2016 AMC 10B Solutions

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1. What is the value of

$$\frac{2a^{-1}+\frac{a^{-1}}{2}}{a}$$

when $a=\frac{1}{2}$?

A 1

B 2

C $\frac{5}{2}$

D 10

E 20

Solution(s):

The expression is equivalent to

$$2a^{-2}+\frac{a^{-2}}{2}=2.5(a^{-1})^2.$$

Then a^{-1} is equal to $\frac{1}{\frac{1}{2}}=2$, so our expression is equal to $2.5\cdot 2^2=10.$

Thus, the correct answer is **D**.

2. If $n \heartsuit m = n^3 m^2$, what is $\frac{2 \heartsuit 4}{4 \heartsuit 2}$?

A $\frac{1}{4}$

B $\frac{1}{2}$

C 1

D 2

E 4

Solution(s):

For any nonzero a, b, we have

$$\frac{a \heartsuit b}{b \heartsuit a} = \frac{a^3 b^2}{b^3 a^2} = \frac{a}{b}.$$

Using $a = 2, b = 4$, we get that $\frac{2}{4} = \frac{1}{2}$.

Thus, the correct answer is **B**.

3. Let $x = -2016$. What is the value of

$$\Bigg|\Big||x| - x\Big| - |x|\Bigg| - x?$$

A -2016

B 0

C 2016

D 4032

E 6048

Solution(s):

Observe that:

$$\begin{aligned}&\Bigg|\Big||x| - x\Big| - |x|\Bigg| - x \\ &= \Bigg|\Big| - x - x\Big| - |x|\Bigg| - x \\ &= \Bigg|\Big| - 2x\Big| - |x|\Bigg| - x\end{aligned}$$

since $x < 0$ implies that $-x = |x|$. Substituting values, we can see that

$$\begin{aligned}&|4032 - 2016| - (-2016) \\ &= 2016 + 2016 \\ &= 4032.\end{aligned}$$

Thus, the correct answer is **D**.

4. Zoey read 15 books, one at a time. The first book took her 1 day to read, the second book took her 2 days to read, the third book took her 3 days to read, and so on, with each book taking her 1 more day to read than the previous book. Zoey finished the first book on a Monday, and the second on a Wednesday. On what day the week did she finish her 15th book?

- A Sunday
- B Monday
- C Wednesday
- D Friday
- E Saturday

Solution(s):

The number of days it takes to read 15 books is $\frac{15 \cdot 16}{2} = 120$. Therefore, it is $120 - 1 = 119$ days after the first book is read. This is a multiple of 7, so the 15th book was finished on the same day as the 1st book. Therefore, it was finished on a Monday.

Thus, the correct answer is **B**.

5. The mean age of Amanda's 4 cousins is 8, and their median age is 5. What is the sum of the ages of Amanda's youngest and oldest cousins?

- A 13
- B 16
- C 19
- D 22
- E 25

Solution(s):

The sum of everyone's age is $4 \cdot 4 = 32$. The median being 5 means that the mean of the middle two cousins is 5. This makes the sum of the middle two cousins $2 + 5 = 10$. Therefore, the sum of the oldest and youngest is $32 - 10 = 22$.

Thus, the correct answer is **D**.

6. Laura added two three-digit positive integers. All six digits in these numbers are different. Laura's sum is a three-digit number S. What is the smallest possible value for the sum of the digits of S?

A 1

B 4

C 5

D 15

E 21

Solution(s):

As $300 < S < 1000$, we know that the sum of the digits must be at least 4. This can be seen in $157 + 243 = 400$.

Thus, the correct answer is **B**.

7. The ratio of the measures of two acute angles is $5:4$, and the complement of one of these two angles is twice as large as the complement of the other. What is the sum of the degree measures of the two angles?

A 75

B 90

C 135

D 150

E 270

Solution(s):

Let the smaller angle be x. Then, the larger angle is $\frac{5}{4}x$. Note that their sum is $\frac{9}{4}x$.

