2018 AMC 10A Solutions

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https://live.poshenloh.com/past-contests/amc10/2018A/solutions

1. What is the value of

$$\left(\left((2+1)^{-1}+1\right)^{-1}+1\right)^{-1}+1?$$

A $\frac{5}{8}$

B $\frac{11}{7}$

C $\frac{8}{5}$

D $\frac{18}{11}$

E $\frac{15}{8}$

Solution(s):

We can simplify this as follows.

$$\begin{aligned}
&\left(\left((2+1)^{-1}+1\right)^{-1}+1\right)^{-1}+1\\
&=\left(\left(\frac{1}{3}+1\right)^{-1}+1\right)^{-1}+1\\
&=\left(\frac{3}{4}+1\right)^{-1}+1\\
&=\frac{4}{7}+1\\
&=\frac{11}{7}
\end{aligned}$$

Thus, **B** is the correct answer.

2. Liliane has 50% more soda than Jacqueline, and Alice has 25% more soda than Jacqueline. What is the relationship between the amounts of soda that Liliane and Alice have?

- A Liliane has 20% more soda than Alice.
- B Liliane has 25% more soda than Alice.
- C Liliane has 45% more soda than Alice.
- D Liliane has 75% more soda than Alice.
- E Liliane has 100% more soda than Alice.

Solution(s):

Let x be the number of gallons of soda that Jacqueline has. Then Alice has $1.25x$ gallons, and Liliane has $1.5x$ gallons.

Therefore, the relationship can be found by dividing the amount of soda that each has to yield $\frac{1.5x}{1.25x} = 1.2$, which mean Liliane has 20% more soda.

Thus, **A** is the correct answer.

3. A unit of blood expires after

$$10! = 10 \cdot 9 \cdot 8 \cdots 1$$

seconds. Yasin donates a unit of blood at noon of January 1. On what day does his unit of blood expire?

- A January 2
- B January 12
- C January 22
- D February 11
- E February 12

Solution(s):

We can divide $10!$ by 60, 60, and 24 to get the number of days that it takes for a unit of blood to expire.

The first division cancels a 6 and 10. The second division cancels $3, 4$, and 5. The final division cancels 8 and turns the 9 into a 3.

This leaves $2, 7$, and a 3, which multiply to 42. There are 31 days in January, so by February 1, the blood only has $42 - 31 = 11$ days left.

11 days from February 1 would make the blood expire on February 12.

Thus, **E** is the correct answer.

4. How many ways can a student schedule 3 mathematics courses — algebra, geometry, and number theory — in a 6-period day if no two mathematics courses can be taken in consecutive periods?

(What courses the student takes during the other 3 periods is of no concern here.)

- A 3
- B 6
- C 12
- D 18
- E 24

Solution(s):

The 3 classes can occupy the following periods:

$$(1,3,5),$$

$$(1,3,6),$$

$$(1,4,6),$$

$$(2,4,6).$$

This means that there are 4 ways to choose which periods the mathematics courses occur.

For each configuration, there are $3!$ ways to determine the order of the courses, for a total of $6 \cdot 4 = 24$ schedules.

Thus, **E** is the correct answer.

5. Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, "We are at least 6 miles away," Bob replied, "We are at most 5 miles away." Charlie then remarked, "Actually the nearest town is at most 4 miles away."

It turned out that none of the three statements were true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d?

A $(0, 4)$

B $(4, 5)$

C $(4, 6)$

D $(5, 6)$

E $(5, \infty)$

Solution(s):

Alice's statement tells us that $d < 6$. Bob's statement tells us that $d > 5$. Charlie's statement tells us that $d > 4$.

Combining all of these tells us that $5 < d$ and $d < 6$, which means d is in the interval $(5, 6)$.

Thus, **D** is the correct answer.

6. Sangho uploaded a video to a website where viewers can vote that they like or dislike a video. Each video begins with a score of 0, and the score increases by 1 for each like vote and decreases by 1 for each dislike vote.

At one point Sangho saw that his video had a score of 90, and that 65% of the votes cast on his video were like votes. How many votes had been cast on Sangho's video at that point?

