2017 AMC 10A Solutions

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https://live.poshenloh.com/past-contests/amc10/2017A/solutions

1. What is the value of

$$(2(2(2(2(2(2+1)+1)$$
$$+1)+1)+1)+1)?$$

A 70

B 97

C 127

D 159

E 729

Solution(s):

Simplifying yields

$$\begin{aligned}
&(2(2(2(2(2(2+1)+1)\\
&\quad+1)+1)+1)+1)\\
=&(2(2(2(2(2(3)+1)\\
&\quad+1)+1)+1)+1)\\
=&(2(2(2(2(7)+1)+1)+1)+1)\\
=&(2(2(2(15)+1)+1)+1)\\
=&(2(2(31)+1)+1)\\
=&2(63)+1\\
=&127.
\end{aligned}$$

Thus, **C** is the correct answer.

2. Pablo buys popsicles for his friends. The store sells single popsicles for $\$1$ each, 3-popsicle boxes for $\$2$ each, and 5-popsicle boxes for $\$3$. What is the greatest number of popsicles that Pablo can buy with $\$8$?

A 8

B 11

C 12

D 13

E 15

Solution(s):

The $3 boxes give us the most popsicles per dollar, so we want to buy as many of those as possible.

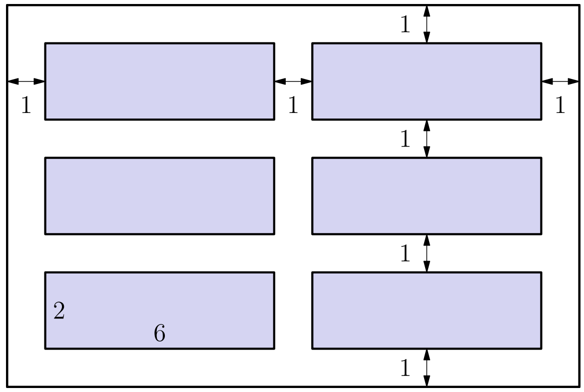
We can buy two of those, getting $5 \cdot 2 = 10$ popsicles with $8 - $6 = $2 remaining.

The $1 single popsicles are the worst deal, so Pablo should spend the rest of his money on the 3-popsicle box.

He then ends up with $10 + 3 = 13$ popsicles.

Thus, **D** is the correct answer.

3. Tamara has three rows of two 6-feet by 2-feet flower beds in her garden. The beds are separated and also surrounded by 1-foot-wide walkways, as shown on the diagram. What is the total area of the walkways, in square feet?



A 72

B 78

C 90

D 120

E 150

Solution(s):

We can see that the width of the garden is

$$2 \cdot 6 + 3 \cdot 1 = 15.$$

We can also see that the height is

$$3 \cdot 2 + 4 \cdot 1 = 10.$$

The total area of the garden is therefore $15 \cdot 10 = 150$. The area of all the flower beds is

$$6 \cdot 2 \cdot 6 = 72.$$

Subtracting this from the area of the garden yields $150 - 72 = 78$, which is the area of the walkways.

Thus, **B** is the correct answer.

4. Mia is "helping" her mom pick up 30 toys that are strewn on the floor. Mia's mom manages to put 3 toys into the toy box every 30 seconds, but each time immediately after those 30 seconds have elapsed, Mia takes 2 toys out of the box. How much time, in minutes, will it take Mia and her mom to put all 30 toys into the box for the first time?

A 13.5

B 14

C 14.5

D 15

E 15.5

Solution(s):

Note that after 30 seconds, there are 3 toys added and 2 removed, leaving a net total of $+1$ toys in the box.

We have to be careful towards the end, however, since it is possible for the box to have 30 toys right after Mia's mom adds the toys and before Mia removes them.

After there are 27 toys in the box, Mia's mom can add $3,$ leaving 30 toys in the box.

It will take $27 \cdot 30$ seconds, plus another 30 seconds, which gives us

$$28 \cdot 30 \div 60 = 14$$

minutes to get 30 toys in the box.

Thus, **B** is the correct answer.

