2015 AMC 10A Solutions

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1. What is the value of

$$(2^0 - 1 + 5^2 - 0)^{-1} \times 5?$$

A -125

B -120

C $\frac{1}{5}$

D $\frac{5}{24}$

E 25

Solution(s):

We can evaluate it as follows.

$$\begin{aligned} &(2^0 - 1 + 5^2 - 0)^{-1} \times 5 \\ &= \frac{1}{1 - 1 + 25} \times 5 \\ &= \frac{5}{25} \\ &= \frac{1}{5} \end{aligned}$$

Thus, **C** is the correct answer.

2. A box contains a collection of triangular and square tiles. There are 25 tiles in the box, containing 84 edges total. How many square tiles are there in the box?

A 3

B 5

C 7

D 9

E 11

Solution(s):

Let x be the number of triangular tiles and y be the number of square tiles. We then have that

$$x + y = 25$$

and

$$3x + 4y = 84.$$

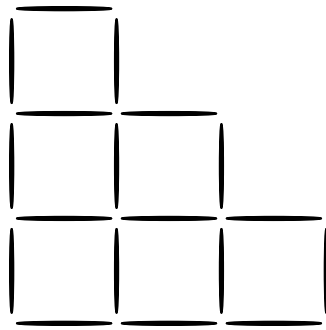
Multiplying the first equation by 3 gives us

$$3x + 3y = 75$$

and subtracting this from the other equation gives us $y = 9$.

Thus, **D** is the correct answer.

3. Ann made a 3-step staircase using 18 toothpicks as shown in the figure. How many toothpicks does she need to add to complete a 5-step staircase?



- A 9
- B 18
- C 20
- D 22
- E 24

Solution(s):

Let us try to find a pattern between the number of toothpicks needed for the staircases.

For a 1-step staircase, we would only need 4 toothpicks (just a square).

For a 2-step staircase, we would need 10 toothpicks according to the diagram.

Similarly, we would need 18 toothpicks for a 3-step staircase.

A 2-step staircase needs $10 - 4 = 6$ more toothpicks than a 1-step staircase. A 3-step staircase needs $18 - 10 = 8$ more toothpicks than a 2-step staircase.

Following this pattern, we can see that a 4-step staircase will need $18 + 10 = 28$ toothpicks, and a 5-step staircase will need $28 + 12 = 40$ toothpicks.

This means that Ann would need to add $40 - 18 = 22$ more toothpicks.

Thus, **D** is the correct answer.

4. Pablo, Sofia, and Mia got some candy eggs at a party. Pablo had three times as many eggs as Sofia, and Sofia had twice as many eggs as Mia. Pablo decides to give some of his eggs to Sofia and Mia so that all three will have the same number of eggs. What fraction of his eggs should Pablo give to Sofia?

A $\frac{1}{12}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution(s):

Let m be the number of candy eggs that Mia had. Then $Sofia$ had $2m$ eggs and $Pablo$ had $6m$ eggs.

The total number of eggs is then

$$m + 2m + 6m = 9m.$$

For all of them to have the same number of eggs, they each must have $9m \div 3 = 3m$ eggs.

Sofia needs $3m - 2m = m$ more eggs. This means Pablo must give $\frac{m}{6m} = \frac{1}{6}$ of his eggs to Sofia.

Thus, **B** is the correct answer.

5. Mr. Patrick teaches math to 15 students. He was grading tests and found that when he graded everyone's test except Payton's, the average grade for the class was 80. After he graded Payton's test, the test average became 81. What was Payton's score on the test?

- A 81
- B 85
- C 91
- D 94
- E 95

Solution(s):

The class's total score with Payton's test is

$$14 \cdot 80 = 1120.$$

Including Payton's test, the total score goes up to

$$15 \cdot 81 = 1215.$$

This means that Payton got a

$$1215 - 1120 = 95$$

on his test.

Thus, **E** is the correct answer.

6. The sum of two positive numbers is 5 times their difference. What is the ratio of the larger number to the smaller number?

