2015 AMC 10B Solutions

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https://live.poshenloh.com/past-contests/amc10/2015B/solutions

1. What is the value of $2-(-2)^{-2}$?

- **A** -2
- **B** $\frac{1}{16}$
- **C** $\frac{7}{4}$
- **D** $\frac{9}{4}$
- **E** 6

Solution(s):

$$\begin{aligned} 2-(-2)^{-2} &= 2-\frac{1}{2}^{2} \\ &= 2-\frac{1}{4} \\ &= \frac{7}{4}. \end{aligned}$$

Thus, the correct answer is **C**.

2. Marie does three equally time-consuming tasks in a row without taking breaks. She begins the first task at $1\!:\!00$ PM and finishes the second task at $2\!:\!40$ PM. When does she finish the third task?

- A 3:10 PM
- B 3:30 PM
- C 4:00 PM
- D 4:10 PM
- E 4:30 PM

Solution(s):

The time it takes to do 2 tasks is 100 minutes. Thus, it takes 50 more minutes after $2:40$, which is $3:30$.

Thus, the correct answer is **B**.

3. Kaashish has written down one integer two times and another integer three times. The sum of the five numbers is 100, and one of the numbers is 28. What is the other number?

- A 8
- B 11
- C 14
- D 15
- E 18

Solution(s):

Kaashish either wrote 28 three times or two times.

If he wrote it twice, then the sum of the other three numbers is:

$$100 - 28 \cdot 2 = 44,$$

which isn't a multiple of 3. Therefore, this case wouldn't work.

If he wrote it three times, then the sum of the other two numbers is

$$100 - 28 \cdot 3 = 16.$$

Therefore, the other number is $\frac{16}{2} = 8$.

Thus, the correct answer is **A**.

4. Four siblings ordered an extra large pizza. Alex ate $\frac{1}{5}$, Beth $\frac{1}{3}$, and Cyril $\frac{1}{4}$ of the pizza. Dan got the leftovers. What is the sequence of the siblings in decreasing order of the part of pizza they consumed?

- A Alex, Beth, Cyril, Dan
- B Beth, Cyril, Alex, Dan
- C Beth, Cyril, Dan, Alex
- D Beth, Dan, Cyril, Alex
- E Dan, Beth, Cyril, Alex

Solution(s):

Since $\frac{1}{3} > \frac{1}{4} > \frac{1}{5}$, we know Beth ate more than Cyril and Cyril ate more than Alex. Thus, those three are in order.

The amount Dan ate is

$$1 - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} = \frac{13}{60}.$$

This is greater than $\frac{1}{5}$ and less than $\frac{1}{4}$, so Dan is in between Cyril and Alex. This makes the order Beth, Cyril, Dan, Alex.

Thus, the correct answer is **C**.

5. David, Hikmet, Jack, Marta, Rand, and Todd were in a 12-person race with 6 other people. Rand finished 6 places ahead of Hikmet. Marta finished 1 place behind Jack. David finished 2 places behind Hikmet. Jack finished 2 places behind Todd. Todd finished 1 place behind Rand. Marta finished in 6th place. Who finished in 8th place?

A David

B Hikmet

C Jack

D Rand

E Todd

Solution(s):

Jack is one place in front of Marta who is in 6th place, so he is in 5th place.

Todd is two places in front of Marta who is in 5th place, so he is in 3rd place.

Rand is one place in front of Todd who is in 3rd place, so he is in 2nd place.

Himket is six places behind Rand who is in 2nd place, so he is in 8th place.

Thus, the correct answer is **B**.

6. Mahdi practices exactly one sport each day of the week. He runs three days a week but never on two consecutive days. On Monday he plays basketball and two days later golf. He swims and plays tennis, but he never plays tennis the day after running or swimming. Which day of the week does Mahdi swim?

- A Sunday
- B Tuesday
- C Thursday
- D Friday
- E Saturday

Solution(s):

There are 4 days between Wednesday and Monday, so he can't fit all of his running days in that time without them being consecutive.

This means he runs on Tuesday. Then, all thats left is running, swimming, or tennis. He must play tennis on Thursday or else he plays tennis after swimming or running.

Therefore, he has Friday, Saturday, and Sunday to run twice and swim once. Since running isn't done on consecutive days, Madhi cannot run on Saturday, leaving that day free for swimming.