5. The sum of two nonzero real numbers is 4 times their product. What is the sum of the reciprocals of the two numbers?

A 1

B 2

C 4

D 8

E 12

Solution(s):

Let x and y be the two numbers. We are given that $x + y = 4xy$.

Note that

$$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = 4.$$

Thus, **C** is the correct answer.

6. Ms. Carroll promised that anyone who got all the multiple choice questions right on the upcoming exam would receive an A on the exam. Which one of these statements necessarily follows logically?

A. If Lewis did not receive an A, then he got all of the multiple choice questions wrong.

B. If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong.

C. If Lewis got at least one of the multiple choice questions wrong, then he did not receive an A.

D. If Lewis received an A, then he got all of the multiple choice questions right.

E. If Lewis received an A, then he got at least one of the multiple choice questions right.

Solution(s):

There is no stipulation on how to get an A other than that getting all the multiple choice right guarantees an A.

This means that it is possible to get an A without getting all the multiple choice questions right.

It is also possible to not get an A even if all but one of the multiple choice questions are answered correctly.

This rules out **A**, **C**, **D**, and **E**.

Thus, **B** is the correct answer.

7. Jerry and Silvia wanted to go from the southwest corner of a square field to the northeast corner. Jerry walked due east and then due north to reach the goal, but Silvia headed northeast and reached the goal walking in a straight line. Which of the following is closest to how much shorter Silvia's trip was, compared to Jerry's trip?

A 30%

B 40%

C 50%

D 60%

E 70%

Solution(s):

Let s be the side length of the field. Then Jerry traveled $2s$ and Silvia traveled $s\sqrt{2}$ from the Pythagorean theorem.

The desired value is

$$\begin{aligned}\frac{2s - s\sqrt{2}}{2s} &= \frac{2-\sqrt{2}}{2} \\ &\approx \frac{2-1.4}{2} \\ &= .3 \\ &= 30\%.\end{aligned}$$

Thus, **A** is the correct answer.

8. At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur within the group?

A 240

B 245

C 290

D 480

E 490

Solution(s):

Each of the 10 people shake hands with each of the 20 people. This results in $10 \cdot 20 = 200$ handshakes.

There are also $\binom{10}{2} = 45$ handshakes within the 10 people (every pair of people shake hands).

Therefore, the total number of handshakes is $200 + 45 = 245.$

Thus, **B** is the correct answer.

9. Minnie rides on a flat road at 20 kilometers per hour (kph), downhill at 30 kph, and uphill at 5 kph. Penny rides on a flat road at 30 kph, downhill at 40 kph, and uphill at 10 kph. Minnie goes from town A to town B, a distance of 10 km all uphill, then from town B to town C, a distance of 15 km all downhill, and then back to town A, a distance of 20 km on the flat. Penny goes the other way around using the same route. How many more minutes does it take Minnie to complete the 45-km ride than it takes Penny?

A 45

B 60

C 65

D 90

E 95

Solution(s):

It will take Minnie $10 \div 5 = 2$ hours to travel the uphill distance. It will take her $15 \div 30 = \frac{1}{2}$ hours to travel the downhill distance.

Finally, it will take her $20 \div 20 = 1$ hour to travel the flat. This will take her a total of

$$\begin{aligned} 60\left(2 + \frac{1}{2} + 1\right) &= 60 \cdot \frac{7}{2} \\ &= 210 \end{aligned}$$

minutes.

It will take Penny $20 \div 30 = \frac{2}{3}$ hours to travel the flat. It will take her another $15 \div 10 = \frac{3}{2}$ hours to travel the uphill.

Finally, it will take her $10 \div 40 = \frac{1}{4}$ hours to travel the downhill. This is a total of

$$\begin{aligned} 60\left(\frac{2}{3} + \frac{3}{2} + \frac{1}{4}\right) &= 60 \cdot \frac{29}{12} \\ &= 145 \end{aligned}$$

minutes. The trip takes Minnie $210 - 145 = 65$ more minutes to travel than Penny.

Thus, **C** is the correct answer.

10. Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?

A 16

B 17

C 18

D 19

E 20

Solution(s):

Note that no one side can be greater than or equal to the sum of the other side lengths.

