The area is therefore

$$\frac{1}{2} \cdot 2\sqrt{2} \cdot 3\sqrt{2} = 6.$$

Thus, **C** is the correct answer.

8. The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?

- some rotation around a point of line ℓ
- some translation in the direction parallel to line ℓ
- the reflection across line ℓ
- some reflection across a line perpendicular to line ℓ

A 0

B 1

C 2

D 3

E 4

Solution(s):

The first transformation works, as we can rotate ℓ 180° around the midpoint between an upward-facing and downward-facing square.

The second also works, as we can just move ell to the right until the squares line up with each other again.

The third fails, as a reflection would cause the line segments to face the opposite direction.

The fourth transformation also doesn't work since the diagonal lines would again be facing in the wrong direction.

Thus, **C** is the correct answer.

9. What is the greatest three-digit positive integer n for which the sum of the first n positive integers is *not* a divisor of the product of the first n positive integers?

- A 995
- B 996
- C 997
- D 998
- E 999

Solution(s):

The sum of the first n numbers is

$$\frac{n(n+1)}{2}.$$

We need this to not divide $n!$.

Note that if $n+1$ is composite, then it can be broken down into factors that divide $n!$.

This means that we need $n+1$ to be prime. The largest three-digit prime is 997, so the largest n value is

$$997 - 1 = 996.$$

Thus, **B** is the correct answer.

10. A rectangular floor that is 10 feet wide and 17 feet long is tiled with 170 one-foot square tiles. A bug walks from one corner to the opposite corner in a straight line. Including the first and the last tile, how many tiles does the bug visit?

- A 17
- B 25
- C 26
- D 27
- E 28

Solution(s):

Note that every time the bug crosses a vertical or horizontal line, the bug visits one new tile.

This means that the number of tiles the bug visits is 1 (the first tile) plus the number of lines it crosses.

The bug never walks over a corner since 10 and 17 are relatively prime, so we don't have to worry about that.

The bug crosses 16 horizontal lines and 9 vertical lines for a total of

$$1 + 16 + 9 = 26$$

tiles.

Thus, **C** see the correct answer.

11. How many positive integer divisors of 201^9 are perfect squares or perfect cubes (or both)?

A 32

B 36

C 37

D 39

E 41

Solution(s):

Taking the prime factorization of 201^9, we get $3^9 \cdot 67^9$.

Note that a perfect square has even exponents for its prime factors, and a cube's exponents are divisible by 3.

There are 5 options for an even exponent $(0$ through $8,)$ and 4 options for multiples of 3 $(0$ through $9)$.

This gives us 5^2 options for the squares and 4^2 options for the cubes. We have to subtract out the powers of 6, however.

Using the same logic, sixth powers have to have exponents of prime factors be divisible by 6. There are 2 options $(0$ and $6)$.

This means that there are $2^2 = 4$ sixth powers. This gives us a total of

$$25 + 16 - 4 = 37$$

perfect squares or perfect cubes.

Thus, **C** is the correct answer.

12. Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, $\ldots$, 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

A $\mu < d < M$

B $M < d < \mu$

C $d = M = \mu$

D $d < M < \mu$

E $d < \mu < M$

Solution(s):

d must have to be less than M because M accounts for the larger numbers that are not modes $(29, 30$ and $31)$.

There are 365 entries, so m is the 183 rd number. The first 15 numbers take up $15 \cdot 12 = 180$ spots, so m is 16.

μ is less than 16 since there are fewer occurrences of the larger numbers, making the distribution left skewed.

d is also less than μ since μ accounts for $29, 30$ and 31. Therefore, we get that

$$d < \mu < M.$$

Thus, **E** is the correct answer.

13. Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40^\circ$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

A 90

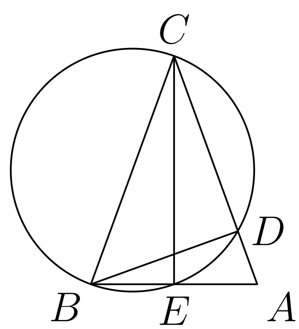
B 100

C 105

D 110

E 120

Solution(s):



Since $\overline{BC}$ is the diameter of the circle, we get that $\angle BDC$ and $\angle BEC$ are right angles.

We know that $\angle ABC = 70^\circ$ from the fact that $\triangle ABC$ is isosceles.