Thus, the correct answer is **E**.

7. Consider the operation "minus the reciprocal of," defined by $a \diamond b = a - \frac{1}{b}$. What is

$$((1 \diamond 2) \diamond 3) - (1 \diamond (2 \diamond 3))?$$

A $-\dfrac{7}{30}$

B $-\dfrac{1}{6}$

C 0

D $\dfrac{1}{6}$

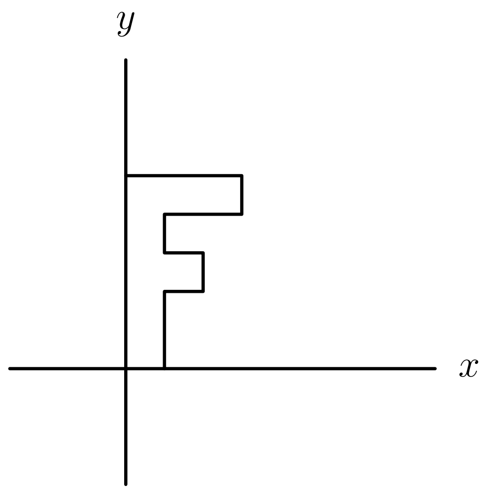
E $\dfrac{7}{30}$

Solution(s):

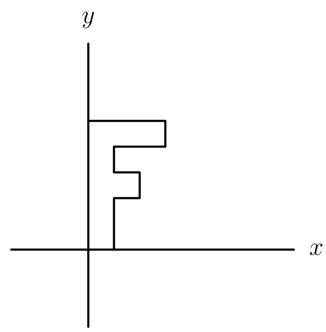
$$\begin{aligned}
&((1 \diamond 2) \diamond 3) - (1 \diamond (2 \diamond 3)) \\
&= \left(\frac{1}{2} - \frac{1}{3}\right) - \left(1 \diamond \frac{5}{3}\right) \\
&= \left(\frac{1}{2} - \frac{1}{3}\right) - \left(1 - \frac{3}{5}\right) \\
&= \frac{1}{6} - \frac{2}{5} \\
&= -\frac{7}{30}
\end{aligned}$$

Thus, the correct answer is **A**.

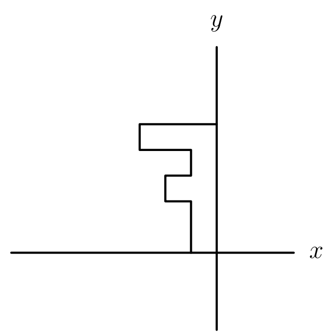
8. The letter F shown below is rotated 90° clockwise around the origin, then reflected in the y-axis, and then rotated a half turn around the origin. What is the final image?



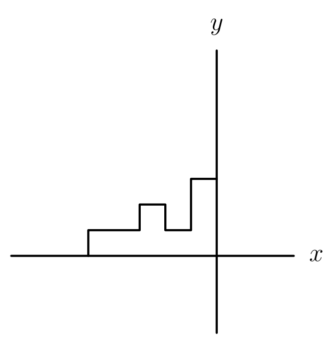
A



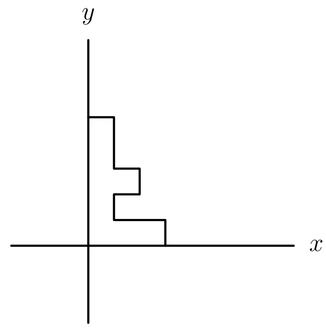
B



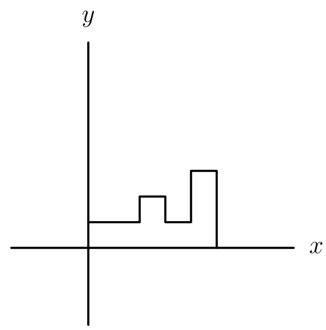
C



D



E



Solution(s):