Let x be the length fourth rod. Then we have that

$$x < 3 + 7 + 15$$

and

$$x + 3 + 7 > 15$$

Simplifying, we know that

$$5 < x < 25.$$

Counting the number of integers in this range, we are left with $25 - 5 - 1 = 19$ values for x.

The rods with length 7 and 15 are already being used, however, so x cannot equal these.

This leaves $19 - 2 = 17$ viable solutions for x.

Thus, **B** is the correct solution.

11. The region consisting of all points in three-dimensional space within 3 units of line segment $\overline{AB}$ has volume 216π. What is the length AB?

A 6

B 12

C 18

D 20

E 24

Solution(s):

Recall that all the points at most a fixed distance r away from a point form a sphere.

At the end points of this line segment, we can visualize two hemispheres being formed at each end.

All the points in the middle also have spheres forming around them, but they get merged into the ones right next to them.

This means that the middle section forms a cylinder with radius 3. The two hemispheres form a sphere with radius 3, and therefore a volume of

$$\frac{4}{3}\pi 3^3 = 27 \cdot \frac{4}{3}\pi = 36\pi.$$

This means that the cylinder has a volume of $216\pi - 36\pi = 180\pi$. We know the base is 3π, so if h is AB, then the volume is

$$\begin{aligned} \pi 3^2 h &= 180\pi \\ 9h\pi &= 180\pi \\ h &= 20. \end{aligned}$$

Thus, **D** is the correct answer.

12. Let S be a set of points (x, y) in the coordinate plane such that two of the three quantities $3, x+2$, and $y-4$ are equal and the third of the three quantities is no greater than this common value. Which of the following is a correct description for S?

A a single point

B two intersecting lines

C three lines whose pairwise intersections are three distinct points

D a triangle

E three rays with a common endpoint

Solution(s):

Let us case on which of the values are equal. If $3 = x + 2$, then $x = 1$. This also tells us that

$$y - 4 \leq 3$$

$$y \leq 7.$$

This describes a ray starting at $(1, 7)$ and extending in the negative y direction.

Similarly, if $3 = y - 4$, then $y = 7$ and

$$x + 2 \leq 3$$

$$x \leq 1.$$

This also describes a ray starting at $(1, 7)$ but instead extending in the negative x direction.

Finally, if $x + 2 = y - 4$, then we have the line $y = x + 6$. Furthermore, we have that

$$3 \leq x + 2$$

$$x \geq 1$$

and

$$3 \leq y - 4$$

$$y \geq 7.$$

Note that if one if these conditions is met, the other is also necessarily true due to the equation of the line.

If $y = 7$, then $x = 1$. The other points are along the line, where $y > 7$ and $x > 1$.

This describes another ray that starts at $(1, 7)$ and goes off in some third direction.

All three cases result in rays originating from $(1, 7)$ that all go in different directions.

Thus, **E** is the correct answer.

13. Define a sequence recursively by $F_0 = 0,\ F_1 = 1$, and $F_n =$ the remainder when $F_{n-1} + F_{n-2}$ is divided by 3, for all $n \geq 2$. Thus the sequence starts $0, 1, 1, 2, 0, 2, \ldots$. What is

$$F_{2017} + F_{2018} + F_{2019} + F_{2020} +$$

$$F_{2021} + F_{2022} + F_{2023} + F_{2024}?$$

A 6

B 7

C 8

D 9

E 10

Solution(s):

Let us list out the first few values to see if we can find a pattern in this sequence.

$$0, 1, 1, 2, 0, 2, 2, 1, 0, 1, \cdots$$

From this we can see that the pattern repeats every 8 terms.

The desired answer is the sum of 8 consecutive numbers, which is fixed. This sum is

$$0 + 1 + 1 + 2 + 0 + 2$$

$$+2 + 1 = 9.$$

Thus, **D** is the correct answer.

14. Every week Roger pays for a movie ticket and a soda out of his allowance. Last week, Roger's allowance was A dollars. The cost of his movie ticket was 20% of the difference between A and the cost of his soda, while the cost of his soda was 5% of the difference between A and the cost of his movie ticket. To the nearest whole percent, what fraction of A did Roger pay for his movie ticket and soda?

