2016 AMC 10A Solutions

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https://live.poshenloh.com/past-contests/amc10/2016A/solutions

1. What is the value of $\dfrac{11! - 10!}{9!}$?

A 99

B 100

C 110

D 121

E 132

Solution(s):

We can factor a $9!$ out the numerator and simplify.

$$\begin{aligned} \frac{9!(11 \cdot 10 - 10)}{9!} &= 11 \cdot 10 - 10 \\ &= 110 - 10 \\ &= 100. \end{aligned}$$

Thus, the correct answer is **B**.

2. For what value of x does

$$10^x \cdot 100^{2x} = 1000^5?$$

A 1

B 2

C 3

D 4

E 5

Solution(s):

We can express the 100 as 10^2 and 1000 as 10^3 to get

$$\begin{aligned} 10^x \cdot (10^2)^{2x} &= (10^3)^5 10^x \cdot 10^{4x} \\ &= 10^{15} 10^{5x} \\ &= 10^{15} \end{aligned}$$

Since the bases are the same, the exponents must also be the same. Therefore,

$$5x = 15$$

$$x = 3$$

Thus, the correct answer is **C**.

3. For every dollar Ben spent on bagels, David spent 25 cents less. Ben paid $\$12.50$ more than David. How much did they spend in the bagel store together?

A $\$37.50$

B $\$50.00$

C $\$87.50$

D $\$90.00$

E $\$92.50$

Solution(s):

This means that for every dollar Ben spent, he spent 25 cents more than David.

This means that Ben spent $12.50 \div 0.25 = 50$ dollars. Therefore, David spent $\$50 - \$12.5 = \$37.5$

The total amount of money they spent is $\$50 + \$37.5 = \$87.5$.

Thus, the correct answer is **C**.

4. The remainder can be defined for all real numbers x and y with $y \neq 0$ by

$$\operatorname{rem}(x, y) = x - y\left\lfloor \frac{x}{y} \right\rfloor$$

where $\left\lfloor \frac{x}{y} \right\rfloor$ denotes the greatest integer less than or equal to $\frac{x}{y}$. What is the value of $\operatorname{rem}\left(\frac{3}{8}, -\frac{2}{5}\right)$?

A $-\dfrac{3}{8}$

B $-\dfrac{1}{40}$

C 0

D $\dfrac{3}{8}$

E $\dfrac{31}{40}$

Solution(s):

Using the formula, we get

$$\begin{aligned}\operatorname{rem}\left(\frac{3}{8}, -\frac{2}{5}\right) &= \frac{3}{8} + \frac{2}{5}\left\lfloor \frac{\frac{3}{8}}{-\frac{2}{5}} \right\rfloor \\ &= \frac{3}{8} + \frac{2}{5}\left\lfloor -\frac{15}{16} \right\rfloor \\ &= \frac{3}{8} + \frac{2}{5} \cdot -1 \\ &= \frac{3}{8} - \frac{2}{5} \\ &= -\frac{1}{40}\end{aligned}$$

Thus, the correct answer is **B**.

5. A rectangular box has integer side lengths in the ratio $1 : 3 : 4$. Which of the following could be the volume of the box?

A 48

B 56

C 64

D 96

E 144

Solution(s):

Let s be the side length of the smallest side. Then the other two sides are $3s$ and $4s$.

The volume is therefore

$$s \cdot 3s \cdot 4s = 12s^3.$$

Testing out values of s, we see that if $s = 2$, then $12s^3 = 96$, which is an answer choice.

Thus, the correct answer is **D**.

6. Ximena lists the whole numbers 1 through 30 once. Emilio copies Ximena's numbers, replacing each occurrence of the digit 2 by the digit 1. Ximena adds her numbers and Emilio adds his numbers. How much larger is Ximena's sum than Emilio's?

A 13

B 26

C 102

D 103

E 110

Solution(s):

Whenever Ximena replaces a units digit of 2 with a 1, her total sum decreases by 1.

Similarly, whenever a tens digit of 2 is replaced with a 1, her total sum decreases by 10.