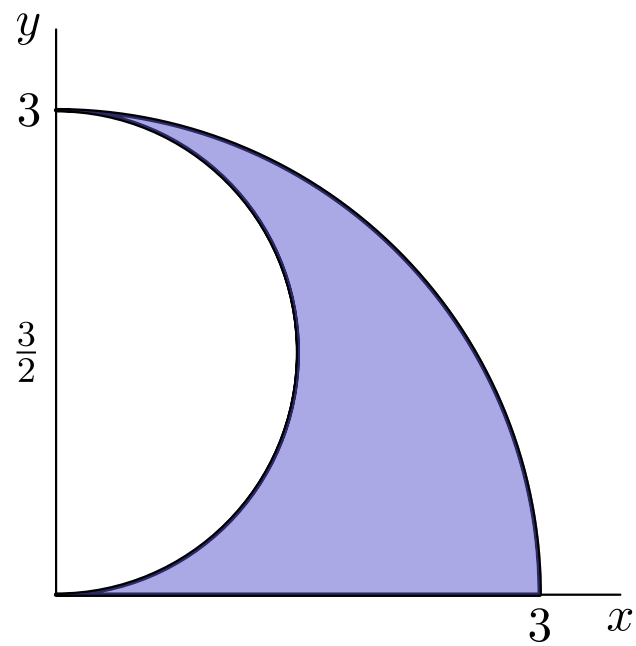
The rotation puts the F under the x-axis with its lines going to the right.

Then, note that a half turn is the same as reflecting upon both axes in any order, so it undoes the reflection upon the y-axis and reflects it upon the x-axis. This means the last two turns just reflects it upon the x-axis.

The reflection puts the F above the x-axis with its lines going to the right.

Thus, the correct answer is **E**.

9. The shaded region below is called a shark's fin falcata, a figure studied by Leonardo da Vinci. It is bounded by the portion of the circle of radius 3 and center $(0, 0)$ that lies in the first quadrant, the portion of the circle with radius $\frac{3}{2}$ and center $(0, \frac{3}{2})$ that lies in the first quadrant, and the line segment from $(0, 0)$ to $(3, 0)$. What is the area of the shark's fin falcata?



A $\frac{4\pi}{5}$

B $\frac{9\pi}{8}$

C $\frac{4\pi}{3}$

D $\frac{7\pi}{5}$

E $\frac{3\pi}{2}$

Solution(s):

The area of the portion of the circle of radius 3 in the first quadrant is equal to:

$$\frac{1}{4} \cdot \pi \cdot 3^2 = \frac{9}{4}\pi$$

Similarly, the are of the portion of the cicle with radius $\frac{3}{2}$ centered at $(0, \frac{3}{2})$ that lies in the first quadrant is equal to:

$$\frac{1}{2} \cdot \pi \cdot \left(\frac{3}{2}\right)^2 = \frac{9}{8}\pi$$

The area of the shark's fin falcata is equal to the difference of these regions:

$$\frac{9}{4}\pi - \frac{9}{8}\pi = \frac{9}{8}\pi$$

Thus, the correct answer is **B**.

10. What are the sign and units digit of the product of all the odd negative integers strictly greater than -2015?

- A It is a negative number ending with a 1.
- B It is a positive number ending with a 1.
- C It is a negative number ending with a 5.
- D It is a positive number ending with a 5.
- E It is a negative number ending with a 0.

Solution(s):

There are 1007 odd numbers greater than -2015.

Our product is of an odd number of negative numbers, so the result is negative.

Also, we multiply by -5 in there, so the product is a multiple of 5, making it end in 5 or 0. None of our factors are even, so the product can't be even.

Therefore, the product must end in 5.

Thus, the correct answer is **C**.

11. Among the positive integers less than 100, each of whose digits is a prime number, one is selected at random. What is the probability that the selected number is prime?

A $\frac{8}{99}$

B $\frac{2}{5}$

C $\frac{9}{20}$

D $\frac{1}{2}$

E $\frac{9}{16}$

Solution(s):

The only digits that are prime are $2, 3, 5$, and 7.

Our number can either be one or two digits. There are 4 one digit numbers that have its digits being prime, and 16 two digit numbers that have its digits being prime. This makes a total of 20.

Each of the one one digit numbers are prime. For the two digit numbers, if it ends in 2 or 5, then it isn't prime since it would be a multiple of 2 or 5.

Thus, we only need to check the numbers that end in 3 or 7.

The possible numbers are:

$$23, 27, 33, 37,$$

$$53, 57, 73, 77.$$

33 and 77 are multiples of 11, so they aren't prime.

Then, 27 and 57 are multiples of 3 so they aren't prime.

This leaves $23, 27, 53$, and 73 as the only two digit primes.

