# 2018 AMC 10B Solutions

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**1.** Kate bakes a $20$-inch by $18$-inch pan of cornbread. The cornbread is cut into pieces that measure $2$ inches by $2$ inches. How many pieces of cornbread does the pan contain?

A $90$

B $100$

C $180$

D $200$

E $360$

## Solution:

The whole pan has area $20 \times 18 = 360$ square inches. Each piece is $2 \times 2 = 4$ square inches. So the number of pieces is $360/4 = 90$. Thus, **A** is the correct answer.

**2.** Sam drove $96$ miles in $90$ minutes. His average speed during the first $30$ minutes was $60$ mph (miles per hour), and his average speed during the second $30$ minutes was $65$ mph. What was his average speed, in mph, during the last $30$ minutes?

A $64$

B $65$

C $66$

D $67$

E $68$

## Solution:

Each leg is half an hour. In the first, Sam drove $60 \cdot \frac{1}{2} = 30$ miles; in the second, $65 \cdot \frac{1}{2} = 32.5$ miles. That's $62.5$ miles so far. That leaves $96 - 62.5 = 33.5$ miles for the last half hour, which is a speed of $33.5/\frac{1}{2} = 67$ mph. Therefore, the answer is **D**.

**3.** In the expression $(\_ \times \_) + (\_ \times \_)$ each blank is to be filled in with one of the digits $1, 2, 3,$ or $4,$ with each digit being used once. How many different values can be obtained?

A $2$

B $3$

C $4$

D $6$

E $24$

## Solution:

Order inside a product doesn't matter, and neither does the order we add the two products. So all that matters is how the four digits split into two pairs. There are three splits: $1 \cdot 2 + 3 \cdot 4 = 14, 1 \cdot 3 + 2 \cdot 4 = 11,$ and $1 \cdot 4 + 2 \cdot 3 = 10.$ That's $3$ different values. Thus, **B** is the correct answer.

**4.** A three-dimensional rectangular box with dimensions $X, Y$, and $Z$ has faces whose surface areas are $24, 24, 48, 48, 72,$ and $72$ square units. What is $X + Y + Z$?

- A $18$
- B $22$
- C $24$
- D $30$
- E $36$

## Solution:

The three distinct face areas are the pairwise products $XY = 24, XZ = 48, YZ = 72$ in some order. Multiply all three: $(XYZ)^2 = 24 \cdot 48 \cdot 72 = 82944$, so $XYZ = 288$. Now divide by each face area. We get $Z = 288/24 = 12, Y = 288/48 = 6$, and $X = 288/72 = 4$, so $X + Y + Z = 22$. Therefore, the answer is **B**.

**5.** How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

- A $128$
- B $192$
- C $224$
- D $240$
- E $256$

## Solution:

Count the complement. The set has $2^8 = 256$ subsets total. A subset avoids every prime exactly when it sticks to the non-primes $\{4, 6, 8, 9\}$, and there are $2^4 = 16$ of those. So $256 - 16 = 240$ subsets contain at least one prime. Thus, **D** is the correct answer.

**6.** A box contains $5$ chips, numbered $1, 2, 3, 4,$ and $5.$ Chips are drawn randomly one at a time without replacement until the sum of the values drawn exceeds $4.$ What is the probability that $3$ draws are required?