Using the fact that the angles of a triangle add up to 180°, we get that

$$\begin{aligned}\angle ECB &= 180^\circ - 70^\circ - 90^\circ \\ &= 20^\circ\end{aligned}$$

and

$$\begin{aligned}\angle DBC &= 180^\circ - 40^\circ - 90^\circ \\ &= 50^\circ.\end{aligned}$$

Now, from $\triangle BFC$, we get that

$$
\begin{aligned}
\angle BFC &= 180^\circ - 50^\circ - 20^\circ \\
&= 110^\circ.
\end{aligned}
$$

Thus, **D** is the correct answer.

14. For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

A 14

B 16

C 18

D 19

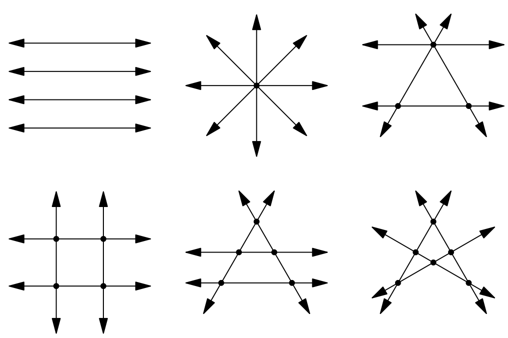
E 21

Solution(s):

It is shown below that

$$0, 1, 3, 4, \text{ and } 5$$

are attainable.



We need to show that it is impossible to attain only 2 intersection points. Let X and Y be the intersection points.

Case 1 : no lines go through both X and Y

This means that there are 2 unique pairs of lines that go through each of X and Y.

Take one line that goes through X and call it ℓ. For ℓ to not intersect the lines that go through Y, it must be parallel to them.

This means that all 3 lines must be parallel, but this is not possible since the other two lines intersect at Y.

Case 2 : one line goes through both X and Y

Let this common line be ℓ. Then the other two lines that go through X and Y must be parallel.

For there to be no other intersections, every other line must also be parallel to this two lines.

This, however, ensures that all the other lines are not parallel with ℓ, which results in more intersections.

In both cases, 2 intersections is not possible. Therefore, the sum of all values of N is

$$1+3+4+5+6=19.$$

Thus, **D** is the correct answer.

15. A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

- A 2020
- B 4039
- C 6057
- D 6061
- E 8078

Solution(s):

We can rewrite the recursive formula as

$$\begin{aligned}\frac{1}{a_n} &= \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}} \\ &= \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}}.\end{aligned}$$

This means that

$$\frac{1}{a_n} - \frac{1}{a_{n-1}} = \frac{1}{a_{n-1}} - \frac{1}{a_{n-2}},$$

which tells us that $\left\{\frac{1}{a_n}\right\}$ is an arithmetic sequence.

Using a_1 and a_2, we get that the common difference is

$$\frac{1}{\frac{3}{7}} - \frac{1}{1} = \frac{7}{3} - 1 = \frac{4}{3}.$$

From this, we get that

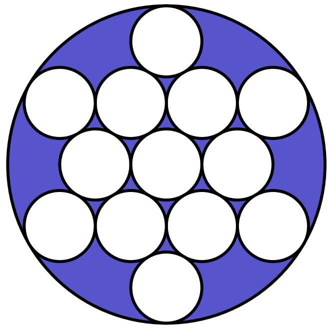
$$\begin{aligned}\frac{1}{a_{2019}} &= 1 + 2918 \cdot \frac{4}{3} \\ &= \frac{8075}{3}.\end{aligned}$$

$p + q$ is therefore

$$8075 + 3 = 8078.$$

Thus, **E** is the correct answer.

16. The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1?



A $4\pi\sqrt{3}$

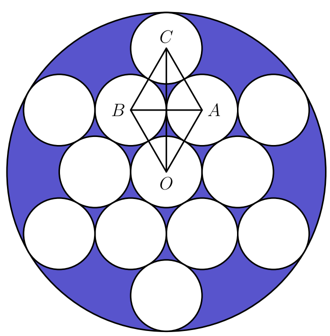
B 7π

C $\pi\left(3\sqrt{3}+2\right)$

D $10\pi\left(\sqrt{3}-1\right)$

E $\pi\left(\sqrt{3}+6\right)$

Solution(s):



We know $\triangle ABC$ and $\triangle ABO$ are equilateral triangles.