A $\frac{5}{4}$

B $\frac{3}{2}$

C $\frac{9}{5}$

D 2

E $\frac{5}{2}$

Solution(s):

Let x and y be the two numbers. Then we have that

$$x + y = 5(x - y).$$

Note that we are assuming $x > y$. This gives us

$$x + y = 5x - 5y$$

$$6y = 4x.$$

Dividing through yields

$$\frac{x}{y} = \frac{6}{4} = \frac{3}{2}.$$

Thus, **B** is the correct answer.

7. How many terms are in the arithmetic sequence $13, 16, 19, \ldots, 70, 73$?

- A 20
- B 21
- C 24
- D 60
- E 61

Solution(s):

Recall that the n th term of an arithmetic sequence is $a+d(n-1)$, where a is the first term and d is the common difference.

For us, $a=13$ and $d=3$. Plugging these in, we get that

$$\begin{aligned} 73 &= 13+3(n-1) \\ 20 &= n-1 \\ n &= 21. \end{aligned}$$

Thus, **B** is the correct answer.

8. Two years ago Pete was three times as old as his cousin Claire. Two years before that, Pete was four times as old as Claire. In how many years will the ratio of their ages be $2:1$?

A 2

B 4

C 5

D 6

E 8

Solution(s):

Let p and c be Pete's and Claire's current ages respectively.

Then we have that

$$p-2=3(c-2)$$

and

$$p-4=4(c-4).$$

Simplifying both equations gives us

$$p=3c-4$$

and

$$p=4c-12.$$

Setting them equal, we have

$$3c-4=4c-12$$

$$c=8.$$

This means that

$$p=3\cdot 8-4=20.$$

Now, we need to find the number of years (y) until

$$20 + y = 2(8 + y),$$

which gives us

$$20 + y = 16 + 2y$$

$$y = 4.$$

Thus, **B** is the correct answer.

9. Two right circular cylinders have the same volume. The radius of the second cylinder is 10% more than the radius of the first. What is the relationship between the heights of the two cylinders?

- A The second height is 10% less than the first.
- B The first height is 10% more than the second.
- C The second height is 21% less than the first.
- D The first height is 21% more than the second.
- E The second height is 80% of the first.

Solution(s):

Let r_1 and h_1 be the radius and height of the first cylinder and similarly define r_2 and h_2 for the second cylinder.

We know that

$$r_2 = \frac{11}{10} r_1$$

and

$$\pi r_1^2 h_1 = \pi r_2^2 h_2.$$

Substituting and simplifying gives us

$$r_1^2 h_1 = \frac{121}{100} r_1^2 h_2,$$

which tells us that

$$h_1 = \frac{121}{100} h_2.$$

Thus, **D** is the correct answer.

10. How many rearrangements of $abcd$ are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either ab or ba.

A 0

B 1

C 2

D 3

E 4

Solution(s):

The initial step would be to choose a letter that could work and build from there.

For instance, if we begin with a, the only letters that can be placed next to it are c or d.

From here, we cannot proceed further since the combinations $acbd$ and $acdb$ are not allowed due to the presence of cb and cd, respectively.

The same issue arises if we start with d, as a b would have to be placed in the middle and end up being adjacent to either an a or a c.

On the other hand, if we start with a b, the next letter would have to be d, and after that, an a and a c can be placed, making this arrangement possible.

The same methodology holds true for starting with a c. This gives us a total of 2 rearrangements.

Thus, **C** is the correct answer.

11. The ratio of the length to the width of a rectangle is $4:3$. If the rectangle has diagonal of length d, then the area may be expressed as kd^2 for some constant k. What is k?

A $\frac{2}{7}$

B $\frac{3}{7}$

C $\frac{12}{25}$

D $\frac{16}{25}$

E $\frac{3}{4}$

Solution(s):

Let the side lengths be $4x$ and $3x$. Then the diagonal has length

$$\sqrt{(4x)^2+(3x)^2}=\sqrt{25x^2}=5x.$$

The area of the rectangle is

$$4x\cdot 3x=12x^2.$$

Then we get that

$$kd^2=12x^2$$

$$k=\frac{12x^2}{25x^2}=\frac{12}{25}.$$

Thus, **C** is the correct answer.