Their complements are then $90-x$ and $90-\frac{5}{4}x$. Since $90-x$ is larger than $90-\frac{5}{4}x$, we know

$$\begin{aligned}90-x&=2\left(90-\frac{5}{4}x\right)\\90-x&=180-\frac{5}{2}x\\x&=60.\end{aligned}$$

Therefore, the sum is

$$\frac{9}{4}\cdot 60=135.$$

Thus, the correct answer is **C**.

8. What is the tens digit of $2015^{2016} - 2017$?

A 0

B 1

C 3

D 5

E 8

Solution(s):

To find the tens digit, we first need to find the number modulo 100. First, we can find $2015^{2016} \mod 100$. We can do this with the Chinese Remainder Theorem by first getting the number $\mod 4$ and then $\mod 25$.

The number

$$2015^{2016} \equiv 0 \mod 25$$

since it is a multiple of 25.

Then, observe that

$$2015^{2016} \equiv (-1)^{2016}$$
$$\equiv 1 \mod 4.$$

By the Chinese remainder theorem, we can get that our number is congruent to $25 \mod 100$.

Therefore,

$$2015^{2016} - 2017$$
$$\equiv 25 - 17 \mod 100$$
$$\equiv 8 \mod 100.$$

This means the tens digit is 0.

Thus, the correct answer is **A**.

9. All three vertices of $\triangle ABC$ lie on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC?

- A 4
- B 6
- C 8
- D 10
- E 16

Solution(s):

Let the points be

$$A = (0,0),$$

$$B = (x_1, y_1),$$

$$C = (x_2, y_2).$$

Then, since BC is parallel with the x-axis, we know $y_1 = y_2$, which we will let be y. Then, $x_1^2 = y = x_2^2$, so $x_1^2 = x_2^2$. This implies that either $x_1 = -x_2$ or $x_1 = x_2$. The second option cannot happen since that would set two points as the same, which would create an area of 0. As such, let $x = -x_1 = x_2$. Then $y = x^2$ also.

Then, the points are

$$A = (0,0),$$

$$B = (-x, x^2),$$

$$C = (x, x^2).$$

With a base of BC, the length is $2x$ and the height is x^2. This would make the area $x^3 = 64$. Therefore, $x = 4$, so $BC = 2 \cdot 4 = 8$.

Thus, the correct answer is **C**.

10. A thin piece of wood of uniform density in the shape of an equilateral triangle with side length 3 inches weighs 12 ounces. A second piece of the same type of wood, with the same thickness, also in the shape of an equilateral triangle, has side length of 5 inches. Which of the following is closest to the weight, in ounces, of the second piece?

- A 14.0
- B 16.0
- C 20.0
- D 33.3
- E 55.6

Solution(s):

The surface area is increased by a factor of $\frac{5}{3}^2$ since the side lengths are increased by a factor of $\frac{5}{3}$. Also since the thickness is constant, the volume is scaled up by that much. Then, the wood having the same density makes the weight increase by a factor of $\frac{5}{3}^2$ as well, so the new volume is

$$12 \cdot \frac{25}{9} = \frac{100}{3}.$$

This is approximately 33.3.

Thus, the correct answer is **D**.

11. Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's garden?

- A 256
- B 336
- C 384
- D 448
- E 512

Solution(s):

Let the number of posts on the longer side be l and let the number of posts on the shorter side be s. Then,

$$l + s + l + s - 4 = 20$$

since we take out the corners. Thus,

$$2l + 2s = 6s = 24,$$

making $s = 4, l = 8$. Thus, on the shorter side, the corners are 3 posts away and on the longers side, they are 7 posts away. The shorter side is then $3 \cdot 4 = 12$, and the longer side is $7 \cdot 4 = 28$. Thus, the area is $12 \cdot 28 = 336$.

Thus, the correct answer is **B**.

12. Two different numbers are selected at random from $\{1, 2, 3, 4, 5\}$ and multiplied together. What is the probability that the product is even?

A 0.2

B 0.4

C 0.5

D 0.7

E 0.8

Solution(s):

The product is odd if an only if both numbers are odd. There are $\binom{3}{2} = 3$ ways to do this out of a possible $\binom{5}{2} = 10$ ways. This makes the probability of it being odd equal to $\frac{3}{10} = 0.3$. This means the probability it is even is $1 - 0.3 = 0.7$.