2 appears as a unit digit 3 times $(2, 12,$ and $22)$ and it appears in the tens digit 10 times $(20 - 29)$.

Her total sum, therefore, will be

$$3 \cdot 1 + 10 \cdot 10 = 103$$

less than Emilio's sum.

Thus, the correct answer is **D**.

7. The mean, median, and mode of the 7 data values

$$60, 100, x, 40, 50, 200, 90$$

are all equal to x. What is the value of x?

A 50

B 60

C 75

D 90

E 100

Solution(s):

The sum of the elements in this set is $540 + x$, making the mean $\frac{1}{7}(540 + x)$. Therefore,

$$\frac{1}{7}(540 + x) = x$$

$$540 + x = 7x$$

$$6x = 540$$

$$x = 90.$$

Thus, the correct answer is **D**.

8. Trickster Rabbit agrees with Foolish Fox to double Fox's money every time Fox crosses the bridge by Rabbit's house, as long as Fox pays 40 coins in toll to Rabbit after each crossing. The payment is made after the doubling. Fox is excited about his good fortune until he discovers that all his money is gone after crossing the bridge three times. How many coins did Fox have at the beginning?

A 20

B 30

C 35

D 40

E 45

Solution(s):

We know that Fox has 0 coins at the end. Then before paying the final toll, Fox had 40 coins.

Then he had $40 \div 2 = 20$ coins before the doubling. Then before paying the toll for the second crossing, he had $20 + 40 = 60$ coins.

Before the doubling on the second crossing, he had $60 \div 2 = 30$ coins. On the first crossing before the toll, Fox had $30 + 40 = 70$ coins.

Finally, before the first doubling, Fox had $70 \div 2 = 35$ coins.

Thus, the correct answer is **C**.

9. A triangular array of 2016 coins has 1 coin in the first row, 2 coins in the second row, 3 coins in the third row, and so on up to N coins in the Nth row. What is the sum of the digits of N?

- A 6
- B 7
- C 8
- D 9
- E 10

Solution(s):

Recall that the sum of the first N number is $\dfrac{N(N+1)}{2}$.

We want to find N such that

$$\frac{N(N+1)}{2} = 2016.$$

Cross-multiplying and simplifying gives us

$$N^2 + N - 4032 = 0.$$

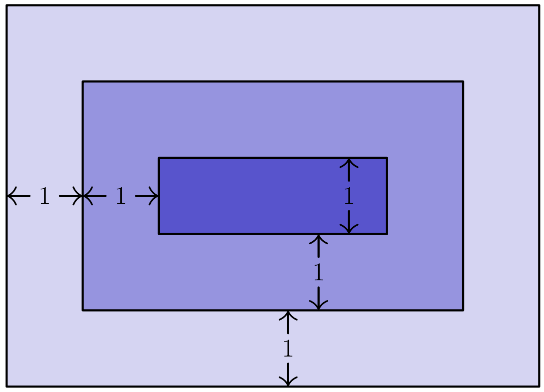
Factoring gives us

$$(N-63)(N+64) = 0$$

We want the positive value so $N = 63$. Adding together the digits gives us 9.

Thus, the correct answer is **D**.

10. A rug is made with three different colors as shown. The areas of the three differently colored regions form an arithmetic progression. The inner rectangle is one foot wide, and each of the two outer regions are 1 foot wide on all four sides. What is the length in feet of the inner rectangle?



A 1

B 2

C 4

D 6

E 8

Solution(s):

Let l be the length of the inner rectangle. Then the area of the inner rectangle is l.