A $\frac{1}{15}$

B $\frac{1}{10}$

C $\frac{1}{6}$

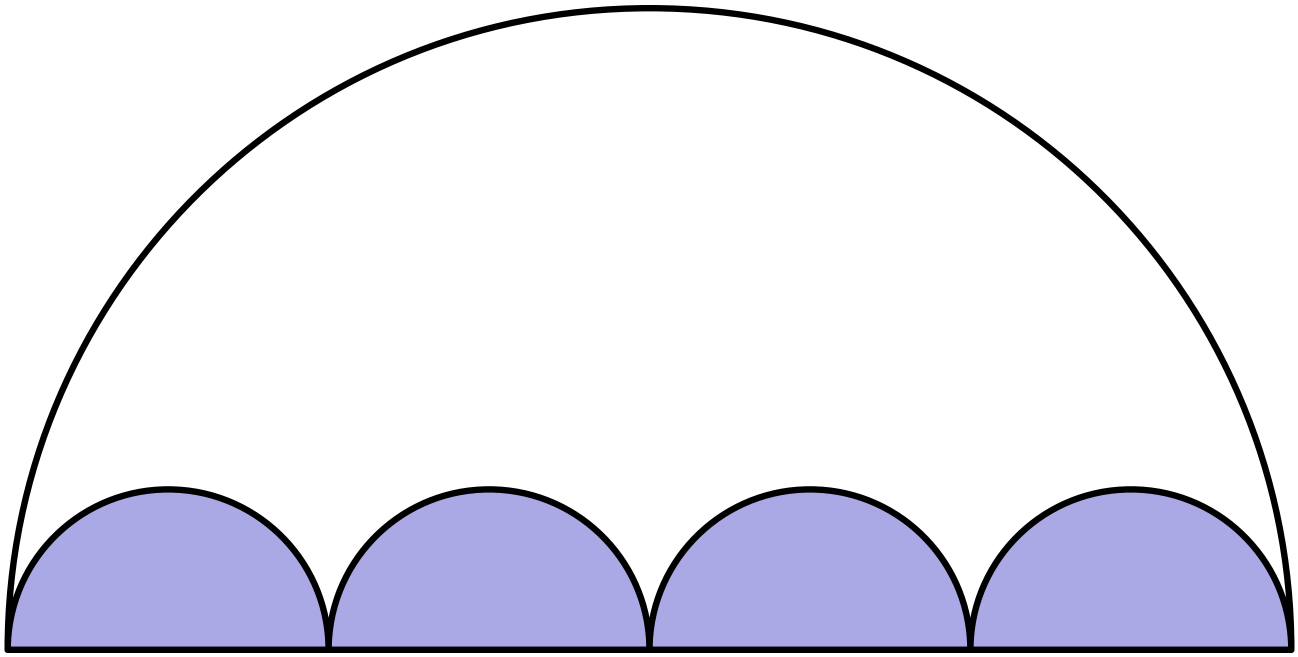
D $\frac{1}{5}$

E $\frac{1}{4}$

## Solution:

We need a third draw exactly when the first two chips still sum to $4$ or less. The only such pairs are $\{1, 2\}$ and $\{1, 3\}$. Each shows up as an ordered pair of first draws in $2$ ways, so there are $4$ favorable sequences out of $5 \cdot 4 = 20$ equally likely ones. The probability is $4/20 = \frac{1}{5}$. Therefore, the answer is **D**.

**7.** In the figure below, $N$ congruent semicircles are drawn along a diameter of a large semicircle, with their diameters covering the diameter of the large semicircle with no overlap. Let $A$ be the combined area of the small semicircles and $B$ be the area of the region inside the large semicircle but outside the small semicircles. The ratio $A : B$ is $1 : 18$. What is $N$?