Thus, the correct answer is **D**.

13. At Megapolis Hospital one year, multiple-birth statistics were as follows: Sets of twins, triplets, and quadruplets accounted for 1000 of the babies born. There were four times as many sets of triplets as sets of quadruplets, and there was three times as many sets of twins as sets of triplets. How many of these 1000 babies were in sets of quadruplets?

- A 25
- B 40
- C 64
- D 100
- E 160

Solution(s):

Let the number of sets of twins, triplets, and quadruplets be w, r, and q respectively. We must find $4q$ which is 4 times the number of sets of children. Then, we know

$$2w + 3r + 4q = 1000$$

as this is the number of total babies. Also, we have $r = 4q$ and $w = 3r$ by the next statments. Thus, $w = 24q$. By substitution, we have $40q = 1000$, so $4q = 100$, which is our answer.

Thus, the correct answer is **D**.

14. How many squares whose sides are parallel to the axes and whose vertices have coordinates that are integers lie entirely within the region bounded by the line $y = \pi x$, the line $y = -0.1$ and the line $x = 5.1$?

A 30

B 41

C 45

D 50

E 57

Solution(s):

The intersection of $x = 5.1$ and the other two lines have an x-value of 5.1. The intersection of $y = \pi x$ and $y = -0.1$ has an x-value of

$$\frac{-0.1}{\pi},$$

which is greater than -1.

Thus, the x value of some point on the square must be between 0 and 5 inclusive. Within this bound, $\pi x < -0.1$, so $-0.1 < y < \pi x$. This makes the largest points with the largest y-value given an x-value is

$$(0, 0),$$

$$(1, 3),$$

$$(2, 6),$$

$$(3, 9),$$

$$(4, 12),$$

$$(5, 15).$$

Now that we know the bounds of the triangle, we shall count the number of squares by looking at the top left corner of the square. We shall case on the size of the square.

If it is a 1×1 square, then the y value must be greater than or equal to 1 and the x value must be less than or equal to 4. The number of possible values for this is $3 + 6 + 9 + 12 = 30$.

If it is a 2×2 square, then the y value must be greater than or equal to 2 and the x value must be less than or equal to 3. The number of possible values for this is $2 + 5 + 8 = 15$.

If it is a 3×3 square, then the y value must be greater than or equal to 3 and the x value must be less than or equal to 2. The number of possible values for this is $1 + 4 = 5$.

The total is then

$$30 + 15 + 5 = 50.$$

Thus, the correct answer is **D**.

15. All the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ are written in a 3×3 array of squares, one number in each square, in such a way that if two numbers are consecutive then they occupy squares that share an edge. The numbers in the four corners add up to 18. What is the number in the center?

A 5

B 6

C 7

D 8

E 9

Solution(s):

We firstly claim that everything that is either in the center or a corner is odd. This is because every number is next to a consecutive number and therefore has the opposite parity as it.

Therefore, all of the points that are the center or a corner are the same parity. The are 5 such points, but only 4 even numbers, so their parity isn't even. Thus, they are all odd. This means the sum of the corners plus the center is

$$1 + 3 + 5 + 7 + 9 = 25.$$

Since the corners have a sum of 18, the center has a value of $25 - 18 = 7$.

Thus, the correct answer is **C**.

16. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

A $\frac{1+\sqrt{5}}{2}$

B 2

C $\sqrt{5}$

D 3

E 4

Solution(s):

Let the first value of the series be a, and let the ratio be r. Thus,

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{ar}{r(1-r)} \\ &= \frac{1}{r(1-r)}. \end{aligned}$$

This means we have to find r that maximizes

$$r(1-r) = 0.25 - (r-0.5)^2.$$

This maximization will happen with $r = 0.5$.

Therefore, $S = \dfrac{1}{0.5(0.5)} = 4.$

Thus, the correct answer is **E**.

17. All the numbers $2, 3, 4, 5, 6, 7$ are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, a product of three numbers is computed, where the three numbers are the numbers assigned to the three faces that include that vertex. What is the greatest possible value of the sum of these eight products?