A 200

B 300

C 400

D 500

E 600

Solution(s):

If 65% of votes were like votes, then 35% of votes are dislike votes. Then Sangho's score is $65\% - 35\% = 30\%$ the total number of votes.

We know that Sangho's score is 90, so the total number of votes is $90 \div 30\% = 300$.

Thus, **B** is the correct answer.

7. For how many (not necessarily positive) integer values of n is the following value an integer?

$$4000 \cdot \left(\frac{2}{5}\right)^n$$

A 3

B 4

C 6

D 8

E 9

Solution(s):

We can rewrite the expression as

$$(2^5 \cdot 5^3) \cdot \left(\frac{2}{5}\right)^n = 2^{5+n} \cdot 5^{3-n}.$$

For this to be an integer, the exponents must be positive. This means that

$$5 + n \geq 0 \Rightarrow n \geq -5$$
$$3 - n \geq 0 \Rightarrow n \leq 3.$$

This gives us $5 + 3 + 1 = 9$ values for n.

Thus, **E** is the correct answer.

8. Joe has a collection of 23 coins, consisting of 5-cent coins, 10-cent coins, and 25-cent coins. He has 3 more 10-cent coins than 5-cent coins, and the total value of his collection is 320 cents. How many more 25-cent coins does Joe have than 5-cent coins?

- A 0
- B 1
- C 2
- D 3
- E 4

Solution(s):

Let x be the number of 5-cent coins that Joe has. Then the number of 10-cent coins he has is $x+3$.

Therefore, Joe has

$$23-x-(x+3)=20-2x$$

25-cent coins.

The total value of all these coins is

$$\begin{aligned}&5x+10(x+3)+25(20-2x)\\&=530-35x.\end{aligned}$$

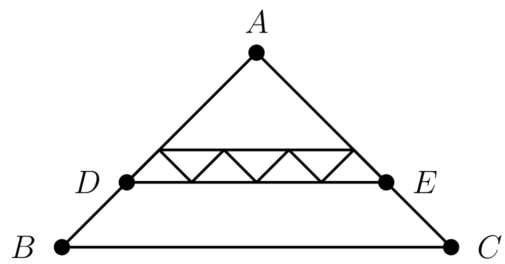
We know that

$$530-35x=320\Rightarrow x=6.$$

This means that Joe has $20-2\cdot 6=8$ 25-cent coins. Therefore, he has $8-6=2$ more 25-cent coins than 5-cent coins.

Thus, **C** is the correct answer.

9. All of the triangles in the diagram below are similar to isosceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?



- A 16
- B 18
- C 20
- D 22
- E 24

Solution(s):

We know that the side length of the smaller triangles is $\sqrt{\frac{1}{40}}$ times the length of the larger triangle from similar triangles.

Then the side length of $\triangle ADE$ is $4\sqrt{\frac{1}{40}}$ times the length of the side length of the larger triangle.

This makes the ratio of the areas

$$\left(4\sqrt{\frac{1}{40}}\right)^2 = 16 \cdot \frac{1}{40} = \frac{2}{5}.$$

Therefore, the area of $\triangle ADE$ is $\frac{2}{5} \cdot 40 = 16$. The area of the trapezoid is then $40 - 16 = 24$.

Thus, **E** is the correct answer.

10. Suppose that real number x satisfies

$$\sqrt{49-x^2}-\sqrt{25-x^2}=3.$$

What is the value of

$$\sqrt{49-x^2}+\sqrt{25-x^2}?$$

- A 8
- B $\sqrt{33}+8$
- C 9
- D $2\sqrt{10}+4$
- E 12

Solution(s):

Note that the left hand side of the equation and the desired expression are conjugates. Multiplying them would remove the square roots.

Multiplying them yields

$$49-x^2-25+x^2=24.$$

This means that the product of the values of the expressions is equal to 24. The desired value is therefore $24 \div 3 = 8$.

Thus, **A** is the correct answer.