The area of the middle region is going to be

$$(l+2)\cdot 3 - l = 2l + 6.$$

The area of the outer region is

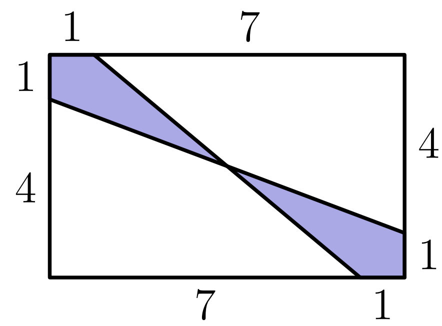
$$(l+4)\cdot 5 - (l+2)\cdot 3 = 2l + 14.$$

We know that these 3 values form an arithmetic sequence. That means that

$$\begin{aligned} l + 2l + 14 &= 2(2l+6) \\ 3l + 14 &= 4l + 12 \\ l &= 2 \end{aligned}$$

Thus, the correct answer is **B**.

11. Find the area of the shaded region.



A $4\frac{3}{5}$

B 5

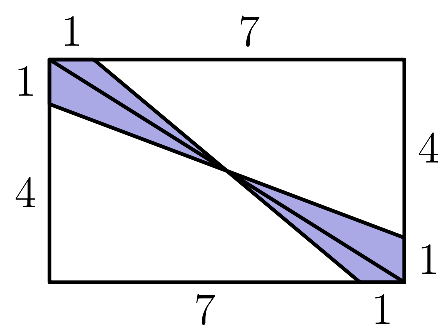
C $5\frac{1}{4}$

D $6\frac{1}{2}$

E 8

Solution(s):

We can split the region into 4 triangles with bases of 1.



Two of the triangles have bases $8 \div 2 = 4$ and the other two have bases $5 \div 2 = \frac{5}{2}$

The sum of the areas of the triangles is

$$2 \cdot \frac{1}{2}\left(1 \cdot \frac{5}{2}\right) + 2 \cdot \frac{1}{2}(1 \cdot 4) = 6\frac{1}{2}$$

Thus, the correct answer is **D**.

12. Three distinct integers are selected at random between 1 and 2016, inclusive. Which of the following is a correct statement about the probability p that the product of the three integers is odd?

A $p<\frac{1}{8}$

B $p=\frac{1}{8}$

C $\frac{1}{8}<p<\frac{1}{3}$

D $p=\frac{1}{3}$

E $p>\frac{1}{3}$

Solution(s):

The only way for the product of the three integers to be odd is if all the numbers themselves are odd.

There are an even number of consecutive integers, which means that there is a $\frac{1}{2}$ chance that a randomly chosen number is odd.

For all 3 numbers to be add, the probability is

$$\left(\frac{1}{2}\right)^3=\frac{1}{8}.$$

But note that all integers must be distinct. This lowers the probability since we have added an extra restriction.

Thus, the correct answer is **A**.

13. Five friends sat in a movie theater in a row containing 5 seats, numbered 1 to 5 from left to right. (The directions "left" and "right" are from the point of view of the people as they sit in the seats.)

During the movie Ada went to the lobby to get some popcorn. When she returned, she found that Bea had moved two seats to the right, Ceci had moved one seat to the left, and Dee and Edie had switched seats, leaving an end seat for Ada. In which seat had Ada been sitting before she got up?

A 1

B 2

C 3

D 4

E 5

Solution(s):

Note that Dee and Edie do not change the answers since their seats remain occupied throughout.

Since Bea moves to the right, this forces Ada and Ceci to move to offset this disruption.

Ceci only moves one seat, so Ada must also move one to cancel out the two seat shift that Bea did.

Bea moved to the right and Ceci moved to the left, so Ada must also move to the left to get a total displacement of 0.

Therefore, Ada must have started off in seat 2 to move one to the left to end up in seat 1.

Thus, the correct answer is **B**.

14. How many ways are there to write 2016 as the sum of twos and threes, ignoring order? (For example, $1008 \cdot 2 + 0 \cdot 3$ and $402 \cdot 2 + 404 \cdot 3$ are two such ways.)

- A 236
- B 336
- C 337
- D 403
- E 672

Solution(s):

The problem can be rewritten as an equation $2x + 3y = 2016$, where x is the number of twos and y is the number of threes.

The goal is to find the number of multiples of 3 that can be subtracted from 2016 to result in an even number.