- A 312
- B 343
- C 625
- D 729
- E 1680

Solution(s):

Suppose that the pairs of opposite sides are labelled

$$(a_1, a_2),$$
$$(b_1, b_2),$$
$$(c_1, c_2).$$

For the numbers on the same side as a_1, the total is

$$a_1(b_1c_1 + b_1c_2 + b_2c_2 + b_2c_1)$$
$$= a_1(b_1 + b_2)(c_1 + c_2).$$

It is similar on the other side, so the total is:

$$(a_1 + a_2)(b_1 + b_2)(c_1 + c_2).$$

Then,

$$9 = \frac{(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)}{3}$$
$$\geq ((a_1 + a_2)(b_1 + b_2)(c_1 + c_2))^{\frac{1}{3}}$$

by AM-GM, so our value is at most $9^3 = 729$. Also, if each number x is opposite to $9 - x$, we have the product as 729, which is a possible value. This makes it, by definition, the maximum.

Thus, the correct answer is **D**.

18. In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

A 1

B 3

C 5

D 6

E 7

Solution(s):

Let the smallest number be x. Then, let the size of the sequence be s. This makes the largest integer $x+s-1$.

Thus, the average of every number is $x+\frac{s-1}{2}$. This makes the sum of all the numbers

$$s\left(x+\frac{s-1}{2}\right)=345.$$

This is equivalent to $s(2x+s-1)=690$, or equivalently:

$$2x+s-1=\frac{690}{s}$$

As x and s are integers, we know that in order for the left hand side of the equation to be an integer, s must be a factor of 690.

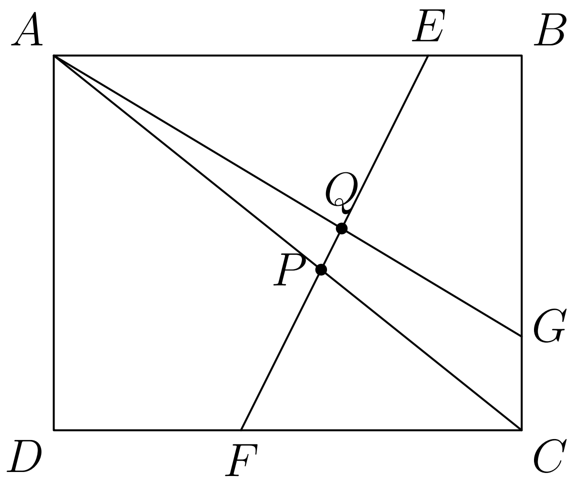
Observing the 16 factors of 690, observe that all factors strictly less than 30 must work, as $\frac{345}{30}=11.5$, suggesting that 11 and 12 are the middle numbers in the sequence. However, a sequence of 30 integers with the middle numbers of 11 and 12 must have negative numbers at the start of the sequence. However, this isn't allowed!

As such, all sequences of length $0<s<30$ work that are factors of 690. These are:

$$2,3,5,6,10,15,23$$

Thus, the correct answer is **E**.

19. Rectangle $ABCD$ has $AB = 5$ and $BC = 4$. Point E lies on $\overline{AB}$ so that $EB = 1$, point G lies on $\overline{BC}$ so that $CG = 1$, and point F lies on $\overline{CD}$ so that $DF = 2$. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P, respectively. What is the value of $\dfrac{PQ}{EF}$?



A $\dfrac{\sqrt{3}}{16}$

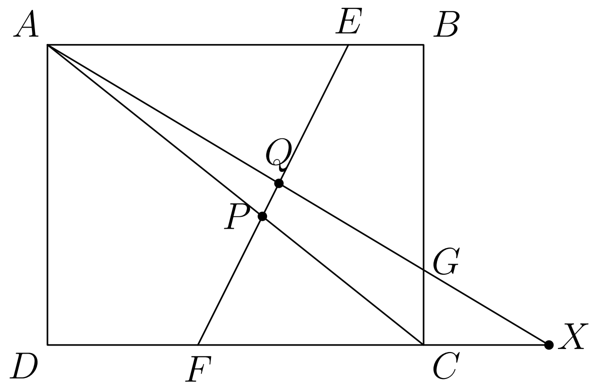
B $\dfrac{\sqrt{2}}{13}$

C $\dfrac{9}{82}$

D $\dfrac{10}{91}$

E $\dfrac{1}{9}$

Solution(s):