A 9%

B 19%

C 22%

D 23%

E 25%

Solution(s):

Let t be the cost of the ticket and s be the cost of the soda. Then we get the following equations.

$$t = \frac{A - s}{5}$$

$$s = \frac{A - t}{20}$$

Cross-multiplying the first equation gives us $5t = A - s$. Substituting in the expression for s yields

$$5t = A - \frac{A - t}{20}.$$

Solving yields

$$\begin{aligned} 5t &= A - \frac{A - t}{20} \\ 100t &= 20A - A + t \\ 99t &= 19A \\ t &= \frac{19A}{99}. \end{aligned}$$

This also gives us

$$s = \frac{A - \frac{19A}{99}}{20} = \frac{4A}{99}.$$

Adding together the costs gives us

$$\frac{19A}{99} + \frac{4A}{99} = \frac{23A}{99} \approx 23\%.$$

Thus, **D** is the correct answer.

15. Chloe chooses a real number uniformly at random from the interval $[0, 2017]$.

Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$.

What is the probability that Laurent's number is greater than Chloe's number?

A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{3}{4}$

D $\frac{5}{6}$

E $\frac{7}{8}$

Solution(s):

If Laurent chooses a number in the interval $(2017, 4034]$, then there is no way that Chloe can have the greater number.

This means that Laurent has a $\frac{1}{2}$ chance of automatically winning.

Otherwise, Laurent chooses a number in the interval $[0, 2017]$. The probability that she gets a greater number than Chloe is the same as Chloe getting a greater number then Laurent.

This means that Laurent has a $\frac{1}{2}$ chance of getting a greater number (when working with real intervals, the probability of a tie is essentially 0 due to the infinite size of the intervals).

Laurent's total chance of getting a greater number is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

Thus, **C** is the correct answer.

16. There are 10 horses, named Horse 1, Horse $2, \ldots,$ Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse k runs one lap in exactly k minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds.

The least time $S > 0$, in minutes, at which all 10 horses will again simultaneously be at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that at least 5 of the horses are again at the starting point. What is the sum of the digits of T?

A 2

B 3

C 4

D 5

E 6

Solution(s):

The time it will take for 5 horses to meet again at the start is the least common multiple of their times.

We want to find the 5 numbers that share lots of prime factors and have small prime factors.

This is because to find the least common multiple, we choose the highest power of a prime that is present among all the numbers.

As such, we can choose Horses $1, 2, 3, 4,$ and 6. These have prime factors of 2 and 3, which the best we can do.

The least common multiple is 12. The sum of its digits is $1 + 2 = 3$.

Thus, **B** is the correct answer.

17. Distinct points P, Q, R, S lie on the circle $x^2 + y^2 = 25$ and have integer coordinates. The distances PQ and RS are irrational numbers.

What is the greatest possible value of the ratio $\dfrac{PQ}{RS}$?

A 3

B 5

C $3\sqrt{5}$

D 7

E $5\sqrt{2}$

Solution(s):

Note that the only integer coordinate pairs on this circle are

$$(\pm 3, \pm 4),$$

$$(\pm 4, \pm 3),$$

$$(\pm 5, 0),$$

and

$$(0, \pm 5).$$

To get the greatest possible ratio, we want to maximize PQ and minimize RS.

We can see that the distance between any of these points is irrational as long as it is not a diameter.

There are only 2 logical candidates for the longest distance: $(-4, 3)$ and $(3, -4)$ or $(-4, 3)$ and $(5, 0)$.

Using the distance formula gives us the two distances as $\sqrt{98}$ and $\sqrt{90}$. The first is greater.

There is only one viable choice for the shortest distance: $(3, 4)$ and $(4, 3)$, which give us a distance of $\sqrt{2}$.

The desired ratio is then

$$\frac{\sqrt{98}}{\sqrt{2}} = \sqrt{49} = 7.$$

Thus, **D** is the correct answer.

18. Amelia has a coin that lands heads with probability $\frac{1}{3}$, and Blaine has a coin that lands on heads with probability $\frac{2}{5}$. Amelia and Blaine alternately toss their coins until someone gets a head; the first one to get a head wins. All coin tosses are independent. Amelia goes first. The probability that Amelia wins is $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $q-p$?