This can be achieved by the pairs of $(1008, 0)$ up to $(0, 672)$ with y being incremented by 2.

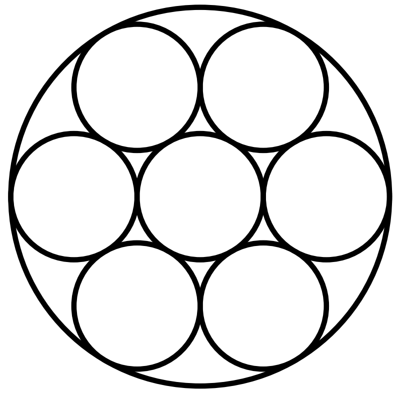
This gives us

$$\frac{672}{2} + 1 = 337$$

solutions for y and x.

Thus, the correct answer is **C**.

15. Seven cookies of radius 1 inch are cut from a circle of cookie dough, as shown. Neighboring cookies are tangent, and all except the center cookie are tangent to the edge of the dough. The leftover scrap is reshaped to form another cookie of the same thickness. What is the radius in inches of the scrap cookie?



A $\sqrt{2}$

B 1.5

C $\sqrt{\pi}$

D $\sqrt{2\pi}$

E π

Solution(s):

The circle of cookie dough has a radius of 3 inches since it is the same as the diameter plus the radius of a cookie.

The area of the cookie dough is $3^2\pi = 9\pi$, and the cookies have an area of $7 \cdot 1^2\pi = 7\pi$.

The area of the leftover scrap is therefore $9\pi - 7\pi = 2\pi$. This means that its radius is $\sqrt{2}$.

Thus, the correct answer is **A**.

16. A triangle with vertices $A(0,2), B(-3,2)$, and $C(-3,0)$ is reflected about the x-axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by 90° to produce $\triangle A''B''C''$. Which of the following transformations will return $\triangle A''B''C''$ to $\triangle ABC$?

A counterclockwise rotation about the origin by 90°

B clockwise rotation about the origin by 90°

C reflection about the x-axis

D reflection about the line $y = x$

E reflection about the y-axis

Solution(s):

To figure out how to reverse the transformations, we can analyze a single point and see what happens to it.

Let (x, y) be the point. After being reflected about the x-axis, the point would go to $(x, -y)$.

Rotating this counterclockwise would put it at (y, x). The only transformation that puts this back at (x, y) is reflection about $y = x$.

Thus, the correct answer is **D**.

17. Let N be a positive multiple of 5. One red ball and N green balls are arranged in a line in random order. Let $P(N)$ be the probability that at least $\frac{3}{5}$ of the green balls are on the same side of the red ball. Observe that $P(5) = 1$ and that $P(N)$ approaches $\frac{4}{5}$ as N grows large. What is the sum of the digits of the least value of N such that $P(N) < \dfrac{321}{400}$?

A 12

B 14

C 16

D 18

E 20

Solution(s):

For the condition to be satisfied, the red ball cannot be placed in the middle fifth of the green balls.

This means that there are $\frac{N}{5} - 1$ spots where the red ball cannot be placed.

Placing the red ball anywhere else works, which means that

$$\begin{aligned} P(N) &= 1 - \frac{\frac{N}{5} - 1}{N+1} \\ &= \frac{4N+10}{5N+5} \\ &< \frac{321}{400} \end{aligned}$$

Cross-multiplying and simplifying gives us

$$2395 < 5N$$

$$479 < N.$$

The smallest value of N is therefore 480. The sum of the digits is $4 + 8 = 12$.

Thus, the correct answer is **A**.

18. Each vertex of a cube is to be labeled with an integer 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?

A 1

B 3

C 6

D 12

E 24

Solution(s):

We have that the sum of the vertices on each face is

$$\frac{1+2+\cdots+8}{2}=18.$$

This is because two opposite faces use all 8 vertices, and their vertices have the sum.

Let a, b, and c be the vertices next to 1. Then the remaining vertices are

$$17-a-b,$$

$$17-a-c,$$

$$17-b-c,$$

and

$$a+b+c-16.$$

We can do casework on a, b, c. The only restrictions we have are that $a+b+c>17$ and all the vertices are distinct.