11. When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

- A 42
- B 49
- C 56
- D 63
- E 84

Solution(s):

We can use stars and bars to find n. It is the same as finding the number of ways to put 10 balls into 7 boxes, where each box has at least one ball.

The formula for such a scenario is

$$\binom{n-1}{k-1},$$

where n is the number of balls and k is the number of boxes.

The desired answer is therefore

$$\binom{9}{6} = \binom{9}{3} = 84.$$

Thus, **E** is the correct answer.

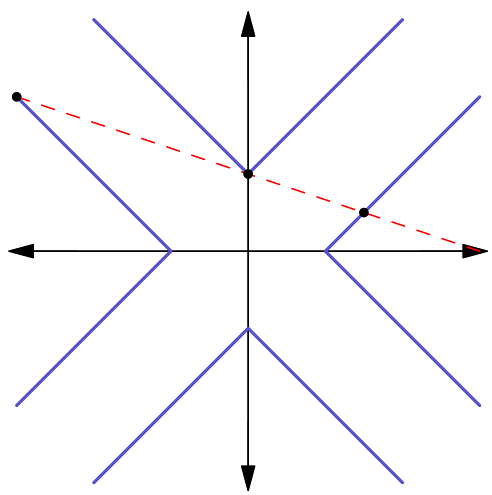
12. How many ordered pairs of real numbers (x, y) satisfy the following system of equations?

$$\begin{cases} x + 3y & = 3 \\ \big||x| - |y|\big| & = 1 \end{cases}$$

- A 1
- B 2
- C 3
- D 4
- E 8

Solution(s):

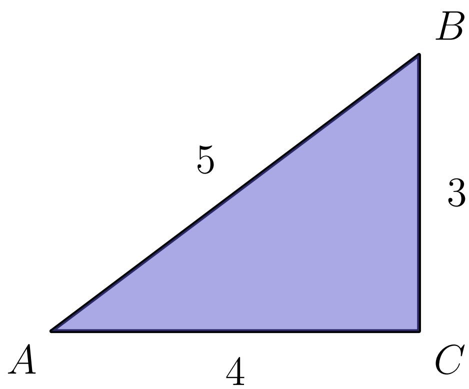
We can graph these equations to easily find out where the intersection points are.



From the graph, we see that the line intersects the other graph three times.

Thus, **C** is the correct answer.

13. A paper triangle with sides of lengths $3, 4,$ and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



A $1+\frac{1}{2}\sqrt{2}$

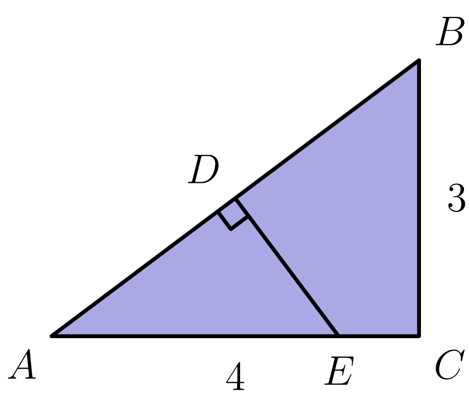
B $\sqrt{3}$

C $\frac{7}{4}$

D $\frac{15}{8}$

E 2

Solution(s):



Note that the crease will the perpendicular bisector of $\overline{AB}$. Let $\overline{DE}$ be the crease.

By AA similarity, we know that $\triangle ADE \sim \triangle ACB$. Therefore,

$$\frac{BC}{AC} = \frac{DE}{AD}$$

Plugging in values:

$$\frac{3}{4} = \frac{DE}{\frac{5}{2}}.$$

Simplifying gets us that

$$DE = \frac{15}{8}.$$

Thus, **D** is the correct answer.