Observe that the value

$$\begin{aligned} AE &= AB - EB \\ &= 5 - 1 \\ &= 4. \end{aligned}$$

Also, the value

$$\begin{aligned} FC &= DC - DF \\ &= 5 - 2 \\ &= 3. \end{aligned}$$

Then, since $AE \mid\mid FC$, we know $\triangle AEP \sim \triangle CFP$ by angle angle symmetry. Thus,

$$\frac{PF}{PE} = \frac{FC}{EA} = \frac{3}{4}.$$

This makes

$$\frac{PF}{EF} = \frac{3}{3+4} = \frac{3}{7}.$$

Then, let X be the extension of AG to meet CD. Since two sides are parallel, we have

$$\frac{AD}{DX} = \frac{BG}{AB}.$$

Thus,

$$\begin{aligned} \frac{4}{DX} &= \frac{3}{5} \\ DX &= \frac{20}{3}. \end{aligned}$$

This makes

$$FX = \frac{20}{3} - 2 = \frac{14}{3}.$$

Then, since $AE \,||\, FX$, we know $\triangle AEQ \sim \triangle XFQ$ by angle angle symmetry. Therefore,

$$\frac{QF}{QE} = \frac{FX}{EA} = \frac{\frac{14}{3}}{4} = \frac{7}{6}.$$

This makes

$$\frac{QF}{EF} = \frac{7}{7+6} = \frac{7}{13}.$$

As such,

$$\frac{QP}{EF} = \frac{QF}{EF} - \frac{PF}{EF}$$

$$= \frac{7}{13} - \frac{3}{7}$$

$$= \frac{10}{91}.$$

Thus, the correct answer is **D**.

20. A dilation of the plane—that is, a size transformation with a positive scale factor—sends the circle of radius 2 centered at $A(2,2)$ to the circle of radius 3 centered at $A'(5,6)$. What distance does the origin $O(0,0)$, move under this transformation?

- A 0
- B 3
- C $\sqrt{13}$
- D 4
- E 5

Solution(s):

The line between the centers must be on the center of dialation as the center is dialated. Also, the dialation factor is 1.5 since the radius goes from 2 to 3. Let the center of dialation be C. Then, $CA : CA' = 1 : 1.5$. Thus,

$$CA : AA' = 1 : 0.5 = 2 : 1.$$

Therefore, we can go in the direction of the vector $A'A$ twice from A. The vector means we go down 4 and left 3. Doing this twice from A is $(-4, -6)$.

Then, $CO : CO' = 1 : 1.5$ since it dialates, meaning $CO : OO' = 1 : 0.5$. This means the distance moved is the length of CO divided by 2, which is equivalent to

$$\frac{\sqrt{52}}{2} = \sqrt{13}.$$

Thus, the correct answer is **C**.

21. What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = |x| + |y|$?

A $\pi + \sqrt{2}$

B $\pi + 2$

C $\pi + 2\sqrt{2}$

D $2\pi + \sqrt{2}$

E $2\pi + 2\sqrt{2}$

Solution(s):

$$|x|^2 + |y|^2 = x^2 + y^2$$
$$= |x| + |y|,$$

so

$$(|x| - 0.5)^2 + (|y| - 0.5)^2 = 0.5$$

This means the graph is symmetric over all quandrants, so we could find the area of the region in just the first quadrant and multiply it by 4.

Thus, we could look at the area of

$$(x - 0.5)^2 + (y - 0.5)^2 = 0.5$$

with $x, y \geq 0$.

If $x + y < 1$, then we have a triangle with base and height of 1, making the area 0.5.

Since the line $x + y = 1$ goes through the center, the region $x + y \geq 1$ creates a semicircle of radius $\sqrt{0.5}$. Thus, the area is

$$\frac{\pi r^2}{2} = \frac{\pi}{4}.$$

This means the total area in each quadrant is

$$0.5 + \frac{\pi}{4}.$$

Therefore, the total area is

$$4\left(0.5 + \frac{\pi}{4}\right) = 2 + \pi.$$

Thus, the correct answer is **C**.