WLOG let $a<b<c$:

$3,7,8: 17-3-7=7$, which is not allowed

$4,6,8:$ all the vertices work

$4, 7, 8$: all the vertices work

$5, 6, 7 : 17 - 5 - 6 = 6$, which is not allowed

$5, 6, 8 : 17 - 5 - 6 = 6$, which is not allowed

$5, 7, 8 : 17 - 5 - 7 = 5$, which is not allowed

$6, 7, 8$: all vertices work

For each triple, the only way we can arrange them to create unique configurations is (x, y, z) and $(z, y, x.)$

These cannot be created with rotations by each other. Therefore, there are $3 \cdot 2 = 6$ arrangements.

Thus, the correct answer is **C**.

19. In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively.

The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of r, s, and t is 1. What is $r + s + t$?

A 7

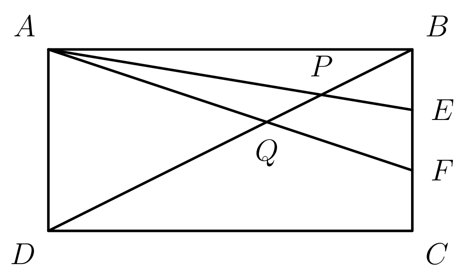
B 9

C 12

D 15

E 20

Solution(s):



Note that $\triangle APD \sim \triangle EPB$ by angle-angle using alternate interior angles.

This gives us

$$\frac{DP}{PB} = \frac{AD}{BE}$$

$$PB = \frac{1}{4}BD.$$

Similarly, we have that $\triangle AQD \sim \triangle FQB$, which gives us

$$\frac{DQ}{BQ} = \frac{AD}{BF}$$

$$BQ = \frac{2}{5}BD.$$

Finally, we get that

$$DQ = \left(1 - \frac{2}{5}\right)BD = \frac{3}{5}BD$$

Then the desired ratio is

$$\left(\frac{1}{4} : \frac{2}{5}\right) - \left(\frac{1}{4} : \frac{3}{5}\right)$$

$$= 5 : 3 : 12$$

The sum of these is $5 + 3 + 12 = 20$.

Thus, **E** is the correct answer.

20. For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables $a, b, c,$ and d, each to some positive power. What is N?

A 9

B 14

C 16

D 17

E 19

Solution(s):

We want to find all the terms that are of the form

$$a^v b^w c^x d^y 1^z.$$

Note that

$$v, w, x, y > 0.$$

These variables must satisfy

$$v + w + x + y + z = N.$$

Since $v, w, x,$ and y must be positive, we can define $x' = x - 1$ and similarly for all the other variables.

Then, we get that

$$v', w', x', y' \geq 0.$$

Using these new variables, we get the new equation

$$v' + w' + x' + y' + z = N - 4.$$

Now we can use stars and bars since all the values are non-negative. There are

$$\binom{N-4+4}{4} = \binom{N}{4}$$

solutions to the equation.

We need to find N such that $\binom{N}{4} = 1001$. Checking all the answer choices yields 14 as the right answer.

Thus, the correct answer is **B**.

21. Circles with centers P, Q and R, having radii $1, 2$ and 3, respectively, lie on the same side of line l and are tangent to l at P', Q' and R', respectively, with Q' between P' and R'. The circle with center Q is externally tangent to each of the other two circles. What is the area of triangle $\triangle PQR$?