A $16$

B $17$

C $18$

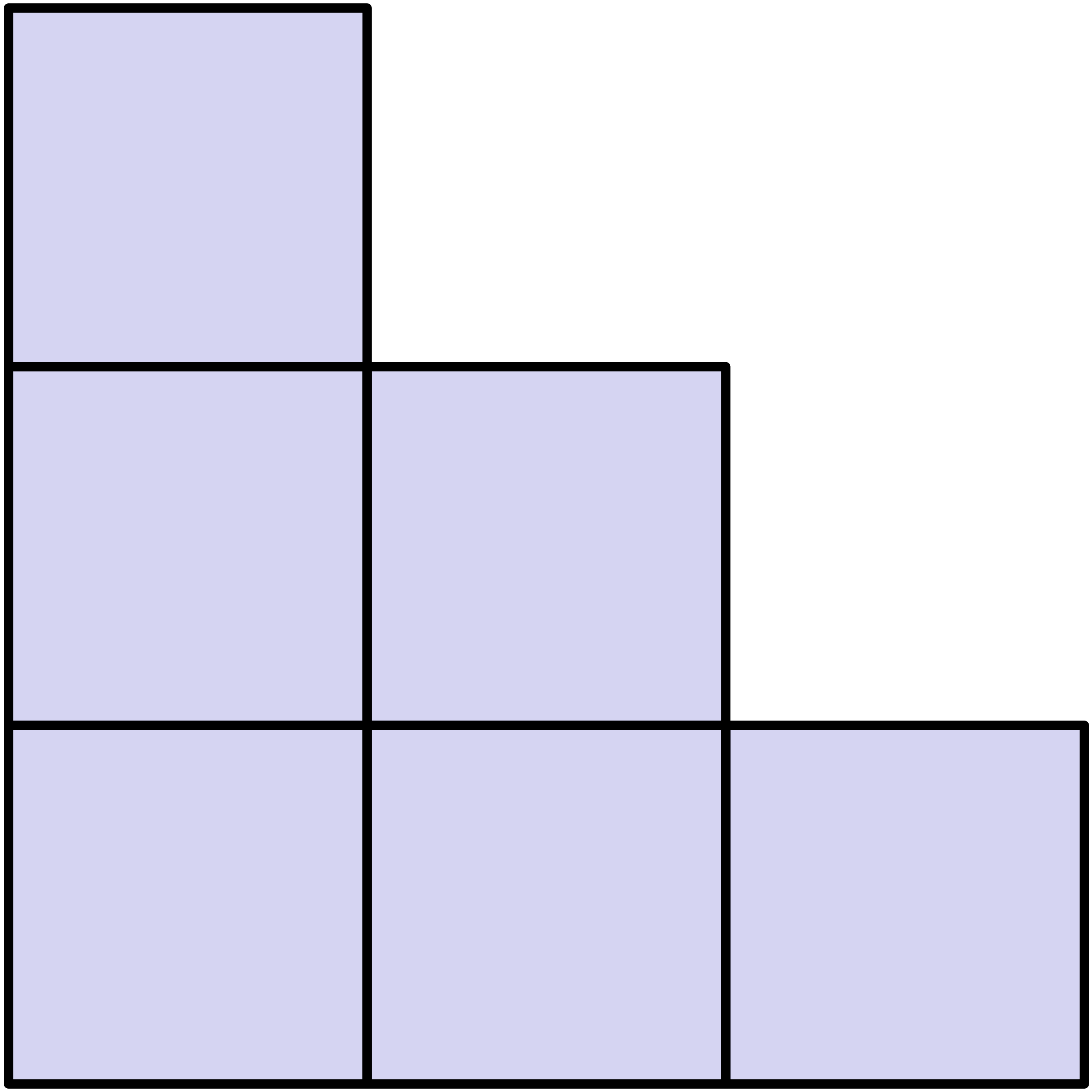
D $19$

E $36$

## Solution:

Let each small semicircle have radius $r$. The $N$ diameters cover the big diameter, so the large radius is $Nr$. Then $A = N \cdot \frac{1}{2}\pi r^2$, and the large semicircle has area $\frac{1}{2}\pi(Nr)^2$, so the leftover region is $B = \frac{1}{2}\pi r^2(N^2 - N)$. This gives $A : B = N : N(N-1) = 1 : (N-1)$. Set $N - 1 = 18$, and $N = 19$. Thus, **D** is the correct answer.

**8.** Sara makes a staircase out of toothpicks as shown:



This is a $3$-step staircase and uses $18$ toothpicks. How many steps would be in a staircase that used $180$ toothpicks?

A $10$

B $11$

C $12$

D 24

E 30

## Solution:

In an $n$-step staircase the vertical toothpicks number $(1+2+\cdots+n)+n=\frac{n(n+1)}{2}+n$, and there are just as many horizontal ones. That's a total of $n(n+1)+2n=n(n+3)$. Check: $n=3$ gives $18$, as it should. Now solve $n(n+3)=180$. This factors as $(n-12)(n+15)=0$, so $n=12$. Therefore, the answer is **C**.

**9.** The faces of each of $7$ standard dice are labeled with the integers from $1$ to $6$. Let $p$ be the probability that when all $7$ dice are rolled, the sum of the numbers on the top faces is $10$. What other sum occurs with the same probability $p$?