Therefore, there are 8 primes out of 20, making the probability $\frac{2}{5}$.

Thus, the correct answer is **B**.

12. For how many integers x is the point $(x, -x)$ inside or on the circle of radius 10 centered at $(5, 5)$?

A 11

B 12

C 13

D 14

E 15

Solution(s):

The distance between $(5, 5)$ and $(x, -x)$ is

$$\sqrt{(x-5)^2 + (x+5)^2}$$

$$= \sqrt{2x^2 + 50}.$$

We must find how many x are such that that value is less than or equal to 10. Therefoe,

$$2x^2 + 50 \leq 100$$

$$x^2 \leq 25.$$

$$-5 \leq x \leq 5.$$

Therefore, we have 11 included tintegers.

Thus, the correct answer is **A**.

13. The line $12x + 5y = 60$ forms a triangle with the coordinate axes. What is the sum of the lengths of the altitudes of this triangle?

A 20

B $\frac{360}{17}$

C $\frac{107}{5}$

D $\frac{43}{2}$

E $\frac{281}{13}$

Solution(s):

The triangle is a right triangle with legs of 12 and 5. This makes the hypotenuse 13.

Two of the altitudes are then 12 and 5. Also, for any side, $A = \frac{bh}{2}$ where b is the base and h is the altitude.

The area is $\frac{12 \cdot 5}{2} = 30$, so the other altitude h can be found with $30 = \frac{13h}{2}$. Thus, this altitude is $\frac{60}{13}$.

Therefore, the sum is

$$12 + 5 + \frac{60}{13} = \frac{281}{13}.$$

Thus, the correct answer is **E**.

14. Let a, b, and c be three distinct one-digit numbers. What is the maximum value of the sum of the roots of the equation

$$(x-a)(x-b)$$
$$+(x-b)(x-c)=0?$$

- A 15
- B 15.5
- C 16
- D 16.5
- E 17

Solution(s):

The equation is equal to

$$(2x-(a+c))(x-b)$$
$$=2(x-b)\left(x-\frac{a+c}{2}\right).$$

This makes the roots equal to:

$$b, \frac{a+c}{2}$$

and the sum is

$$\frac{2b+a+c}{2}.$$

Therefore, we want to maximize a, b, c, while making b the highest.

As such, we can have $b=9, a=8, c=7$ and get a sum:

$$9+7.5=16.5$$

Thus, the correct answer is **D**.

15. The town of Hamlet has 3 people for each horse, 4 sheep for each cow, and 3 ducks for each person. Which of the following could not possibly be the total number of people, horses, sheep, cows, and ducks in Hamlet?

A 41

B 47

C 59

D 61

E 66

Solution(s):

Let the number of horses be h and let the number of cows be c.

Then, the number of people is $3h$ and the number of ducks is $9h$.

Also, the total number of sheep is $4c$, so the number of total number of animals or people is $13h + 5c$.

If we take the total and subtract $5c$, then we get a multiple of 13. Thus, any valid number is such that there is a multiple of 13 that is lower than the number that has the same remainder when divided by 5.

For 41, the lowest multiple of 13 that has the same remainder when divided by 5 is 26, so this is a valid solution.

For 47, the lowest multiple of 13 that has the same remainder when divided by 5 is 52, **so this isn't valid solution.**

For 59, the lowest multiple of 13 that has the same remainder when divided by 5 is 39, so this is a valid solution.

For 61, the lowest multiple of 13 that has the same remainder when divided by 5 is 26, so this is a valid solution.

For 66, the lowest multiple of 13 that has the same remainder when divided by 5 is 26, so this is a valid solution.

Therefore, 47 cannot possible be the total.

Thus, the correct answer is **B**.

16. Al, Bill, and Cal will each randomly be assigned a whole number from 1 to 10, inclusive, with no two of them getting the same number. What is the probability that Al's number will be a whole number multiple of Bill's and Bill's number will be a whole number multiple of Cal's?

A $\frac{9}{1000}$

B $\frac{1}{90}$

C $\frac{1}{80}$

D $\frac{1}{72}$

E $\frac{2}{121}$

Solution(s):

Let their numbers be a, b, and c. Then, if b is a multiple of a and they aren't the same number, then $b \geq 2a$. Similarly, $c \geq 2b$.