12. Points $(\sqrt{\pi}, a)$ and $(\sqrt{\pi}, b)$ are distinct points on the graph of

$$y^2 + x^4 = 2x^2y + 1.$$

What is $|a - b|$?

A 1

B $\frac{\pi}{2}$

C 2

D $\sqrt{1+\pi}$

E $1 + \sqrt{\pi}$

Solution(s):

Plugging in $\sqrt{\pi}$ into the equation, we get that

$$y^2 + \pi^2 = 2\pi y + 1,$$

which can be rearranged to get

$$\begin{aligned} y^2 - 2\pi y + \pi^2 &= 1 \\ (y - \pi)^2 &= 1 \\ y - \pi &= \pm 1 \\ y &= \pi \pm 1. \end{aligned}$$

This gives us the values for a and b. We then get that

$$|a - b| = |\pi + 1 - \pi + 1| = 2.$$

Thus, **C** is the correct answer.

13. Claudia has 12 coins, each of which is a 5-cent coin or a 10-cent coin. There are exactly 17 different values that can be obtained as combinations of one or more of his coins. How many 10-cent coins does Claudia have?

A 3

B 4

C 5

D 6

E 7

Solution(s):

Let the number of 5-cent coins be x and the number of 10-cent coins be $12 - x$.

Then we have that any multiple of 5 between 5 and

$$5x + 10(12 - x) = 120 - 5x$$

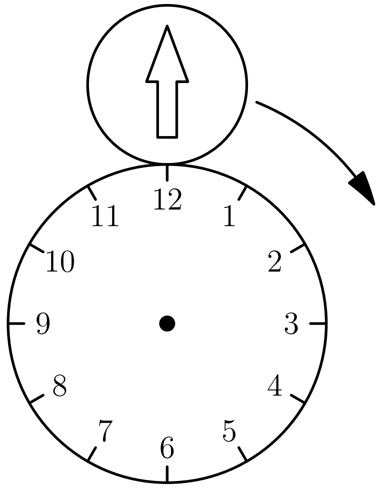
can be achieved by a combination of coins.

There are $24 - x$ such multiples of 5, which means that $x = 7$ to get 17 possible different values.

The number of 10-cent coins is therefore $12 - 7 = 5$.

Thus, **C** is the correct answer.

14. The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock face at 12 o' clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. Let the disk roll clockwise around the clock face. At what point on the clock face will the disk be tangent when the arrow is next pointing in the upward vertical direction?



A 2 o' clock

B 3 o' clock

C 4 o' clock

D 6 o' clock

E 8 o' clock

Solution(s):

Note that the circumference of the clock is twice the circumference of the disk.

This means that at $3,$ the disk would have traveled half way around, which means that the arrow would be pointing left.

This is because the tip of the arrow is half way around, and that is the point that is tangent to the clock.

Now, we see that in 3 hours the disk rotated $\frac{3}{4}$ of the way.

To get the final quarter rotation, the disk needs to travel $\frac{3}{3} = 1$ more hour.

Thus, **C** is the correct answer.

15. Consider the set of all fractions $\frac{x}{y}$, where x and y are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by 1, the value of the fraction is increased by 10%?

- A 0
- B 1
- C 2
- D 3
- E infinitely many

Solution(s):

We get that

$$\frac{x+1}{y+1} = \frac{11x}{10y}.$$

Cross-multiplying and simplifying yields

$$xy + 11x - 10y = 0.$$

We can add -110 to add side and then factor to get

$$(x-1)(y+11) = -110.$$

We are restricted by $x > 0$ and $y > 0$ to get $x - 10 > 10$ and $y + 11 > 11.$

The only factor pairs of -110 that satisfy the above constraints are

$$(-1, 110), (-2, 55),$$

and

$$(-5, 22).$$

Solving these get the following (x, y) pairs:

$$(9, 99), (8, 44), (5, 11).$$

Only one of these pairs gives values for x and y that are relatively prime.

Thus, **B** is the correct answer.