22. A set of teams held a round-robin tournament in which every team played every other team exactly once. Every team won 10 games and lost 10 games; there were no ties. How many sets of three teams $\{A, B, C\}$ were there in which A beat B, B beat C, and C beat A?

- A 385
- B 665
- C 945
- D 1140
- E 1330

Solution(s):

The total number of teams is $10 + 10 + 1 = 21$. The total number of sets is therefore $\binom{21}{3} = 1330$.

Now, we must subtract the total number of sets such that there is no cycle. This only happens if one team beats the other two teams. There are 21 choices for the team that beat the other two and $\binom{10}{2} = 45$ ways to choose the teams they beat. Thus, the total of non-cycles is $21 \cdot 45 = 945$. This means the total number of cycles is $1330 - 945 = 385$.

Thus, the correct answer is **A**.

23. In regular hexagon $ABCDEF$, points W, X, Y, and Z are chosen on sides $\overline{BC}$, $\overline{CD}$, $\overline{EF}$, and $\overline{FA}$ respectively, so lines AB, ZW, YX, and ED are parallel and equally spaced. What is the ratio of the area of hexagon $WCXYFZ$ to the area of hexagon $ABCDEF$?

A $\frac{1}{3}$

B $\frac{10}{27}$

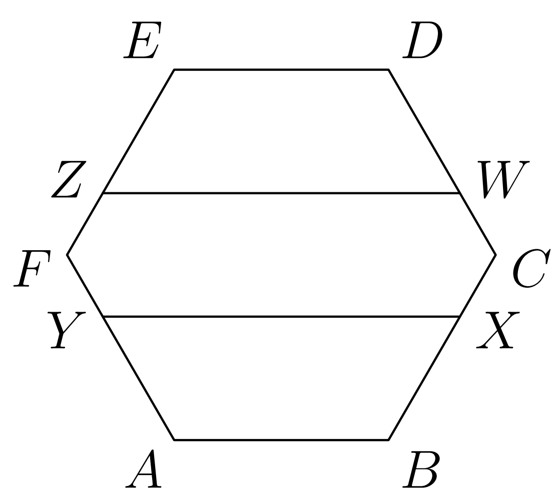
C $\frac{11}{27}$

D $\frac{4}{9}$

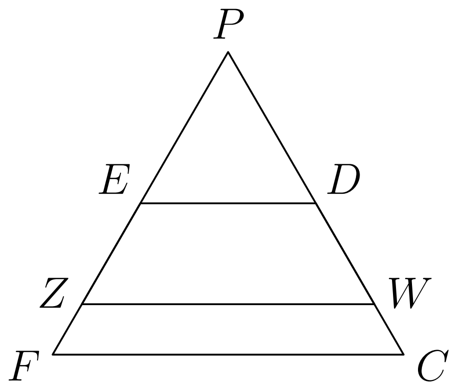
E $\frac{13}{27}$

Solution(s):

Consider the following diagram:



The shape is symmetric, so we can find the ratio of the areas of $EFCD$ to the area of $ZFCW$. For this, we can extend FE and CD until they meet each other, and let this point be P.



Also, the distance from ED to ZW is the same as the distance from ZW to YX, and the distance from ZW to YX is twice the distance as the distance from ZW to FC.

Thus, the distance from ED to ZW is twice the distance as the distance from ZW to FC. Let the altitude from P to ED be s. Then, if the altitude from P to ZW is $s+d$, the altitude from P to CF is $s+1.5d$ because of the ratio. Then, since

$$\triangle FPC \sim \triangle PED,$$

$$\frac{FP}{EP} = \frac{s+1.5d}{s}.$$

Also, since $FE = ED = PE$ as PED is equilateral, we have $2s = s+1.5d$, and so $s = 1.5d$.

Thus, the ratio between side lengths of $\triangle PED$ and $\triangle PZW$ is $\frac{5}{3}$ which makes the ratio between the area $\frac{25}{9}$. Also the ratio between side lengths of $\triangle PED$ and $\triangle PFC$ is 2, which makes the ratio between the area 4.