Thus, $4a \leq c \leq 10$, making $a \leq 2.5$. Thus, $a = 1$ or $a = 2$. Also, $b \leq 0.5c \leq 5$, creating an upper bound of b.

Now, casing:

If $a = 2$, then $b \leq 5$ means $b = 4$. Then, $c = 8$. This makes exactly one case.

If $a = 1$, then $b \leq 5$ means $b = 2, 3, 4$ or 5.

If $b = 2$, then c can be one of $4, 6, 8, 10$ making 4 cases.

If $b = 3$, then c can be one of $6, 9$ making 2 cases.

If $b = 4$, then c can is 8 making 1 case.

If $b = 5$, then c can is 10 making 1 case.

Thus, $a = 1$ makes 8 cases.

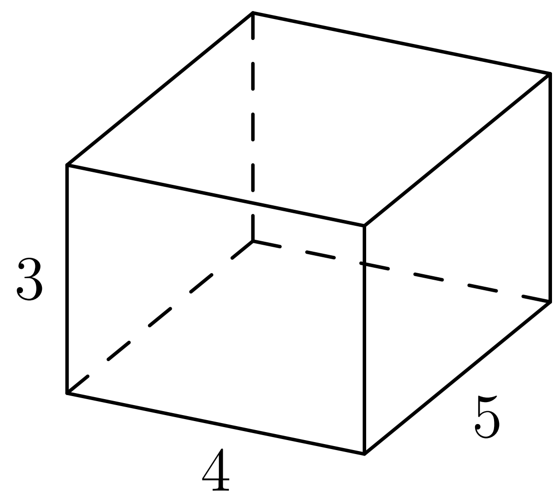
There are then 9 total values of a, b, c that work. The total number of pairs that are possible is $10 \cdot 9 \cdot 8 = 720$.

Therefore, the probability is

$$\frac{9}{720} = \frac{1}{80}.$$

Thus, the correct answer is **C**.

17. The centers of the faces of the right rectangular prism shown below are joined to create an octahedron. What is the volume of this octahedron?



A $\frac{75}{12}$

B 10

C 12

D $10\sqrt{2}$

E 15

Solution(s):

If we connect all the centers of the prism, we get an octohedron made of two different pyramids. The area of the base of the pyramid is half of the area of the rectangle it is parallel with. As such, the volume of each pyramid is $\frac{1}{3}Ah$, so the combined area is:

$$\frac{A(h_1 + h_2)}{3}.$$

The sum of the height is the full side length, so this scales the volume down by a factor of $\frac{1}{3}$.

Therefore, the ratio of the the volume of the prism to the octahedron is $\frac{1}{6}$. The volume of the prism is $3 \cdot 4 \cdot 5 = 60$, so the volume of the octohedron is 10.

Thus, the correct answer is **B**.

18. Johann has 64 fair coins. He flips all the coins. Any coin that lands on tails is tossed again. Coins that land on tails on the second toss are tossed a third time. What is the expected number of coins that are now heads?

- A 32
- B 40
- C 48
- D 56
- E 64

Solution(s):

A coin ends as tails if and only if it has 3 flips that are tails, which happens with probability $\frac{1}{8}$. Thus, the probability of any coin being heads is $\frac{7}{8}$.

As the probability that a given coin flip is $\frac{7}{8}$, and there are 64 coin flips in total, the expected number of coins that are now heads is:

$$64 \cdot \frac{7}{8} = 56.$$

Thus, the correct answer is **D**.

19. In $\triangle ABC$, $\angle C = 90^\circ$ and $AB = 12$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. What is the perimeter of the triangle?

A $12 + 9\sqrt{3}$

B $18 + 6\sqrt{3}$

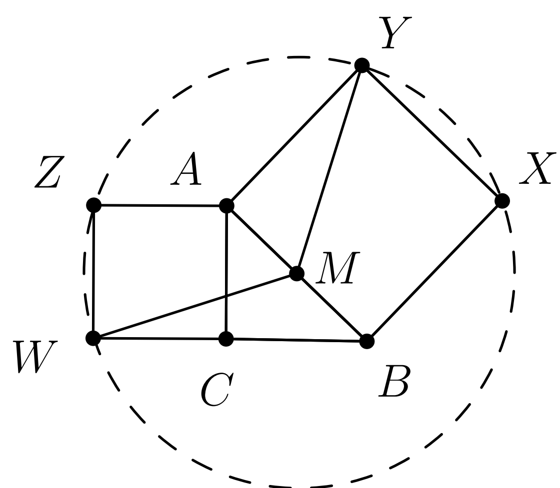
C $12 + 12\sqrt{2}$

D 30

E 32

Solution(s):

Consider the following diagram:



The points X, Y are on the circle, so the center goes through its perpendicular bisector, which is the same as the perpendicular bisector of A, B.