A 0

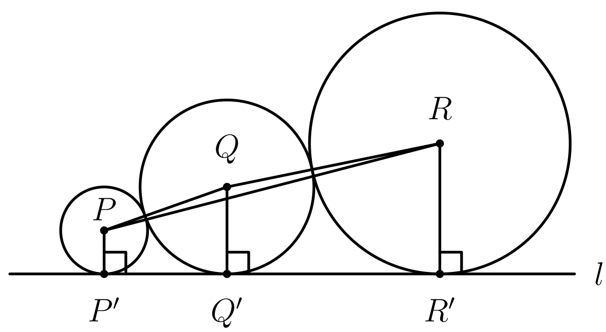
B $\sqrt{\frac{2}{3}}$

C 1

D $\sqrt{6}-\sqrt{2}$

E $\sqrt{\frac{3}{2}}$

Solution(s):



Using the Pythagorean theorem, we get that

$$P'Q' = \sqrt{3^2 - 1^2} = 2\sqrt{2}$$

and

$$Q'R' = \sqrt{5^2 - 1^2} = 2\sqrt{6}.$$

This follows from $PQ = 1 + 2 = 3$ and $QR = 2 + 3 = 5$. The heights of the triangles are also just 1.

Then, we get that

$$[Q'QPP'] = \frac{1}{2}(1+2)2\sqrt{2}$$

$$= 3\sqrt{2}.$$

We also get that

$$[R'RQQ'] = \frac{1}{2}(2+3)2\sqrt{6}$$

$$= 5\sqrt{6}.$$

Finally, we have that

$$[R'RPP'] =$$

$$\frac{1}{2}(1+3)(2\sqrt{2}+2\sqrt{6})$$

$$= 4\sqrt{2}+4\sqrt{6}.$$

Now, we can express $[PQR]$ as

$$[Q'QPP'] + [R'RQQ']$$

$$-[R'RPP'].$$

This evaluates to

$$3\sqrt{2}+5\sqrt{6}-4\sqrt{2}-4\sqrt{6}$$

$$= \sqrt{6}-\sqrt{2}.$$

Thus, the correct answer is **D**.

22. For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

A 110

B 191

C 261

D 325

E 425

Solution(s):

Note that the prime factorization of 110 is $2 \cdot 5 \cdot 11$.

Recall that if a number is expressed as

$$p_1^{e_1} p_2 e_2 \cdots p_n e_n,$$

then it has

$$(e_1 + 1)(e_2 + 1) \cdots (e_n + 1)$$

factors.

We know that $110n^3$ has at least 3 factors, namely $2, 5$, and 11.

The only way to express 110 as the product of at least 3 numbers is $2 \cdot 5 \cdot 11$.

This means that $110n^3$ has no other prime factors. Then the exponents must be $1, 4$, and 10 from the above formula.

Let 2^3 be a factor of n. Then n^3 has a factor of 2^9, which makes $110n^3$ have a factor of 2^{10}.

Then let 5 be a factor of n. Then n^3 has a factor of 5^3, which means that $110n^3$ has a factor of 5^4.

We do not have to alter the number of 11s since it already has 1 as its exponent.

This means that we can let $n = 2^3 \cdot 5$. Then

$$81n^4 = 3^4 \cdot 2^1 2 \cdot 5^4.$$

This number has

$$5 \cdot 13 \cdot 5 = 325$$

factors.

Thus, the correct answer is **D**.

23. A binary operation $\diamondsuit$ has the properties that

$$a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$$

and that $a \diamondsuit a = 1$ for all nonzero real numbers a, b, and c. (Here $\cdot$ represents multiplication). The solution to the equation

$$2016 \diamondsuit (6 \diamondsuit x) = 100$$

can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

A 109

B 201

C 301

D 3049

E $33,601$

Solution(s):

Let us see what properties we can gather from the given conditions. First, we see that

$$a \diamondsuit (a \diamondsuit a) = (a \diamondsuit a) \cdot a,$$

which tells us that

$$a \diamondsuit 1 = a.$$

Now, let's see what happens if we set $b = c$ and apply the second property. We get

$$a \diamondsuit (b \diamondsuit b) = (a \diamondsuit b) \cdot b,$$

which simplifies to

$$a = (a \diamondsuit b) \cdot b$$

$$\frac{a}{b} = a \diamondsuit b.$$

We can divide by b since b is nonzero.