A 13

B 26

C 32

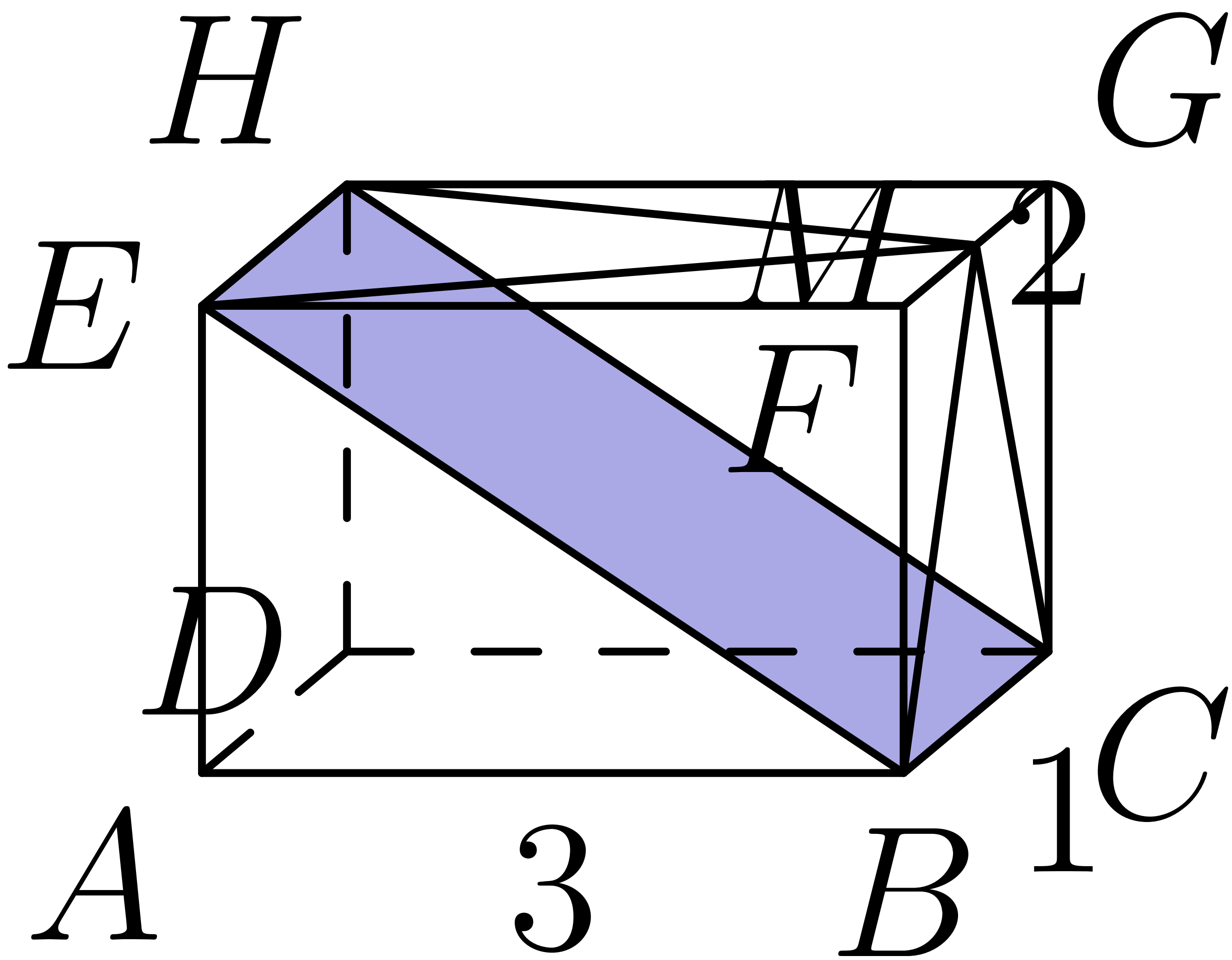
D 39

E 42

## Solution:

Replace each die's value $k$ by $7-k$. This pairs up outcomes one-to-one and keeps their probabilities, and it sends a total of $s$ to $7\cdot 7-s=49-s$. So the sums $s$ and $49-s$ are equally likely. The partner of $10$ is $49-10=39$. Thus, **D** is the correct answer.

**10.** In the rectangular parallelepiped shown, $AB = 3, BC = 1,$ and $CG = 2.$ Point $M$ is the midpoint of $FG.$ What is the volume of the rectangular pyramid with base $BCHE$ and apex $M$?




A $1$

B $\frac{4}{3}$

C $\frac{3}{2}$

D $\frac{5}{3}$

E $2$

## Solution:

Put $A$ at the origin with edges along the axes: $A = (0,0,0), B = (3,0,0), C = (3,1,0), E = (0,0,2), H = (0,1,2), F = (3,0,2), G = (3,1,2)$, so $M = (3, \frac{1}{2}, 2)$. The base $BCHE$ is a rectangle with $BC = 1$ and $BE = \sqrt{3^2 + 2^2} = \sqrt{13}$, hence area $\sqrt{13}$. Its plane is $2x + 3z = 6$, and $M$ sits at distance $\frac{|2\cdot3+3\cdot2-6|}{\sqrt{13}} = \frac{6}{\sqrt{13}}$ from it. The volume is $\frac{1}{3} \cdot \sqrt{13} \cdot \frac{6}{\sqrt{13}} = 2$. Therefore, the answer is **E**.