14. What is the greatest integer less than or equal to

$$\frac{3^{100}+2^{100}}{3^{96}+2^{96}}?$$

- A 80
- B 81
- C 96
- D 97
- E 625

Solution(s):

Let $a = 3^{96}$ and $b = 2^{96}$. Then the expression can be rewritten as

$$\frac{81a+16b}{a+b} = \frac{16a+16b}{a+b} + \frac{65a}{a+b}$$

$$= 16 + \frac{65a}{a+b}.$$

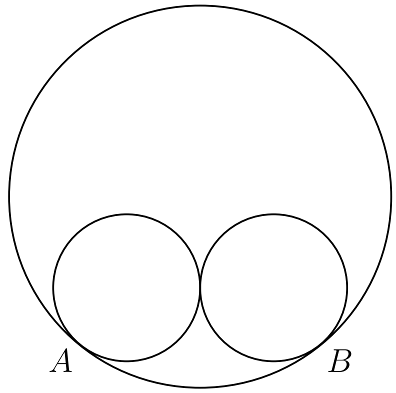
Note that

$$\frac{65a}{a+b} < \frac{65a}{a} = 65.$$

Therefore, the answer is less than $65 + 16 = 81$. 80 is the only choice that satisfies this condition.

Thus, **A** is the correct answer.

15. Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?



A 21

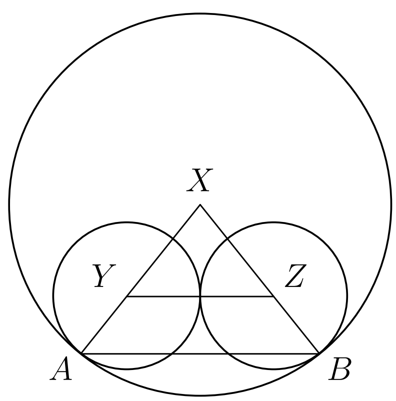
B 29

C 58

D 69

E 93

Solution(s):



Let Y and Z be the centers of the small circles and X be the center of the large circle.

Then

$$AX = BX = 13,$$

$$XY = XZ = 13 - 8 = 5,$$

and

$$YZ = 2 \cdot 5 = 10.$$

We also know that $\triangle XYZ \sim \triangle XAB$. Therefore,

$$\frac{XY}{YZ} = \frac{XA}{AB}$$

Plugging in values:

$$\frac{8}{13} = \frac{10}{AB}.$$

Multiplying yields $AB = \dfrac{65}{4}$.

$$m + n = 65 + 4 = 69$$

Thus, **D** is the correct answer.

16. Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

A 5

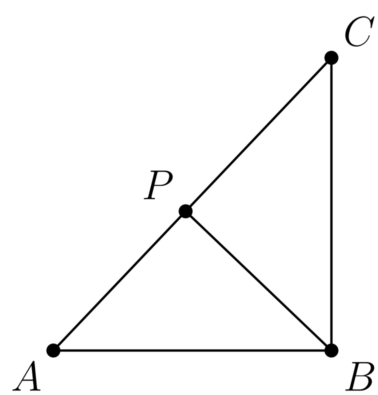
B 8

C 12

D 13

E 15

Solution(s):



Let P be the foot of the altitude from B to $\overline{AC}$. We also get that $AC = 29$.

This tells us that

$$\frac{29 \cdot PB}{2} = \frac{20 \cdot 21}{2}$$

$$PB = \frac{20 \cdot 21}{29},$$

by calculating the area in two ways. This value is between 14 and 15.

Note that as we move the line segment from $\overline{AB}$ to $\overline{PB}$, the line segment's length ranges from AB to PB.

The integer values it covers therefore goes from 20 to 15. Similarly, as the line segment moves from $\overline{PB}$ to $\overline{CB}$, its takes on the values from 15 to 21.

This gives us 13 unique line segments that have an integer value length.

Thus, **D** is the correct answer.

17. Let S be a set of 6 integers taken from $\{1, 2, \ldots, 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a. What is the least possible value of an element in S?

- A 2
- B 3
- C 4
- D 5
- E 7

Solution(s):

We proceed by casing on possible values for S :

1 cannot be the smallest element since that would mean that no other number can be in the set.

2 cannot be the smallest element since we would have to include every odd number except 1. This would make 3 and 9 violate the rule.

Let 3 be the smallest element. Then we can include 7 and 11. We can finally include either 4 or 8 and 5 or 10.