The points W, Z are on the circle, so the center goes through its perpendicular bisector, which is the same as the perpendicular bisector of A, C.

Thus, the center is on the perpendicular bisector of A, B and A, C, which is the circumcenter. The circumcenter of a right triangle is the midpoint of the hypotenuse, so it is the midpoint of AB.

Then, let M be the midpoint of AB. This would make

$$YM^2 = MA^2 + AY^2$$

$$= 6^2 + 12^2$$

$$= 180.$$

Therefore, the radius is $\sqrt{180}$, so $MW = \sqrt{180}$.

Then, the distance from M to W is also equal to:

$$\sqrt{\left(WC + \frac{BC}{2}\right)^2 + \left(\frac{AC}{2}\right)^2}$$

Which simplifies to equal

$$\sqrt{AC^2 + AC \cdot BC + 36.}$$

Therefore,

$$AC^2 + AC \cdot BC = 12$$

$$= AC^2 + BC^2.$$

Since $BC \neq 0$, we have $AC = BC$, making an isoceles right triangle. Therefore $AC = BC = 6\sqrt{2}$.

As such, the perimeter is

$$2 \cdot 6\sqrt{2} + 12$$

$$= 12\sqrt{2} + 12.$$

Thus, the correct answer is **C**.

20. Erin the ant starts at a given corner of a cube and crawls along exactly 7 edges in such a way that she visits every corner exactly once and then finds that she is unable to return along an edge to her starting point. How many paths are there meeting these conditions?

- A 6
- B 9
- C 12
- D 18
- E 24

Solution(s):

Suppose we have the unit cube where the points are ordered triples with coordinates that are 0 or 1.

Then, after an odd number of moves, the sum of the coordinates is odd. Therefore, the only valid last point is $(1, 1, 1)$.

Then, there are 3 choices for the first move and 2 for the second. These choices are all identical, so their next moves would be the same.

From here everything is forced to prevent us from going to $(1, 1, 1)$ until the last move, so there are just 6 moves.

Thus, the correct answer is **A**.

21. Cozy the Cat and Dash the Dog are going up a staircase with a certain number of steps. However, instead of walking up the steps one at a time, both Cozy and Dash jump.

Cozy goes two steps up with each jump (though if necessary, he will just jump the last step).

Dash goes five steps up with each jump (though if necessary, he will just jump the last steps if there are fewer than 5 steps left).

Suppose Dash takes 19 fewer jumps than Cozy to reach the top of the staircase. Let s denote the sum of all possible numbers of steps this staircase can have. What is the sum of the digits of s?

A 9

B 11

C 12

D 13

E 15

Solution(s):

The amount Cozy jumps is $\lceil \frac{s}{2} \rceil$. The amount Dash jumps is $\lceil \frac{s}{5} \rceil$. Thus, $\lceil \frac{s}{2} \rceil - \lceil \frac{s}{5} \rceil = 19$. As such, we can case on whether s is even or odd.

If it is even, then we have:

$$\frac{s}{2} - \left\lceil \frac{s}{5} \right\rceil = 19.$$

Which implies that,

$$\frac{s}{5} + 1 \geq \frac{s}{2} - 19 = \left\lceil \frac{s}{5} \right\rceil \geq \frac{s}{5}$$

Therefore,

$$20 \geq \frac{3s}{10} \geq 19.$$

Simplifying the inequality, we get

$$\frac{200}{3} \geq s = \frac{190}{3},$$

so the only possible even s are $66, 64$.

Both these cases satisfy the equation $\frac{s}{2} - \left\lceil \frac{s}{5} \right\rceil = 19$, so they are both valid answers.