**11.** Which of the following expressions is never a prime number when $p$ is a prime number?

A $p^2 + 16$

B $p^2 + 24$

C $p^2 + 26$

D $p^2 + 46$

E $p^2 + 96$

## Solution:

Look at $p^2 + 26$. When $p = 3$, it's $35 = 5 \cdot 7$. For any other prime, $p$ isn't divisible by $3$, so $p^2 \equiv 1 \pmod 3$ and $p^2 + 26 \equiv 1 + 2 \equiv 0 \pmod 3$. Either way it's a multiple of $3$ bigger than $3$, hence composite. So it's never prime. Thus, **C** is the correct answer.

**12.** Line segment $AB$ is a diameter of a circle with $AB = 24$. Point $C$, not equal to $A$ or $B$, lies on the circle. As point $C$ moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

A $25$

B $38$

C $50$

D $63$

E $75$

## Solution:

Put the center $O$ at the origin, so $A = (-12, 0)$ and $B = (12, 0)$, while $C$ runs over the circle of radius $12$. Then $A + B = 0$, so the centroid is $\frac{1}{3}(A + B + C) = \frac{1}{3}C$. As $C$ circles, $\frac{1}{3}C$ traces a circle of radius $\frac{12}{3} = 4$ (minus the two points where $C = A$ or $B$). Its area is $\pi \cdot 4^2 = 16\pi \approx 50$. Therefore, the answer is **C**.

**13.** How many of the first $2018$ numbers in the sequence $101, 1001, 10001, 100001, \ldots$ are divisible by $101$?

- A $253$
- B $504$
- C $505$
- D $506$
- E $1009$

## Solution:

The $k$-th term is $10^{k+1}+1$, which $101$ divides iff $10^{k+1} \equiv -1 \pmod{101}$. Notice $10^2 = 100 \equiv -1 \pmod{101}$. So $10^m \equiv -1$ exactly when $m \equiv 2 \pmod 4$, meaning $k+1 \equiv 2$, that is $k \equiv 1 \pmod 4$. Among $k = 1, 2, \ldots, 2018$, the values $1, 5, \ldots, 2017$ number $505$. Thus, **C** is the correct answer.

**14.** A list of $2018$ positive integers has a unique mode, which occurs exactly $10$ times. What is the least number of distinct values that can occur in the list?