Either way, the maximum number of elements that we can include is 5, so 3 cannot be the smallest element.

Starting with 4, we can include $6, 7, 9$ and 11. Finally, we can add either 5 or 10, creating a 6-element set.

Thus, **C** is the correct answer.

18. How many nonnegative integers can be written in the form

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5$$
$$+a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2$$
$$+a_1 \cdot 3^1 + a_0 \cdot 3^0,$$

where $a_i \in \{-1, 0, 1\}$ for $0 \leq i \leq 7$?

A 512

B 729

C 1094

D 3281

E $59,048$

Solution(s):

Note that every number formed by this sum is either positive, negative, or zero.

The number of positive numbers equals the number of negative numbers due to symmetry (flip the 1 s to -1 s and -1 s to 1 s).

The only way for the sum to be 0 is if all the coefficients are 0.

The total number of numbers is $3^8 = 6561$. Because each power of 3 is larger than the sum of all previous powers of three, each combination of coefficients yields different numbers.

Therefore, there are

$$\frac{6561 - 1}{2} + 1 = 3281$$

distinct nonnegative integers.

Thus, **D** is the correct answer.

19. A number m is randomly selected from the set

$$\{11, 13, 15, 17, 19\},$$

and a number n is randomly selected from

$$\{1999, 2000, 2001, \ldots, 2018\}.$$

What is the probability that m^n has a units digit of 1?

A $\frac{1}{5}$

B $\frac{1}{4}$

C $\frac{3}{10}$

D $\frac{7}{20}$

E $\frac{2}{5}$

Solution(s):

Since we only care about the units digit, we can turn the set

$$\{11, 13, 15, 17, 19\}$$

into

$$\{1, 3, 5, 7, 9\}.$$

Then we can case on the value of m.

$m = 1$

Any value of n works. This occurs with a $\frac{1}{5}$ probability.

$m = 3$

Looking at powers of 3, we see that this sequence of units digits repeats:

$$3, 9, 7, 1, \ldots$$

This means that n must be a multiple of 4. There are 5 such values. This means that n works $\frac{5}{20} = \frac{1}{4}$ of the time. The total probability is

$$\frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20}.$$

$m = 5$

Powers of 5 always end in 5, which means that this case will never work.

$m = 7$

The units digits repeat in this pattern:

$$7, 9, 3, 1, \ldots$$

This means that n must be a multiple of 4 to work. As when $m = 3$, this case works with a probability of $\frac{1}{20}$.

$m = 9$

The units digit alternates between 1 and 9. This means that n has to be even. This happens with a $\frac{1}{2}$ chance. The total probability is then

$$\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}.$$

The total probability is therefore

$$\frac{1}{5} + 2 \cdot \frac{1}{20} + \frac{1}{10} = \frac{2}{5}.$$

Thus, **E** is the correct answer.

20. A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares.

A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides.

What is the total number of possible symmetric scanning codes?

A 510

B 1022

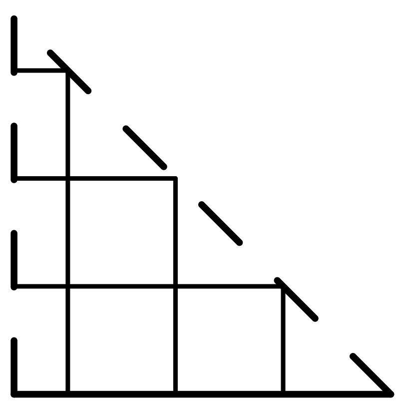
C 8190

D 8192

E $65,534$

Solution(s):

Note that all the line of symmetry split the grid into 8 congruent regions that like look the figure below.



If we analyze one of these pieces, we can see that what we color this region determines the coloring of the other 7 regions.

There are 10 sections in this figure, and we have 2 options for each of them, for a total of $2^{10} = 1024$ colorings.

We have to subtract 2 since all the colors cannot be the same ($1024 - 2 = 1022$).

Thus, **B** is the correct answer.

21. Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy-plane intersect at exactly 3 points?