We get that $OC = 2\sqrt{3}$ using special right triangles to find the altitudes of the triangles.

The radius of the larger circle is therefore $2\sqrt{3}+1$, since there is the extra unit radius after $\overline{OC}$.

The area of the larger circle is

$$(2\sqrt{3}+1)^2\pi = (13+4\sqrt{3})\pi.$$

The area of all the inner circles is 13π.

The area of the shaded region is

$$(13+4\sqrt{3})\pi - 13\pi = 4\pi\sqrt{3}.$$

Thus, **A** is the correct answer.

17. A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

A 24

B 288

C 312

D $1,260$

E $40,320$

Solution(s):

Every tower of height 8 could have been formed by creating a tower of height 9 and removing the top cube.

This shows that there is a one-to-one correspondence between towers of height 8 and 9.

There are $9!$ ways to make a tower of height 9, but we are overcounting since there are multiple cubes of the same color.

We have to divide through by $2!$ ways to arrange the red cubes, $3!$ for the blue cubes, and $4!$ for the green cubes.

Therefore, the number of valid arrangements is

$$\frac{9!}{2!\cdot 3!\cdot 4!} = 1,260.$$

Thus, **D** is the correct answer.

18. For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is

$$0.\overline{23}_k = 0.232323\ldots_k.$$

What is k?

A 13

B 14

C 15

D 16

E 17

Solution(s):

We can expand the repeating fraction as

$$0.\overline{23}_k = 2 \cdot k^{-1} + 3 \cdot k^{-2} + \cdots.$$

We can rearrange this to get the sum of two infinite sequences:

$$2(k^{-1} + k^{-3} + k^{-5} + k^{-7} \cdots)$$

and

$$3(k^{-2} + k^{-4} + k^{-6} + k^{-8} \cdots).$$

These sums evaluate to

$$2 \cdot \frac{k^{-1}}{1 - k^{-2}} = \frac{2k}{k^2 + 1}$$

and

$$3 \cdot \frac{k^{-2}}{1 - k^{-2}} = \frac{3}{k^2 + 1}.$$

Adding these together yields

$$\frac{2k}{k^2+1} + \frac{3}{k^2+1} = \frac{2k+3}{k^2+1}.$$

We know that these equals $\frac{7}{51}$, which gives us the quadratic

$$51(2k+3) = 7(k^2-1)$$
$$7k^2 - 102k - 160 = 0$$
$$(k-16)(7k+10) = 0.$$

k can't be negative, so we get that $k = 16$.

Thus, **D** is the correct answer.

19. What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4)$$
$$+2019,$$

where x is a real number?

A 2017

B 2018

C 2019

D 2020

E 2021

Solution(s):

Multiplying the first two terms and the last terms yields

$$(x^2+5x+4)(x^2+5x+6).$$

Note that these two terms differ by 2. We can try to express this as a difference of squares, which is

$$(x^2+5x+5)^2-1.$$

Adding 2019 to this gets us

$$(x^2+5x+5)^2+2018.$$

Squares are non-negative, so as long as we find a way to make the inner expression 0, we can make the square 0.

The discriminant is $5^2-4\cdot 5=5$, which is positive meaning that there is a value that makes the square 0.

This means that the minimum value would be

$$0^2+2018=2018.$$

Thus, **B** is the correct answer.

20. The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

A $\frac{1}{21}$

B $\frac{1}{14}$

C $\frac{5}{63}$

D $\frac{2}{21}$

E $\frac{1}{7}$

Solution(s):

Note that the only way to get an odd sum is if there are either 0 or 2 even numbers in the row or column.

The only way for this to happen is if the 4 even numbers form a rectangle with sides parallel to the large square.

The way to see this is we choose a spot for the first even number. Then we need to choose another square in the same row, x, and column, y, to be even.

The final even has to be in same column as x and the same row as y. This forms the aforementioned rectangle.

There are four 2×2 rectangles, two 3×2 rectangles, two 2×3 rectangles, and one 3×3 rectangle.

This gives us a total of

$$4 + 2 + 2 + 1 = 9$$

rectangles, are arrangements for the even numbers.

There are $4!$ ways to arrange the even numbers and $5!$ ways to arrange the odd numbers.