A $202$

B $223$

C $224$

D $225$

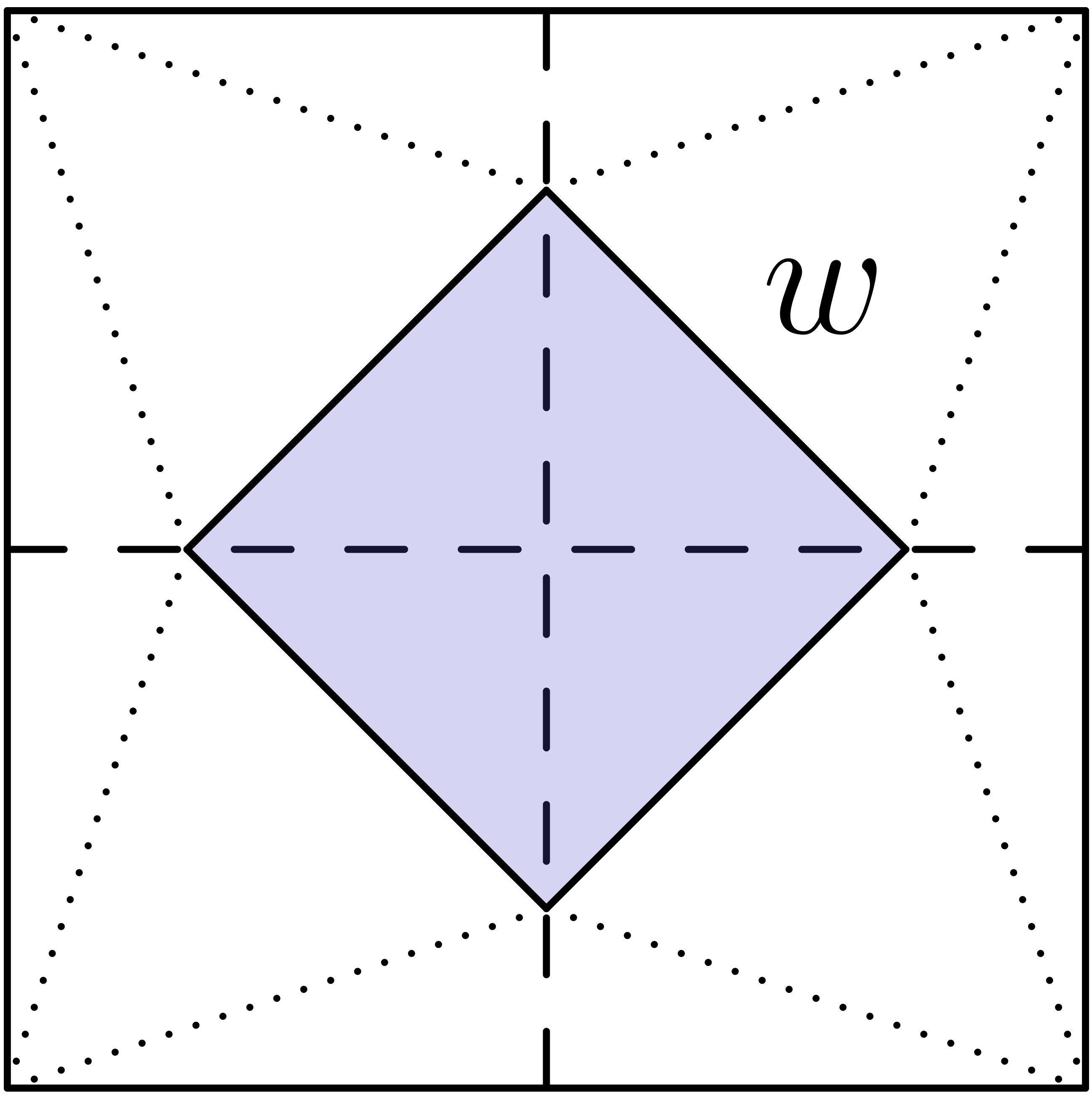
E $234$

## Solution:

The mode shows up $10$ times. To keep the number of distinct values small, let every other value repeat as much as the rules allow, which is $9$ times each (any more would tie the mode). With $d$ distinct values the list holds at most $10 + 9(d-1)$ entries. We need $10 + 9(d-1) \geq 2018$, so $d - 1 \geq 223.1$, giving $d \geq 225$. Therefore, the answer is **D**.

15.

A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure. The four corners of the wrapping paper are folded up over the sides and brought together to meet at the center of the top of the box. The box has base length $w$ and height $h$. What is the area of the sheet of wrapping paper?




A $2(w+h)^2$

B $\frac{(w+h)^2}{2}$

C $2w^2 + 4wh$

D $2w^2$

E $w^2h$

## Solution:

Let the sheet have side $s$. The base sits as a square of side $w$ turned $45^\circ$, so the center is $\frac{w}{2}$ from each base edge. A corner of the sheet lies $\frac{s}{\sqrt{2}}$ from the center. Folding that corner up to the top center traces a straight line: $\frac{w}{2}$ out to the base edge, then $h$ up the side, then $\frac{w}{2}$ across the top. So $\frac{s}{\sqrt{2}} = \frac{w}{2} + h + \frac{w}{2} = w + h$. Then $s = \sqrt{2}\,(w + h)$, and the area is $s^2 = 2(w + h)^2$. Thus, **A** is the correct answer.

**16.** Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by $6$?

A $0$

B $1$

C $2$

D $3$

E $4$

## Solution:

For any integer $n$, $n^3 - n = (n-1)n(n+1)$ is a product of three consecutive integers, so it's divisible by $6$. That means $n^3 \equiv n \pmod 6$. Summing, $\sum a_i^3 \equiv \sum a_i = 2018^{2018} \pmod 6$. Now $2018 \equiv 2 \pmod 6$, and powers of $2$ mod $6$ alternate $2, 4, 2, 4, \ldots$. The exponent $2018$ is even, so $2^{2018} \equiv 4 \pmod 6$. The remainder is $4$. Therefore, the answer is **E**.

**17.** In rectangle $PQRS, PQ = 8$ and $QR = 6.$ Points $A$ and $B$ lie on $PQ,$ points $C$ and $D$ lie on $QR,$ points $E$ and $F$ lie on $RS,$ and points $G$ and $H$ lie on $SP$ so that $AP = BQ < 4$ and the convex octagon $ABCDEFGH$ is equilateral. The length of a side of this octagon can be expressed in the form $k + m\sqrt{n},$ where $k, m,$ and $n$ are integers and $n$ is not divisible by the square of any prime. What is $k + m + n$?