A $a = \frac{1}{4}$

B $\frac{1}{4} < a < \frac{1}{2}$

C $a > \frac{1}{4}$

D $a = \frac{1}{2}$

E $a > \frac{1}{2}$

Solution(s):

We can substitute y from the second curve into the first curve.

$$x^2 + (x^2 - a)^2 = a^2$$
$$x^2 + x^4 - 2ax^2 = 0$$
$$x^2(x^2 - (2a - 1)) = 0$$

From this, we see that $x = 0$ always yields a solution (which is in fact a double root).

For the other two roots to exist and be unique, we need

$$x^2 - (2a - 1) = 0$$

to have a positive discriminant.

For this to happen,

$$4(2a - 1) > 0$$

or equivalently

$$a > \frac{1}{2}.$$

Thus, **E** is the correct answer.

22. Let $a, b, c,$ and d be positive integers such that

$$\gcd(a, b) = 24,$$

$$\gcd(b, c) = 36,$$

$$\gcd(c, d) = 54,$$

and

$$70 < \gcd(d, a < 100.$$

Which of the following must be a divisor of a?

A 5

B 7

C 11

D 13

E 17

Solution(s):

The problem statement tells us that 24 divides a and b, 36 divides b and c, and 54 divides c and d.

Note that these are the following prime factorizations:

$$24 = 2^3 \cdot 3$$
$$36 = 2^2 \cdot 3^2$$
$$54 = 2 \cdot 3^3.$$

This means that we can express $a, b,$ and c as follows:

$$a = 2^3 \cdot 3 \cdot w$$
$$b = 2^3 \cdot 3^2 \cdot x$$
$$c = 2^2 \cdot 3^2 \cdot y$$
$$d = 2 \cdot 3^3 \cdot z.$$

Then we get that

$$\gcd(a, d) = 2 \cdot 3 \cdot \gcd(w, z).$$

Note that w cannot have a factor of 3 since that would mean $\gcd(a, b)$ has an extra factor of 3.

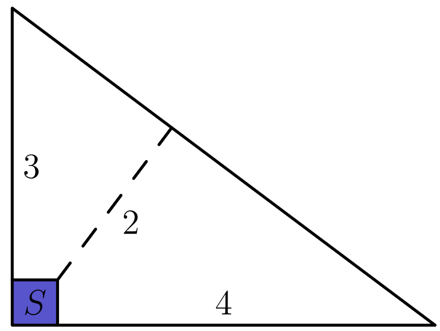
Similarly, we can see that z does not have a factor of 2.

This means that $\gcd(w, z)$ only has prime factors that are greater than or equal to 5.

The only number of the form $6p$, where p satisfies the above criteria, between 70 and 100 is $78 = 6 \cdot 13$.

Thus, **D** is the correct answer.

23. Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



A $\frac{25}{27}$

B $\frac{26}{27}$

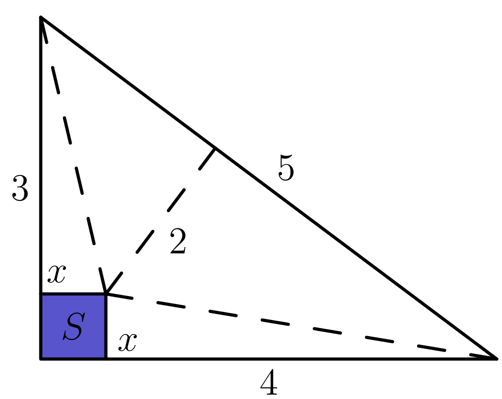
C $\frac{73}{75}$

D $\frac{145}{147}$

E $\frac{74}{75}$

Solution(s):

Let x be the side length of S. Then we can split the field up into the following shapes.