A 1

B 2

C 3

D 4

E 5

Solution(s):

Let x be the probability that Amelia wins.

There is a $\frac{1}{3}$ chance Amelia wins off her first flip.

If she gets a tails, we want Blaine to lose, which happens with a $\frac{3}{5}$ chance.

The total probability of this case is

$$\frac{2}{3}\cdot\frac{3}{5}=\frac{2}{5}.$$

The game then goes back to Amelia, who then again has a x chance of winning.

Therefore, we get the following equation.

$$x=\frac{1}{3}+\frac{2}{5}x$$
$$\frac{3}{5}x=\frac{1}{3}$$
$$x=\frac{5}{9}.$$

The difference between the denominator and numerator is $9-5=4$.

Thus, **D** is the correct answer.

19. Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

A 12

B 16

C 28

D 32

E 40

Solution(s):

If Alice sets on an edge, then the person next to her cannot be Bob or Carla. This means that it must be Derek or Eric.

WLOG, let the person be Eric. Then the person next to Eric has to be Bob or Carla. After that there are no more restrictions.

This gives us a total of

$$2 \cdot 2 \cdot 2 \cdot 2 = 16.$$

The first 2 is for both edges. The second 2 is for Derek or Eric. The third 2 is for Bob or Carla. The final 2 is just for the 2 people that are remaining.

Otherwise, let Alice be in the middle. Then the two people next to her have to be Derek and Eric. Bob and Carla are forced to be in the last 2 seats.

There are 3 choices for Alice sets. The side on which Derek is sat has 2 options, and then there are 2 options for where Bob and Carla go.

This gives us

$$3 \cdot 2 \cdot 2 = 12$$

configurations.

Therefore, there are a total of $12 + 16 = 28$ total seating arrangements.

Thus, **C** is the correct answer.

20. Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$.

Which of the following could be the value of $S(n+1)$?

A 1

B 3

C 12

D 1239

E 1265

Solution(s):

Recall that a number is divisible by 9 if and only if the sum of its digits is also divisible by 9.

This means that looking at $S(n)$ mod 9 would also give us n mod 9.

Let us prove this. If we add x to n without carrying, it is clear that the sum of the digits increases by x and that n itself increases by x.

This would increase both their values mod 9 by x.

Now, if it does carry, we would be subtracting 10 from some digit and adding on 1 to the next digit.

This would keep the value mod 9 constant. We did, however, add x in there, so the value mod 9 still increased by x.

These are the only two cases, and in both we have shown that the value mod 9 for both n and $S(n)$ increased by x.

Therefore, we have that

$$n \equiv S(n) \equiv 5 \pmod{9}.$$

From this, we can see that

$$n + 1 \equiv 6 \equiv S(n+1) \pmod{9}.$$

The only answer choice that leaves a remainder of 6 when divided by 9 is 1239.

Thus, **D** is the correct answer.

21. A square with side length x is inscribed in a right triangle with sides of length $3, 4,$ and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length $3, 4,$ and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

A $\frac{12}{13}$

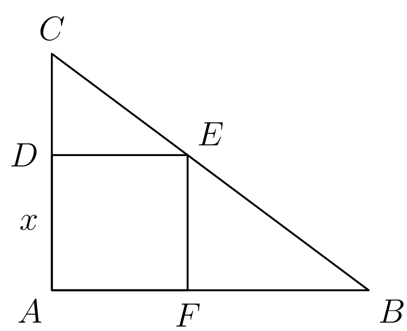
B $\frac{35}{37}$

C 1

D $\frac{37}{35}$

E $\frac{13}{12}$

Solution(s):



We can see that $\triangle ABC$ and $\triangle FBE$ are similar (angle-angle). This gives us

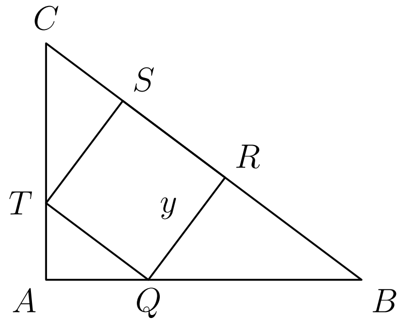
$$\frac{BF}{FE} = \frac{AB}{AC}$$

$$\frac{4-x}{x} = \frac{4}{3}.$$

Cross-multiplying yields

$$12 - 3x = 4x$$

$$x = \frac{12}{7}.$$



Here, we have that $\triangle ABC$, $\triangle RBQ$, and $\triangle STC$ are similar (angle-angle).

