2015 AMC 10A Solutions

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https://live.poshenloh.com/past-contests/amc10/2015A/solutions

1. What is the value of

A

B

C

D

E

Solution(s):

We can evaluate it as follows.

Thus, **C** is the correct answer.

2. A box contains a collection of triangular and square tiles. There are tiles in the box, containing edges total. How many square tiles are there in the box?

A

B

C

D

E

Solution(s):

Let be the number of triangular tiles and be the number of square tiles. We then have that

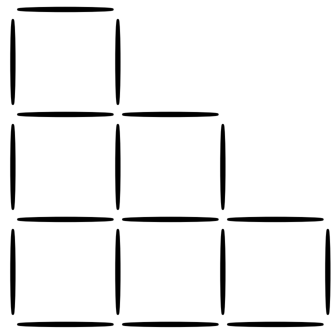
and

Multiplying the first equation by gives us

and subtracting this from the other equation gives us

Thus, **D** is the correct answer.

3. Ann made a -step staircase using toothpicks as shown in the figure. How many toothpicks does she need to add to complete a -step staircase?



A

B

C

D

E

Solution(s):

Let us try to find a pattern between the number of toothpicks needed for the staircases.

For a -step staircase, we would only need toothpicks (just a square).

For a -step staircase, we would need toothpicks according to the diagram.

Similarly, we would need toothpicks for a -step staircase.

A -step staircase needs more toothpicks than a -step staircase. A -step staircase needs more toothpicks than a -step staircase.

Following this pattern, we can see that a -step staircase will need toothpicks, and a -step staircase will need toothpicks.

This means that Ann would need to add more toothpicks.

Thus, **D** is the correct answer.

4. Pablo, Sofia, and Mia got some candy eggs at a party. Pablo had three times as many eggs as Sofia, and Sofia had twice as many eggs as Mia. Pablo decides to give some of his eggs to Sofia and Mia so that all three will have the same number of eggs. What fraction of his eggs should Pablo give to Sofia?

A —

B —

C —

D —

E —

Solution(s):

Let be the number of candy eggs that Mia had. Then had eggs and had eggs.

The total number of eggs is then

For all of them to have the same number of eggs, they each must have eggs.

Sofia needs more eggs. This means Pablo must give — — of his eggs to Sofia.

Thus, **B** is the correct answer.

5. Mr. Patrick teaches math to students. He was grading tests and found that when he graded everyone's test except Payton's, the average grade for the class was After he graded Payton's test, the test average became What was Payton's score on the test?

A

B

C

D

E

Solution(s):

The class's total score with Payton's test is

Including Payton's test, the total score goes up to

This means that Payton got a

on his test.

Thus, **E** is the correct answer.

6. The sum of two positive numbers is times their difference. What is the ratio of the larger number to the smaller number?

A —

B —

C —

D

E —

Solution(s):

Let and be the two numbers. Then we have that

Note that we are assuming This gives us

Dividing through yields

— — —

Thus, **B** is the correct answer.

7. How many terms are in the arithmetic sequence

A

B

C

D

E

Solution(s):

Recall that the th term of an arithmetic sequence is where is the first term and is the common difference.

For us, and Plugging these in, we get that

Thus, **B** is the correct answer.

8. Two years ago Pete was three times as old as his cousin Claire. Two years before that, Pete was four times as old as Claire. In how many years will the ratio of their ages be : ?

A

B

C

D

E

Solution(s):

Let and be Pete's and Claire's current ages respectively.

Then we have that

and

Simplifying both equations gives us

and

Setting them equal, we have

This means that

Now, we need to find the number of years ( ) until which gives us

Thus, **B** is the correct answer.

9. Two right circular cylinders have the same volume. The radius of the second cylinder is more than the radius of the first. What is the relationship between the heights of the two cylinders?

- A The second height is less than the first.
- B The first height is more than the second.
- C The second height is less than the first.
- D The first height is more than the second.
- E The second height is of the first.

Solution(s):

Let and be the radius and height of the first cylinder and similarly define and for the second cylinder.

We know that

and

Substituting and simplifying gives us

which tells us that

Thus, **D** is the correct answer.