This means that there are a total of

$$9 \cdot 4! \cdot 5!$$

configurations of squares that satisfy the condition.

There are a total of $9!$ arrangements with no restrictions. The probability is therefore

$$\frac{9 \cdot 4! \cdot 5!}{9!} = \frac{1}{14}.$$

Thus, **B** is the correct answer.

21. A sphere with center O has radius 6. A triangle with sides of length $15, 15,$ and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

A $2\sqrt{3}$

B 4

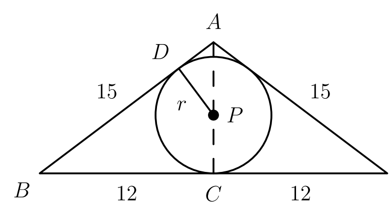
C $3\sqrt{2}$

D $2\sqrt{5}$

E 5

Solution(s):

We get the following diagrams by taking the cross-section of the plane of the triangle.



Note that $AC = 9$ by the Pythagorean theorem. We also get that $\triangle ADP \sim \triangle ACB$.

We can see that $PCBD$ is a kite, so we know that

$$DB = BC = 12,$$

which makes $AD = 15 - 12 = 3$. Using the similar triangles, above, we get that

$$\frac{r}{3} = \frac{12}{9}$$

$$r = 4.$$

Let d be the distance from the sphere to this plane. d is also the distance from O to P.

Once again using the Pythagorean theorem, we get that

$$d = \sqrt{6^2 - 4^2} = 2\sqrt{5}.$$

Thus, **D** is the correct answer.

22. Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads, and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \frac{1}{2}$?

A $\frac{1}{3}$

B $\frac{7}{16}$

C $\frac{1}{2}$

D $\frac{9}{16}$

E $\frac{2}{3}$

Solution(s):

We can case on whether x and y are chosen from the interval or from 0 and 1. Each case has a $\frac{1}{4}$ chance of happening, since they depend on two coin flips.

Case $1 : x$ and y are either 0 or 1

x and y need to be different, which happens with a $\frac{1}{2}$ probability.

Case $2 : x$ is either 0 or 1, and y is chosen from $[0, 1]$

If $x = 0$, then y has to be chosen from $\left(\frac{1}{2}, 1\right]$, and if $x = 1$, then y has to be chosen from $\left[0, \frac{1}{2}\right)$.

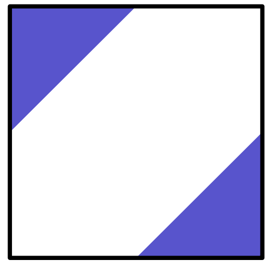
This means that y always has a $\frac{1}{2}$ probability of being chosen from the correct interval.

Case $3 : x$ is chosen from $[0, 1]$, and y is either 0 or 1

This has the same probability as case 2 due to symmetry.

$4 : x$ **and** y **are chosen from** $[0, 1]$

We can use geometric probability since we are working with an infinite number of (x, y) pairs. We graph $|x - y| > \frac{1}{2}$.



The shaded area covers $\frac{1}{4}$ of the graph, showing that there is a $\frac{1}{4}$ probability of this case working.

Adding up all the probabilities, we get

$$\begin{aligned}\frac{1}{4}\left(3 \cdot \frac{1}{2} + \frac{1}{4}\right) &= \frac{1}{4} \cdot \frac{7}{4} \\ &= \frac{7}{16}.\end{aligned}$$

Thus, **B** is the correct answer.

23. Travis has to babysit the terrible Thompson triplets. Knowing that they love big numbers, Travis devises a counting game for them. First Tadd will say the number 1, then Todd must say the next two numbers (2 and 3), then Tucker must say the next three numbers (4, 5, 6), then Tadd must say the next four numbers (7, 8, 9, 10), and the process continues to rotate through the three children in order, each saying one more number than the previous child did, until the number $10,000$ is reached. What is the 2019th number said by Tadd?