This means that $RB = \dfrac{4}{3}y$ and $CS = \dfrac{3}{4}y$. This gives us the equation

$$\frac{4}{3}y + \frac{3}{4}y + y = 5$$

$$\frac{37}{12}y = 5.$$

Finally, we get that $y = \dfrac{60}{37}$. The desired ratio is

$$\frac{\frac{12}{7}}{\frac{60}{37}} = \frac{37}{35}.$$

Thus, **D** is the correct answer.

22. Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle?

A $\dfrac{4\sqrt{3}\pi}{27} - \dfrac{1}{3}$

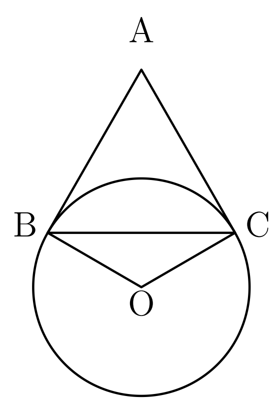
B $\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{8}$

C $\dfrac{1}{2}$

D $\sqrt{3} - \dfrac{2\sqrt{3}\pi}{9}$

E $\dfrac{4}{3} - \dfrac{4\sqrt{3}\pi}{27}$

Solution(s):



Let the radius of the circle be r.

To find the area of the triangle outside of the circle, we can find the area of the triangle inside the circle and subtract it.

We get that $\angle BOC = 120^\circ$ since $\angle ABO$ and $\angle ACO$ are right angles.

This means that the area of sector OBC is

$$\frac{120}{360} \cdot \pi r^2 = \frac{\pi r^2}{3}.$$

Now, we need to find the area of $\triangle BOC$. Using the formula for the area of a triangle with sine, we get the area to be

$$\frac{1}{2}\sin(120^\circ)\cdot r^2 = \frac{r^2\sqrt{3}}{4}.$$

Then the area of the triangle inside the circle is

$$\frac{\pi r^2}{3} - \frac{r^2\sqrt{3}}{4} = \frac{r^2(4\pi - 3\sqrt{3})}{12}.$$

The area of $\triangle ABC$ is

$$\frac{(r\sqrt{3})^2\sqrt{3}}{4} = \frac{3r^2\sqrt{3}}{4}.$$

The desired fraction is then

$$\begin{aligned} 1 - \frac{\frac{r^2(4\pi-3\sqrt{3})}{12}}{\frac{3r^2\sqrt{3}}{4}} &= 1 - \frac{4\pi - 3\sqrt{3}}{9\sqrt{3}} \\ &= 1 - \frac{4\pi\sqrt{3}}{27} + \frac{1}{3} \\ &= \frac{4}{3} - \frac{4\pi\sqrt{3}}{27}. \end{aligned}$$

Thus, **E** is the correct answer.

23. How many triangles with positive area have all their vertices at points (i, j) in the coordinate plane, where i and j are integers between 1 and 5, inclusive?

A 2128

B 2148

C 2160

D 2200

E 2300

Solution(s):

We can use complementary counting to find the total number of triangles and subtract out the ones that don't work.

There are a total of $5^2 = 25$ points, so there are $\binom{25}{3} = 2300$ possible triangles.

Note that the only way a triangle doesn't work is if all the 3 points are in a straight line.

There are 5 rows, 5 columns, and 2 long diagonals. Each of these 12 lines have 5 points, which means they contribute

$$12 \cdot \binom{5}{3} = 12 \cdot 10 = 120$$

degenerate triangles.

There are also the diagonal lines with 4 points, such as $(0, 1)$ to $(4, 5)$. There are 4 of these lines, so they have

$$4 \cdot \binom{4}{3} = 4 \cdot 4 = 16$$

degenerate triangles.