A $1$

B $7$

C $21$

D $92$

E $106$

## Solution:

By symmetry the four cut corners are congruent right triangles, with legs $x$ along the sides of length $8$ and $y$ along the sides of length $6.$ The octagon's sides come in three types, $8 - 2x, 6 - 2y,$ and $\sqrt{x^2 + y^2},$ and they're all equal. From $8 - 2x = 6 - 2y$ we get $y = x - 1.$ Substitute into $8 - 2x = \sqrt{x^2 + (x-1)^2}$ and square: $2x^2 - 30x + 63 = 0,$ so $x = \frac{15-3\sqrt{11}}{2}$ (taking the root with $x < 4$). The side length is $8 - 2x = -7 + 3\sqrt{11},$ so $k + m + n = -7 + 3 + 11 = 7.$ Thus, **B** is the correct answer.

**18.** Three young brother-sister pairs from different families need to take a trip in a van. These six children will occupy the second and third rows in the van, each of which has three seats. To avoid disruptions, siblings may not sit right next to each other in the same row, and no child may sit directly in front of his or her sibling. How many seating arrangements are possible for this trip?

- A $60$
- B $72$
- C $92$
- D $96$
- E $120$

## Solution:

Suppose some family put both children in one row. They'd have to take the non-adjacent seats $1$ and $3$, which forces the middle family's two children into the same column. Not allowed. So each row holds exactly one child from each family. The second row is a permutation of the three families, $3! = 6$ ways. The third row needs a different family in every column, a derangement of the second row's order, and there are $2$ of those. Finally, each pair can swap its two children between their seats, $2^3 = 8$ ways. The total is $6 \cdot 2 \cdot 8 = 96.$ Therefore, the answer is **D**.

**19.** Joey and Chloe and their daughter Zoe all have the same birthday. Joey is $1$ year older than Chloe, and Zoe is exactly $1$ year old today. Today is the first of the $9$ birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

A $7$

B $8$

C $9$

D $10$

E $11$

## Solution:

Let Chloe be $n$ today; Zoe is $1.$ In $t$ years her age over Zoe's is $\frac{n+t}{1+t} = 1 + \frac{n-1}{1+t}$, an integer exactly when $1+t$ divides $n-1.$ So the number of such birthdays is the number of divisors of $n-1.$ Nine of them means $n-1$ has $9$ divisors, forcing $n-1 = 2^2 \cdot 3^2 = 36$ (the only two-digit choice). So Chloe is $37$ and Joey is $38.$ Now Joey's age $38+t$ is a multiple of $1+t$ iff $1+t$ divides $37.$ The next such time is $t = 36,$ when Joey is $74.$ Its digit sum is $7+4=11.$ Thus, **E** is the correct answer.

**20.** A function $f$ is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

- A $2016$
- B $2017$
- C $2018$
- D $2019$
- E $2020$

## Solution:

Notice $f(n) = n + 1$ solves the recurrence on its own, so write $f(n) = (n+1) + g(n)$. Then $g$ satisfies the homogeneous version $g(n) = g(n-1) - g(n-2)$. With $g(1) = -1$ and $g(2) = -2$, it cycles with period $6$: $-1, -2, -1, 1, 2, 1, \ldots$. Since $2018 \equiv 2 \pmod 6$, we get $g(2018) = -2$, so $f(2018) = 2019 - 2 = 2017$. Therefore, the answer is **B**.

**21.** Mary chose an even $4$-digit number $n$. She wrote down all the divisors of $n$ in increasing order from left to right: $1, 2, \ldots, \frac{n}{2}, n$. At some moment Mary wrote $323$ as a divisor of $n$. What is the smallest possible value of the next divisor written to the right of $323$?

A $324$

B $330$

C $340$

D $361$

E $646$

## Solution:

Factor $323 = 17 \cdot 19$. Since it divides the even number $n$, $n$ is a multiple of $2 \cdot 17 \cdot 19 = 646$. For the next divisor $d$, $n$ has to be a multiple of $\operatorname{lcm}(323, d)$. Something like $324$ or $330 = 2 \cdot 3 \cdot 5 \cdot 11$ shares no factor with $323$, which pushes $n \geq 323 \cdot 324 > 9999$, too big. But $d = 340 = 2^2 \cdot 5 \cdot 17$ gives $\operatorname{lcm}(323, 340) = 2^2 \cdot 5 \cdot 17 \cdot 19 = 6460$, an even $4$-digit number whose divisor list jumps straight from $323$ to $340$. So the smallest possible next divisor is $340$. Thus, **C** is the correct answer.