16. If we have that

$$y + 4 = (x - 2)^2,$$

$$x + 4 = (y - 2)^2,$$

and $x \neq y$, what is the value of

$$x^2 + y^2?$$

A 10

B 15

C 20

D 25

E 30

Solution(s):

Adding the two equations gives us

$$x^2 + y^2 - 4x - 4y + 8$$

$$= x + y + 8.$$

We can rearrange this equation to get

$$x^2 + y^2 = 5(x + y).$$

We can then subtract them to get

$$x^2 - y^2 - 4x + 4y = y - x.$$

Once again rearranging, we can find

$$x^2 - y^2 = 3(x - y).$$

We have that $x \neq y$, which means that we can divide both sides by $x - y$. This gives us

$$x + y = 3$$

$$x^2 + y^2 = 15.$$

Thus, **B** is the correct answer.

17. A line that passes through the origin intersects both the line $x = 1$ and the line

$$y = 1 + \frac{\sqrt{3}}{3}x.$$

The three lines create an equilateral triangle. What is the perimeter of the triangle?

A $2\sqrt{6}$

B $2 + 2\sqrt{3}$

C 6

D $3 + 2\sqrt{3}$

E $6 + \frac{\sqrt{3}}{3}$

Solution(s):

Since one of the sides of the equilateral triangle is a vertical line, the line of symmetry perpendicular to this side must be horizontal.

This means that the slope of the third side must be opposite the slope of the second side, which would be $-\frac{\sqrt{3}}{3}$.

To find the perimeter, we only need to find the length of one of the sides of the triangle.

We can plug in $x = 1$ into the two other equations to get the two vertices on the vertical line.

The two y-values are then

$$1 + \frac{\sqrt{3}}{3} \text{ and } -\frac{\sqrt{3}}{3}.$$

The distance between the two is $1 + \frac{2\sqrt{3}}{3}$, which makes the perimeter

$$3 \cdot \left(1 + \frac{2\sqrt{3}}{3}\right) = 3 + 2\sqrt{3}.$$

Thus, **D** is the correct answer.

18. Hexadecimal (base-16) numbers are written using numeric digits 0 through 9 as well as the letters A through F to represent 10 through 15. Among the first 1000 positive integers, there are n whose hexadecimal representation contains only numeric digits. What is the sum of the digits of n?

A 17

B 18

C 19

D 20

E 21

Solution(s):

Note that 1000 converted to hexadecimal is $3E8$. Now we need to count the number of numbers that have only numerical digits in their hexadecimal.

The first digit can be $0, 1, 2$ or 3. The second and third digits can be any number from $0 - 9$. This gives us

$$4 \cdot 10 \cdot 10 = 400$$

numbers. This, however, includes 0, which is not a positive integer so we have to subtract one.

The sum of the digits in 399 is 21.

Thus, **E** is the correct answer.

19. The isosceles right triangle ABC has right angle at C and area 12.5. The rays trisecting $\angle ACB$ intersect AB at D and E. What is the area of $\triangle CDE$?

A $\dfrac{5\sqrt{2}}{3}$

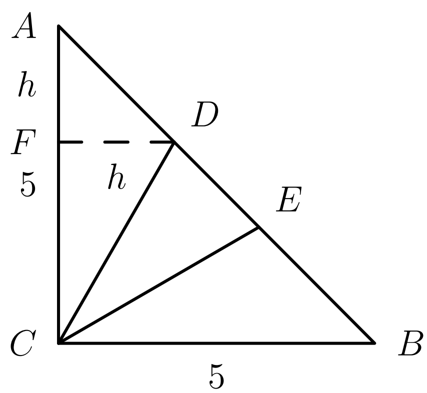
B $\dfrac{50\sqrt{3}-75}{4}$

C $\dfrac{15\sqrt{3}}{8}$

D $\dfrac{50-25\sqrt{3}}{2}$

E $\dfrac{25}{6}$

Solution(s):



Note that $\angle ACD = 30^\circ$ since it trisects a right angle. This means that we can split $\triangle ACD$ into a $45-45-90$ right triangle and a $30-60=90$ right triangle.