If it is odd, then

$$\frac{s}{2} + \frac{1}{2} - \left\lceil \frac{s}{5} \right\rceil = 19.$$

Which implies that,

$$\frac{s}{5} + 19.5 \geq \frac{s}{2} = \left\lceil \frac{s}{5} \right\rceil \geq \frac{s}{5}$$

Therefore,

$$19.5 \geq \frac{3s}{10} = 18.5.$$

Simplifying the inequality, we get,

$$\frac{195}{3} \geq s = \frac{185}{3},$$

so the only possible odd s are $65, 63$.

Only 63 satisfies

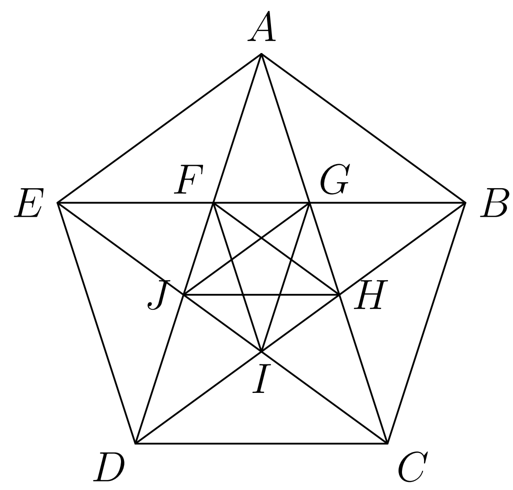
$$\frac{s}{2} + \frac{1}{2} - \left\lceil \frac{s}{5} \right\rceil = 19,$$

so it is the only valid answer.

Therefore, the sum of the answers is $63 + 64 + 66 = 193$, making the sum of the digits $1 + 9 + 3 = 13$.

Thus, the correct answer is **D**.

22. In the figure shown below, $ABCDE$ is a regular pentagon and $AG = 1$. What is $FG + JH + CD$?



A 3

B $12 - 4\sqrt{5}$

C $\dfrac{5 + 2\sqrt{5}}{3}$

D $1 + \sqrt{5}$

E $\dfrac{11 + 11\sqrt{5}}{10}$

Solution(s):

Due to rotational symmetry,

$$\angle EBD = \angle BDA = \angle JHD,$$

so DJH is isoceles. Thus, $JH = DJ$. This is equal to $AG = BG = 1$ by rotational symmetry.

Then, since $FD||BC$ and $BF||DC$, $FBCD$ is a parallelogram, making

$$\begin{aligned} DC &= BF \\ &= BG + FG \\ &= 1 + FG. \end{aligned}$$

Thus, we need to just find $2(1 + FG)$.

Since $AFG \sim AJH$,

$$\frac{AG}{AH} = \frac{FG}{JH}.$$

Therefore,

$$\frac{1}{FG+1} = FG.$$

This means $FG^2 + FG - 1 = 0$, so,

$$FG = \frac{-1+\sqrt{5}}{2}.$$

This means $2(1+FG) = 1+\sqrt{5}$.

Thus, the correct answer is **D**.

23. Let n be a positive integer greater than 4 such that the decimal representation of $n!$ ends in k zeros and the decimal representation of $(2n)!$ ends in $3k$ zeros. Let s denote the sum of the four least possible values of n. What is the sum of the digits of s?

A 7

B 8

C 9

D 10

E 11

Solution(s):

The number of zeros to end a number is equal to the power of 5 in the $n!$.

Since we are trying to find the lowest numbers, we can inspect small numbers where $n \leq 60$, so that the number of zeros to end $n!$ is

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor$$

and the number of zeros to end $2n!$ is

$$\left\lfloor \frac{2n}{5} \right\rfloor + \left\lfloor \frac{2n}{25} \right\rfloor.$$

This means

$$3\left(\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor\right)$$
$$= \left\lfloor \frac{2n}{5} \right\rfloor + \left\lfloor \frac{2n}{25} \right\rfloor.$$

Then, we can inspect each value of $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor$ as follows:

Case 1: If $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor = 0$, then $\left\lfloor \frac{n}{5} \right\rfloor = 0$ implying that $0 \leq n < 4$ which has no valid solutions.

Case 2: If $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor = 1$, then $\left\lfloor \frac{n}{5} \right\rfloor = 1$ so $5 \leq n < 10$.

Then, we also know

$$\left\lfloor \frac{2n}{5} \right\rfloor + \left\lfloor \frac{2n}{25} \right\rfloor = 3.$$

The only solutions are when $\left\lfloor \frac{2n}{5} \right\rfloor = 3$, so $15 \leq 2n < 20$, making $7.5 \leq n < 10$.