We can express the area of the field in two ways:

$$\frac{3 \cdot 4}{2} = x^2 + \frac{x(3-x)}{2}$$

$$+ \frac{x(4-x)}{2} + \frac{2 \cdot 5}{2}.$$

Simplifying yields

$$6 = \frac{7x}{2} + 5$$

$$x = \frac{2}{7}.$$

The desired fraction is

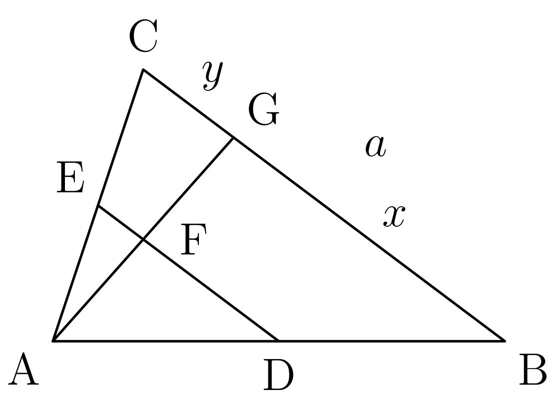
$$\frac{6 - x^2}{6} = \frac{6 - \frac{4}{49}}{6} = \frac{145}{147}.$$

Thus, **D** is the correct answer.

24. Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

- A 60
- B 65
- C 70
- D 75
- E 80

Solution(s):



Let $BC = a, BG = x, GC = y$, and h be the length of the altitude through A.

By the angle bisector theorem, we get that

$$\frac{50}{x} = \frac{10}{y},$$

where $y = a - x$. Substituting yields $BG = \frac{5a}{6}$. We also know that $DF = \frac{5a}{12}$ due to similar triangles.

Note that the height of the trapezoid is $\frac{1}{2}h$, and $\frac{ah}{2} = 120$. The area of the trapezoid is

$$\frac{5a}{8} \cdot \frac{h}{2} = \frac{5}{8} \cdot \frac{ah}{2} = 75.$$

Thus, **D** is the correct answer.

25. For a positive integer n and nonzero digits a, b, and c, let A_n be the n-digit integer each of whose digits is equal to a; let B_n be the n-digit integer each of whose digits is equal to b; and let C_n be the $2n$-digit (not n-digit) integer each of whose digits is equal to c. What is the greatest possible value of $a+b+c$ for which there are at least two values of n such that

$$C_n - B_n = A_n^2?$$

A 12

B 14

C 16

D 18

E 20

Solution(s):

We can use the formula for the sum of a geometric sequence to rewrite A_n, B_n, and C_n.

$$\begin{aligned} A_n &= a(11\cdots 11) \\ &= a(1+10+10^2+\cdots+10^{n-1}) \\ &= a\cdot\frac{10^n-1}{9} \end{aligned}$$

Similarly, we get that

$$B_n = b\cdot\frac{10^n-1}{9}$$

and

$$C_n = c\cdot\frac{10^{2n}-1}{9}.$$

We can substitute these expressions into our condition to get

$$c\cdot\frac{10^{2n}-1}{9} - b\cdot\frac{10^n-1}{9}$$

$$= a^2\left(\frac{10^n - 1}{9}\right)^2.$$

Simplifying yields

$$\begin{aligned} c(10^n + 1) - b &= a^2 \cdot \frac{10^n - 1}{9} \\ 9c(10^n + 1) - 9b &= a^2 \cdot (10^n - 1) \\ (9c - a^2)10^n &= 9b - 9c - a^2. \end{aligned}$$

From the last line, we see that $9c - a^2$ and $9b - 9c - a^2$ are constants.

For there to be at least 2 unique values of n that satisfy the equation, both sides must equal zero.

We can see this by realizing that this equation is linear with respect to 10^n. If both sides are non-zero, then there cannot exist 2 unique solutions to a linear equation.

This tell us that

$$9c - a^2 = 0$$

and

$$9b - 9c - a^2 = 0.$$

The first equation gives us

$$c = \frac{a^2}{9}.$$

Plugging this into the second equation gives us

$$b = \frac{2a^2}{9}.$$

This tells us that a must be divisible by 3. This gives us the following triples:

$$(3, 2, 1), (6, 8, 4), (9, 18, 9).$$

The last triple is not allowed, so the maximum sum is

$$6 + 8 + 4 = 18.$$

Thus, **D** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2018A