Similarly, they are 4 diagonal lines with 3 points. These give us $4 \cdot 1 = 4$ extra triangles that don't work.

Now, we have to look at the lines with slopes of $\frac{1}{2}, 2, -\frac{1}{2}$, and -2.

There are 3 such lines for each slope, and they all have 3 points on them. Therefore, they contribute

$$4 \cdot 3 \cdot 1 = 12$$

more triangles to discount.

The total number of working triangles is then

$$2300 - 120 - 16 - 4 - 12$$

$$= 2148.$$

Thus, **B** is the correct answer.

24. For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

- A -9009
- B -8008
- C -7007
- D -6006
- E -5005

Solution(s):

We know that $f(x)$ has 4 roots, 3 of which are the roots of $g(x)$. This means that we can express as $f(x)$ as

$$f(x) = g(x)(x - r),$$

for some complex number r that is the other root of $f(x)$.

Plugging in $g(x)$, we get $f(x)$ equals:

$$(x^3 + ax^2 + x + 10)(x - r)$$

$$= x^4 + (a - r)x^3 + (1 - ar)x^2$$

$$+(10 - r)x - 10r.$$

Comparing coefficients, we get

$$10 - r = 100$$

$$r = -90.$$

We also know that

$$a - r = 1$$

$$a = -89.$$

Finally, we have that $f(1)$ equals:

$$1^4 + (a - r)1^3 + (1 - ar)1^2$$

$$+(10 - r)1 - 10r$$

$$= 1 + (-89 + 90) + (1 - 89 \cdot 90)$$

$$+(10 + 90) + 10 \cdot 90$$

$$= 1 + 1 - 8009 + 100 + 900$$

$$= -7007.$$

Thus, **C** is the correct answer.

25. How many integers between 100 and 999, inclusive, have the property that some permutation of its digits is a multiple of 11 between 100 and $999?$ For example, both 121 and 211 have this property.

A 226

B 243

C 270

D 469

E 486

Solution(s):

We can analyze all the multiple so 11 and see how many permutations each of them contribute. We can do this by casing on the number of unique digits in the number.

Case 1 : all the digits are the same

This cannot happen. We can see this by the divisibility rule for 11, which says that the sum of the first and last digit minus the middle digit must be divisible by 11.

If all the digits are the sum, then the above expression evaluates to that digit, which cannot be divisible by 11.

Case 2 : two of the digits are the same

We can split this up into the numbers that have the digit 0 and those that don't.

There are 8 multiples of 11 that do not have the digit 0 :

$$121, 242, 363, 484, 616, 737, 858,$$

and 979.

Each of these numbers contributes 3 permutations, so this scenario has $8 \cdot 3 = 24$ numbers.

There are 9 multiples of 11 that have the digit 0 :

$$110, 220, 330, 440, 550, 660, 770,$$

880, and 990.

For these numbers, 0 cannot be the hundreds digit, so each of them only contributes 2 permutations, for a total of $9 \cdot 2 = 18$.

Case 3 : all the digits are different

There are a total of 81 multiples of 11 between 100 and 999. The number of these with all different digits is

$$81 - 8 - 9 = 64.$$

As in case 2, we have to specially account for the numbers with 0 as a digit. There are 8 :

$$209, 308, 407, 506, 605, 704, 803,$$

and 902.

Each of these gives us $2 \cdot 2 = 4$ permutations, but we overcount by a factor of 2 since flipping the first and last digits creates another number already in the set.

Therefore, these numbers provide a total of

$$8 \cdot 4 \div 2 = 16$$

unique permutations.

There are now $64 - 8 = 56$ multiples of 11 that we need to account for.

We know that each of these provides $3! = 6$ permutations. As above, however, note that flipping the first and last digit of any number in this set produces another number in this set.

We can see this by using the divisibility rule for 11. If ABC is divisible by 11, then we have that $A + C - B$ is divisible by 11.

This means that $C + A - B$ is divisible by 11, which means that CBA is also divisible by 11.

Therefore, these numbers contribute

$$56 \cdot 6 \div 2 = 168$$

more permutations.

Over all the cases, we have a total of

$$24 + 18 + 16 + 168 = 226$$

numbers.

Thus, **A** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2017A