Thus, we have the solutions $n = 8, 9$.

Case 3: If $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor = 2$, then $\left\lfloor \frac{n}{5} \right\rfloor = 2$ so $10 \leq n < 15$.

Then, we also know

$$\left\lfloor \frac{2n}{5} \right\rfloor + \left\lfloor \frac{2n}{25} \right\rfloor = 6.$$

The only solutions are when $\left\lfloor \frac{2n}{5} \right\rfloor = 5$, so $25 \leq 2n < 30$, making $12.5 \leq n < 15$.

Thus, we have the solutions $n = 13, 14$.

Since $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor$, is non-decreasing, we found the four smallest solutions.

As such, their sum is $8 + 9 + 13 + 14 = 44$, making the sum of the digits 8.

Thus, the correct answer is **B**.

24. Aaron the ant walks on the coordinate plane according to the following rules.

He starts at the origin $p_0 = (0, 0)$ facing to the east and walks one unit, arriving at $p_1 = (1, 0)$.

For $n = 1, 2, 3, \ldots$, right after arriving at the point p_n, if Aaron can turn 90° left and walk one unit to an unvisited point p_{n+1}, he does that. Otherwise, he walks one unit straight ahead to reach p_{n+1}. Thus the sequence of points continues

$$p_2 = (1, 1),$$
$$p_3 = (0, 1),$$
$$p_4 = (-1, 1),$$
$$p_5 = (-1, 0),$$
$$\vdots$$

and so on in a counterclockwise spiral pattern. What is p_{2015}?

A $(-22, -13)$

B $(-13, -22)$

C $(-13, 22)$

D $(13, -22)$

E $(22, -13)$

Solution(s):

When walking around, Aaron always walks in a counter-clockwise spiral centered at $(0, 0)$.

Thus, he always makes a square length s in as few points as possible, which would be $(s+1)^2$. Thus, after $45^2 = 2025$ points, he would make a square of side length 44. This would have corners at the points

$$(\pm 22, \pm 22).$$

Since it is a counter-clockwise spiral, the square the 2025th point is at $(22, -22)$. Note that p_0 is one of the points, so

$$p_{2024} = (22, -22).$$

Then, since we have to go back 9 points and it is a counter-clockwise spiral, we subtract 9 from the x-value, making

$$p_{2015} = (13, -22).$$

Thus, the correct answer is **D**.

25. A rectangular box measures $a \times b \times c$, where a, b, and c are integers and

$$1 \leq a \leq b \leq c.$$

The volume and the surface area of the box are numerically equal. How many ordered triples (a, b, c) are possible?

A 4

B 10

C 12

D 21

E 26

Solution(s):

Our problem statment is equivalent to the number of solutions for

$$abc = 2(ab + bc + ac).$$

Then,

$$abc = 2(ab + bc + ac) \leq 6bc,$$

so $a \leq 6$.

Also,

$$abc = 2(ab + bc + ac) > 2bc,$$

so $a > 2$, so $2 < a \leq 6$.

We case on a as follows:

If $a = 3$, we have

$$3bc = 2(3b + 3c + bc),$$

so

$$bc - 6b - 6c = 0.$$

Thus,

$$(b-6)(c-6)=36,$$

which has 5 solutions since 36 has 9 factors, making 5 of them less than or equal to 6 and thus possible values for b.

If $a=4$, we have

$$4bc=2(4b+4c+bc),$$

so

$$bc-4b-4c=0.$$

Thus,

$$(b-4)(c-4)=16,$$

which has 3 solutions since 16 has 5 factors, making 3 of them less than or equal to 4 and thus possible values for b.

If $a=5$, we have

$$5bc=2(5b+5c+bc),$$

so

$$3bc-10b-10c=0.$$

Thus,

$$(3b-10)(3c-10)=100,$$

which has 1 solution since we can only get $(a,b,c)=(5,5,10)$ since $b\geq a$.

If $a=6$, we have

$$6bc=2(6b+6c+bc),$$

so

$$bc-3b-3c=0.$$

Thus,

$$(b-3)(c-3)=9,$$

which has 1 solution since we can only get $(a,b,c)=(6,6,6)$ since $b\geq a$. Thus, we have 10 solutions.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc10/2015B