**22.** Real numbers $x$ and $y$ are chosen independently and uniformly at random from the interval $[0, 1]$. Which of the following numbers is closest to the probability that $x, y,$ and $1$ are the side lengths of an obtuse triangle?

- A $0.21$
- B $0.25$
- C $0.29$
- D $0.50$
- E $0.79$

## Solution:

The three lengths $x, y, 1$ make a triangle iff $x + y > 1$. Since $1$ is the longest side, that triangle is obtuse iff $x^2 + y^2 < 1$. So in the unit square we want the region inside the quarter circle $x^2 + y^2 = 1$ but above the line $x + y = 1$. That's the quarter disk with the right triangle under the chord removed: $\frac{\pi}{4} - \frac{1}{2} \approx 0.285$. The closest choice is $0.29$. Therefore, the answer is **C**.

**23.** How many ordered pairs $(a, b)$ of positive integers satisfy the equation

$$a \cdot b + 63 = 20 \cdot \operatorname{lcm}(a, b) + 12 \cdot \gcd(a, b),$$

where $\gcd(a, b)$ denotes the greatest common divisor of $a$ and $b$, and $\operatorname{lcm}(a, b)$ denotes their least common multiple?

- A $0$
- B $2$
- C $4$
- D $6$
- E $8$

## Solution:

Recall $ab = \gcd(a, b) \cdot \operatorname{lcm}(a, b)$. Let $x = \operatorname{lcm}(a, b)$ and $y = \gcd(a, b)$. The equation turns into $xy + 63 = 20x + 12y$, which factors as $(x - 12)(y - 20) = 177 = 3 \cdot 59$. The positive factorizations give $(x, y) = (13, 197), (189, 21), (15, 79), (71, 23)$. But we also need $y \mid x$, and only $(189, 21)$ passes. So $\gcd = 21, \operatorname{lcm} = 189$, and $\{a, b\} = \{21, 189\}$. That's the $2$ ordered pairs $(21, 189)$ and $(189, 21)$. Thus, **B** is the correct answer.

**24.** Let $ABCDEF$ be a regular hexagon with side length $1$. Denote by $X$, $Y$, and $Z$ the midpoints of sides $AB, CD,$ and $EF,$ respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

A $\frac{3}{8}\sqrt{3}$

B $\frac{7}{16}\sqrt{3}$

C $\frac{15}{32}\sqrt{3}$

D $\frac{1}{2}\sqrt{3}$

E $\frac{9}{16}\sqrt{3}$

## Solution:

Center the hexagon at the origin. Then $\triangle ACE$ is equilateral with side $AC = \sqrt{3}$, so its area is $\frac{\sqrt{3}}{4}(\sqrt{3})^2 = \frac{3\sqrt{3}}{4}$. And $\triangle XYZ$ is equilateral with side $\frac{3}{2}$, area $\frac{\sqrt{3}}{4}\left(\frac{3}{2}\right)^2 = \frac{9\sqrt{3}}{16}$. The two are concentric and rotated $30^\circ$ apart, so their intersection is $\triangle XYZ$ with three congruent corners (each of area $\frac{\sqrt{3}}{32}$) cut off: $\frac{9\sqrt{3}}{16} - 3 \cdot \frac{\sqrt{3}}{32} = \frac{15}{32}\sqrt{3}.$ Therefore, the answer is **C**.

**25.** Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to $x$. How many real numbers $x$ satisfy the equation $x^2 + 10{,}000\lfloor x \rfloor = 10{,}000x$?

A $197$

B $198$

C $199$

D $200$

E $201$

## Solution:

Let $a = \lfloor x \rfloor$. The equation reads $x^2 = 10{,}000(x - a) = 10{,}000\{x\}$, and since $0 \leq \{x\} < 1$, this forces $0 \leq x^2 < 10{,}000$, so $-100 < x < 100$. On each interval $[a, a+1)$ the quantity $10{,}000x - x^2$ climbs across $[10{,}000a - a^2,\ 10{,}000(a+1) - (a+1)^2)$, and it hits $10{,}000a$ exactly once precisely when $(a+1)^2 < 10{,}000$. That holds for the integers $-100 \leq a \leq 98$, which is $199$ solutions. Thus, **C** is the correct answer.

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