This makes the area of $EDCF$ 3 times the area of $\triangle PED$ and the area of $EDWZ$ $\frac{16}{9}$ times the area of $\triangle PED$ Therefore, the area of $ZWCF$ is $\frac{11}{9}$ times the area of $\triangle PED$.

Thus, the ratio between $ZWCF$ and $EDCF$ is

$$\frac{\frac{11}{9}}{3} = \frac{11}{27}.$$

Thus, the correct answer is **C**.

24. How many four-digit integers $abcd$, with $a \neq 0$, have the property that the three two-digit integers $ab < bc < cd$ form an increasing arithmetic sequence?

One such number is 4692, where $a = 4$, $b = 6$, $c = 9$, and $d = 2$.

A 9

B 15

C 16

D 17

E 20

Solution(s):

We know $a \leq b \leq c$ by analyzing $ab < bc < cd$. Also, bc is the average of ab and cd so

$$\frac{10a + b + 10c + d}{2} = 10b + c.$$

This means that

$$10(c - 2b + a) = -b + 2c - d,$$

making the right hand side a multiple of 10. Thus, it must be 0 or 10 since it is digits that satisfy $a \leq b \leq c$. Thus, we can case on that value.

Case 1:

$$-b + 2c - d = 10$$

$$2b - a - c = 1$$

We can look at the possible values of c.

$c = 6 : b + d = 2, 2b - 5 = a$. Thus, $b \leq 2$ from the first equation, but can't work for the second equation.

$c = 7 : b + d = 4, 2b - 6 = a$. Thus, $b \leq 4$ from the first equation, and $b > 3$ from the second eqution. This makes one case for $b = 4$.

$c = 8 : b + d = 6, 2b - 7 = a$. Thus, $b \leq 6$ from the first equation, and $b > 3$ from the second eqution. This makes three cases for $b = 4, 5, 6$.

$c = 9 : b + d = 8, 2b - 8 = a$. Thus, $b \leq 8$ from the first equation, and $b > 4$ from the second eqution. This makes four cases for $b = 5, 6, 7, 8$. This case has 8 solutions.

Case 2:

$$-b + 2c - d = 0,$$

$$2b - a - c = 0,$$

which means the digits are an arithmetic sequence.

If the difference is 1, then $1 \leq a \leq 6$ makes 6 solutions.

If the difference is 2, then $1 \leq a \leq 3$ makes 3 solutions. This case makes 9 solutions.

In total, the number of solutions is $8 + 9 = 17$.

Thus, the correct answer is **D**.

25. Let

$$f(x) = \sum_{k=2}^{10} (\lfloor kx \rfloor - k\lfloor x \rfloor),$$

where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r. How many distinct values does $f(x)$ assume for $x \geq 0$?

A 32

B 36

C 45

D 46

E infinitely many

Solution(s):

First, note that $x = \lfloor x \rfloor + \{x\}$ where $\{x\}$ is the fractional part of x.

Thus,

$$f(x) = \sum_{k=2}^{10} \lfloor k(\lfloor x \rfloor + \{x\}) \rfloor - k\lfloor x \rfloor$$

$$= \sum_{k=2}^{10} \lfloor k\{x\} \rfloor$$

since $\lfloor kx \rfloor$ is an integer. This means $f(x)$ can only increase when the fractional part of kx becomes greater than $\frac{i}{k}$ for some $i < k$.

Thus, we must count all possible fractions that are $\frac{i}{k}$ for some k less than or equal to 10. Note that we could exclusively count $k \geq 6$ as all $k \leq 5$ would have every fraction equal to a fraction where the denominator is greater than 6.

There are $k - 1$ fractions for each k, so we have

$$9 + 8 + 7 + 6 + 5 = 35$$

fractions.

However, $\frac{4}{8}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}$ are all duplicates from other greater k, so we subtract 4.

As such, we have $35 - 4 = 31$ possible values of f. Then, we add one more for $f(x) = 0$. This makes the total count equal to $31 + 1 = 32$.

Thus, the correct answer is **A**.

Problems: https://live.poshenloh.com/past-contests/amc10/2016B