- A 5743
- B 5885
- C 5979
- D 6001
- E 6011

Solution(s):

We can find how many numbers each triplet says in one round.

$$\begin{aligned}&\text{Tadd: } 1,4,7,10,13\cdots\\&\text{Todd: } 2,5,8,11,14\cdots\\&\text{Tucker: } 3,6,9,12,15\cdots\end{aligned}$$

Now we can find a general formula for the number of numbers Tadd says after the n th round.

$$\begin{aligned}\sum_{i=1}^{n} 3n-2 &= -2n+3\sum_{i=1}^{n}\\&= -2n+\frac{3n(n+1)}{2}\\&= \frac{3n^2-n}{2}\end{aligned}$$

Now we to find the largest n such that

$$\frac{3n^2-n}{2} \leq 2019.$$

We can guess and check to find that 36 is the largest such value.

Note that Todd and Tucker also go through 36 turns each before Tadd says the 2019 th number.

The number of numbers that Todd and Tucker go through, plus the 2019 numbers that Tadd says, is

$$\begin{aligned}&\sum_{i=1}^{36} 3n + \sum_{i=1}^{36} (3n-1) + 2019 \\ &= \sum_{i=1}^{36} (6n-1) + 2019 \\ &= (5 + 11 + \cdots + 215) + 2019 \\ &= \frac{36 \cdot 220}{2} + 2019 \\ &= 5979.\end{aligned}$$

Thus, **C** is the correct answer.

24. Let p, q, and r be the distinct roots of the polynomial

$$x^3 - 22x^2 + 80x - 67.$$

It is given that there exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} =$$

$$\frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \notin \{p, q, r\}$. What is

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C}?$$

- A 243
- B 244
- C 245
- D 246
- E 247

Solution(s):

We can multiply each side by

$$(s-p)(s-q)(s-r)$$

to get

$$A(s-q)(s-r) + B(s-p)(s-r)$$

$$+C(s-p)(s-q) = 1.$$

We can expand to get

$$s^2(A+B+C) - s\cdot$$

$$(Aq + Ar + Bp + Br + Cp + Cq)$$

$$+(Aqr + Bpr + Cpq - 1) = 0.$$

Note that the coefficients of s and s^2 must both be 0.

From

$$A + B + C = 0, \tag{1}$$

we get

$$A = -(B + C),$$

$$B = -(A + C),$$

and

$$C = -(A + B).$$

Plugging this into the coefficient of s, we get

$$Ap + Bq + Cr = 0.$$

Subtracting $(1) \cdot r$ form this above equation, we get

$$A(p - r) + B(q - r) = 0(2)$$

We also know that

$$Aqr + Bpr + Cpq = 1.$$

Subtracting $(1) \cdot pq$ from this equation, we get

$$Aq(r - p) + Bp(r - q) = 1.$$

Adding $(2) \cdot p$ to this equation, we get

$$A(r - p)(q - p) = 1,$$

which gets us

$$A = \frac{1}{(r - p)(q - p)}.$$

Similarly, we get

$$B = \frac{1}{(r - q)(p - q)}$$

and

$$C = \frac{1}{(q-r)(p-r)}.$$

This gives us

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} =$$

$$p^2 + q^2 + r^2 - pq - qr - pr.$$

Using Vieta's formulas, we get

$$p^2 + q^2 + r^2 = (p+q+r)^2 -$$

$$2(pq + qr + pr) = 324.$$

Finally, since

$$pq + qr + pr = 80,$$

we get our desired value of $324 - 80 = 244.$

Thus, **B** is the correct answer.

25. For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0! = 1$.)

A 31

B 32

C 33

D 34

E 35

Solution(s):

One fact that greatly helps with this problem is realizing that

$$\frac{(n^2)!}{(n!)^{n+1}}$$

is always an integer.

This is because it is the number of ways to split up n^2 objects into n unordered groups of size n.

Now, we get that

$$\frac{(n^2-1)!}{(n!)^n} = \frac{(n^2)!}{(n!)^{n+1}} \cdot \frac{n!}{n^2}.$$

Therefore, we need to find when $n^2 \div n!$, or when $n \div (n-1)!$.

This condition is false if $n = 4$, or if n is prime. $n = 4$ is too large for it to divide $3!$.

n cannot be prime because $(n-1)!$ does not contain any numbers where n could be a factor.

If n is not n and not prime, this works since n can be decomposed into 2 numbers both less than n that are found in $(n-1)!$.

There are 15 primes less than 50, and adding on 4, we get that there are 16 values for n that do not work.

Therefore, the desired answer is $50 - 16 = 34$.

Thus, **D** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2019A