We have that $AF = DF$ since $\triangle AFD$ is isosceles. Using this, we have that

$$\frac{CF}{DF} = \frac{5-h}{h} = \frac{\sqrt{3}}{1}.$$

Cross-multiplying gives us

$$5-h = h\sqrt{3}$$

$$h = \frac{5}{1+\sqrt{3}}.$$

Rationalizing the denominator gives us

$$\begin{aligned} h &= \frac{5}{1+\sqrt{3}} \cdot \frac{\sqrt{3}-1}{\sqrt{3}-1} \\ &= \frac{5\sqrt{3}-5}{2}. \end{aligned}$$

The area of $\triangle ACD$ is then

$$\frac{1}{2} \cdot 5 \cdot \frac{5\sqrt{3}-5}{2} = \frac{25\sqrt{3}-25}{4}.$$

Note that the area of $\triangle ABC$ is $\frac{25}{2}$. We also have that

$$[ACD] = [BCE].$$

Finally, we get that

$$\begin{aligned} [CDE] &= [ABC] - 2 \cdot [ACD] \\ &= \frac{25}{2} - \frac{25\sqrt{3}-25}{2} \\ &= \frac{50-25\sqrt{3}}{2}. \end{aligned}$$

Thus, **D** is the correct answer.

20. A rectangle with positive integer side lengths in cm has area $A\,\mathrm{cm}^2$ and perimeter $P\,\mathrm{cm}$. Which of the following numbers cannot equal $A+P$?

- A 100
- B 102
- C 104
- D 106
- E 108

Solution(s):

Let x and y be the side lengths of the rectangle. Then we have that

$$A = xy \text{ and } P = 2(x+y).$$

Adding, we get

$$\begin{aligned} A+P &= xy + 2(x+y) \\ &= (x+2)(y+2) - 4. \end{aligned}$$

This means that $A+P-4$ must be the product of two numbers which are both greater than 2.

The only answer choice that cannot be expressed as such is 102.

Thus, **B** is the correct answer.

21. Tetrahedron $ABCD$ has $AB = 5$, $AC = 3$, $BC = 4$, $BD = 4$, $AD = 3$, and $CD = \frac{12}{5}\sqrt{2}$. What is the volume of the tetrahedron?

A $3\sqrt{2}$

B $2\sqrt{5}$

C $\dfrac{24}{5}$

D $3\sqrt{3}$

E $\dfrac{24}{5}\sqrt{2}$

Solution(s):

We claim that triangles ABC and ABD are perpendicular to each other.

We can show this be dropping the altitudes from C to AB and from D to AB in each triangle.

Since $AC = AD$ and $BC = BD$, we have that the feet of these altitudes will coincide at point P.

Then we have that

$$CP = DP = \frac{3 \cdot 4}{5} = \frac{12}{5}.$$

We then have that $CD = CP\sqrt{2}$, which shows that CPD is an isosceles right triangle.

This proves the above claim. Finally, the volume of the tetrahedron is

$$\frac{1}{3}[ABD] \cdot CP = \frac{6}{3} \cdot \frac{12}{5} = \frac{24}{5}.$$

Thus, **C** is the correct answer.

22. Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

A $\frac{47}{256}$

B $\frac{3}{16}$

C $\frac{49}{256}$

D $\frac{25}{128}$

E $\frac{51}{256}$

Solution(s):

We can case on the number of people standing.

Case $1 : 0$ people standing

There is only 1 way for this to happen: everyone sits.

Case $2 : 1$ person stands

There are 8 choices for which person stands.

Case $3 : 2$ people stand

There are $\binom{8}{2} = 28$ ways to choose the pair of people that stand. There are, however, 8 pairs of adjacent people, so we have to subtract those out for a total of $28 - 8 = 20$ configurations.

Case $4 : 3$ people stand

There are 8 choices for the first person. This rules out the person standing and their two neighbors, leaving 5 people to choose from.

There are $\binom{5}{2} = 10$ ways to choose the other 2 people, but 4 of these pairs involve adjacent people standing.

Therefore, there are a total of $8 \cdot 6 = 48$ arrangements for this case. Note, however, that we have divide by 3 since we overcounted which person we choose first.

This case only has $48 \div 3 = 16$ configurations then.

Case 5 : 4 people stand

There are only 2 choices depending on which half of the people stand up.

These cases together contribute

$$1+8+20+16+2=47$$

configurations. There are $2^{8}=256$ possibilities for the coin flips, which makes the desired probability $\dfrac{47}{256}$.