We can now calculate the value of x directly.

$$\frac{2016}{\frac{6}{x}} = 100$$
$$2016 = \frac{600}{x}$$
$$x = \frac{600}{2016} = \frac{25}{84}$$

Then we have that $p + q = 109$.

Thus, the correct answer is **A**.

24. A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

A 200

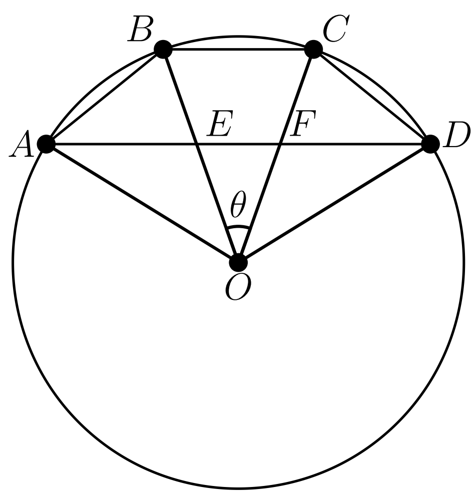
B $200\sqrt{2}$

C $200\sqrt{3}$

D $300\sqrt{2}$

E 500

Solution(s):



Let $\overline{AD}$ intersect $\overline{BO}$ and $\overline{CO}$ at E and F respectively.

Since $AB = BC = CD$, we get that

$$\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \theta.$$

Then we get that $\angle BAD$ is an interior angle with value

$$\frac{1}{2}\overset{\frown}{BCD} = \frac{1}{2} \cdot 2\theta = \theta = \angle AOB.$$

Then, by angle-angle, we get that $\triangle OAB \sim \triangle ABE$. This gives us

$$\frac{OA}{AB} = \frac{AB}{BE} = \frac{OB}{AE}.$$

We have that $OA = OB$ as they are both radii, which means that $AB = AE$. Similarly, we have that $CD = DF$.

Now, we get that $BE = 100\sqrt{2}$ from the above ratios, which means that $\frac{BE}{BO} = \frac{1}{2}$.

This tells us that

$$EF = \frac{BC}{2} = \frac{200}{2} = 100.$$

Then we get that

$$AD = AE + EF + FD$$
$$= AB + EF + CD = 500.$$

Thus, the correct answer is **E**.

25. How many ordered triples (x, y, z) of positive integers satisfy

$$\operatorname{lcm}(x, y) = 72,$$

$$\operatorname{lcm}(x, z) = 600,$$

and

$$\operatorname{lcm}(y, z) = 900?$$

A 15

B 16

C 24

D 27

E 64

Solution(s):

We can prime factorize 72 into $2^3 \cdot 3^2$, 600 into $2^3 \cdot 3 \cdot 5^2$, and 900 into $2^2 \cdot 3^2 \cdot 5^2$.

Note that the lcm of x and y does not have a factor of 5, so will neither x nor y. This means z must have a factor of 5^2.

Then we can express $x = 2^a \cdot 3^b$, $y = 2^c \cdot 3^d$, and $z = 2^e \cdot 3^f \cdot 5^2$.

By definition of lcm, we get that

$$\begin{gathered}
\max(a, c) = 3, \\
\max(b, d) = 2, \\
\max(a, e) = 3, \\
\max(b, f) = 1, \\
\max(c, e) = 2, \\
\max(d, f) = 2.
\end{gathered}$$

Since the max of b and f is 1, we have that $d = 2$. Similarly, since the max of c and e is 2, we have that $a = 3$.

We are left with a few redundant equations, so we can trim them down to

$$\max(b, f) = 1$$

and

$$\max(c, e) = 2.$$

For the first equation, at least one of them must be 1, giving us 3 options: $(0, 1), (1, 0), (1, 1)$.

For the second equation, at least one of them must be 2, giving us 5 options:

$$(2, 0), (2, 1), (2, 2), (0, 2), (1, 2).$$

There are then $3 \cdot 5 = 15$ possible combinations for all the variables.

Thus, the correct answer is **A**.

Problems: https://live.poshenloh.com/past-contests/amc10/2016A