Thus, **A** is the correct answer.

23. The zeroes of the function

$$f(x) = x^2 - ax + 2a$$

are integers. What is the sum of the possible values of a?

A 7

B 8

C 16

D 17

E 18

Solution(s):

Let the zeroes be r and s. Using Vieta's formulas, we have that $a = r + s$ and $2a = rs$. Then we get that

$$rs = 2(r+s),$$

which rearranges to

$$rs - 2r - 2s = 0$$

$$rs - 2r - 2s + 4 = 4$$

$$(r-2)(s-2) = 4.$$

The only possible pairs $(r-2, s-2)$ that work are

$$(1,4), (-1,-4), (4,1), (-4,-1),$$

$$(2,2), (-2,-2).$$

For any of these pairs, we have that

$$a = r - 2 + s - 2 + 4.$$

We want all the such unique values of a. We get that they are

$$-1, 0, 8, 9.$$

The sum of these values is $-1 + 8 + 9 = 16.$

Thus, **C** is the correct answer.

24. For some positive integers p, there is a quadrilateral $ABCD$ with positive integer side lengths, perimeter p, right angles at B and C, $AB = 2$, and $CD = AD$. How many different values of $p < 2015$ are possible?

A 30

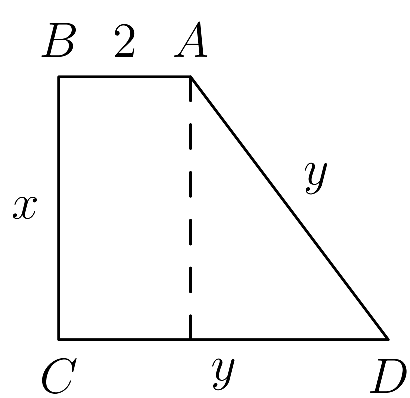
B 31

C 61

D 62

E 63

Solution(s):



Drop the altitude from A down to $\overline{CD}$. Then we have that

$$(y-2)^2 + x^2 = y^2,$$

which simplifies to

$$x^2 = 4(y-1).$$

Since x and y are both integers, we have that y must be one more than a perfect square.

We know that

$$p = 2 + 2y + 2\sqrt{y-1}.$$

We have that $p < 2015$. Guessing and checking tells us that $y = 31^2+!$ is the max value, whereas $y = 1^1 + 1$ is the minimum value.

This means that there are 31 values of y for which all the problem constraints are satisfied.

Thus, **B** is the correct answer.

25. Let S be a square of side length 1. Two points are chosen independently at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is $\frac{a-b\pi}{c}$, where $a, b,$ and c are positive integers with $\gcd(a,b,c)=1$. What is $a+b+c$?

A 59

B 60

C 61

D 62

E 63

Solution(s):

Fix one of the points. Then the probability the other point is on the same side is $\frac{1}{4}$.

The probability that the other point is on an adjacent side is $\frac{1}{2}$ and $\frac{1}{4}$ for the opposite side.

Case 1 : the other point is one the same side

Let the two points be a and b. If we view them as points in the unit square, then we see that the area such that

$$|a-b| > \frac{1}{2}$$

forms a triangle with area

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Case 2 : the points are on adjacent sides

WLOG, let the two points be on the bottom and left sides. Then the two points are $(0, a)$ and $(b, 0)$.

The distance between the two points is $\sqrt{a^2+b^2}$. We want this to be greater than $\frac{1}{2}$.

When graphed as above, we get that the points fill out the unit square except for a quarter circle of radius $\frac{1}{2}$.

This means that the probability that the distance is greater than $\frac{1}{2}$ is

$$1-\frac{1}{4}\cdot\pi\cdot\frac{1}{2}^{2}=1-\frac{\pi}{16}.$$

Case 3 : the points are on opposite sides

The length will always be greater than $\frac{1}{2}$, which means that the probability is 1.

The total probability is therefore

$$\begin{aligned}&\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\left(1-\frac{\pi}{16}\right)+\frac{1}{4}\cdot 1\\ &=\frac{26-\pi}{32}.\end{aligned}$$

The desired sum is

$$26+1+32=59.$$

Thus, **A** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2015A