10. How many rearrangements of are there in which no two adjacent letters are also adjacent letters in the alphabet? For example, no such rearrangements could include either or

A

B

C

D

E

Solution(s):

The initial step would be to choose a letter that could work and build from there.

For instance, if we begin with the only letters that can be placed next to it are or

From here, we cannot proceed further since the combinations and are not allowed due to the presence of and respectively.

The same issue arises if we start with as a would have to be placed in the middle and end up being adjacent to either an or a

On the other hand, if we start with a the next letter would have to be and after that, an and a can be placed, making this arrangement possible.

The same methodology holds true for starting with a This gives us a total of rearrangements.

Thus, **C** is the correct answer.

11. The ratio of the length to the width of a rectangle is : If the rectangle has diagonal of length then the area may be expressed as for some constant What is

A —

B —

C —

D —

E —

Solution(s):

Let the side lengths be and Then the diagonal has length

The area of the rectangle is

Then we get that

Thus, **C** is the correct answer.

12. Points and are distinct points on the graph of

What is

A

B

C

D

E

Solution(s):

Plugging in into the equation, we get that

which can be rearranged to get

This gives us the values for and We then get that

Thus, **C** is the correct answer.

13. Claudia has coins, each of which is a -cent coin or a -cent coin. There are exactly different values that can be obtained as combinations of one or more of his coins. How many -cent coins does Claudia have?

A

B

C

D

E

Solution(s):

Let the number of -cent coins be and the number of -cent coins be

Then we have that any multiple of between and

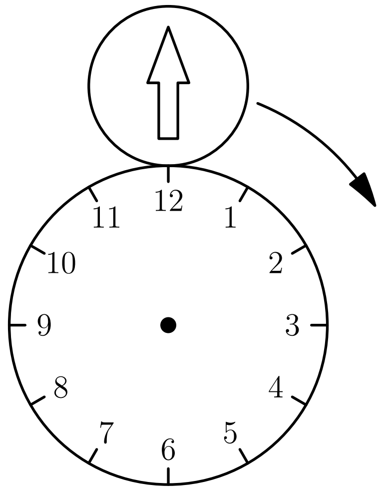
can be achieved by a combination of coins.

There are such multiples of which means that to get possible different values.

The number of -cent coins is therefore

Thus, **C** is the correct answer.

14. The diagram below shows the circular face of a clock with radius cm and a circular disk with radius cm externally tangent to the clock face at o′ clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. Let the disk roll clockwise around the clock face. At what point on the clock face will the disk be tangent when the arrow is next pointing in the upward vertical direction?



A o′ clock

B o′ clock

C o′ clock

D o′ clock

E o′ clock

Solution(s):

Note that the circumference of the clock is twice the circumference of the disk.

This means that at the disk would have traveled half way around, which means that the arrow would be pointing left.

This is because the tip of the arrow is half way around, and that is the point that is tangent to the clock.

Now, we see that in hours the disk rotated — of the way.

To get the final quarter rotation, the disk needs to travel — more hour.

Thus, **C** is the correct answer.

15. Consider the set of all fractions — where and are relatively prime positive integers. How many of these fractions have the property that if both numerator and denominator are increased by the value of the fraction is increased by

A

B

C

D

E

Solution(s):

We get that

Cross-multiplying and simplifying yields

We can add to add side and then factor to get

We are restricted by and to get and
The only factor pairs of that satisfy the above constraints are

and

Solving these get the following pairs:

Only one of these pairs gives values for and that are relatively prime.

Thus, **B** is the correct answer.

16. If we have that

and what is the value of

A

B

C

D

E

Solution(s):

Adding the two equations gives us

We can rearrange this equation to get

We can then subtract them to get

Once again rearranging, we can find

We have that which means that we can divide both sides by This gives us

Thus, **B** is the correct answer.

17. A line that passes through the origin intersects both the line and the line

The three lines create an equilateral triangle. What is the perimeter of the triangle?

A

B

C

D

E

Solution(s):

Since one of the sides of the equilateral triangle is a vertical line, the line of symmetry perpendicular to this side must be horizontal.

This means that the slope of the third side must be opposite the slope of the second side, which would be

To find the perimeter, we only need to find the length of one of the sides of the triangle.

We can plug in into the two other equations to get the two vertices on the vertical line.

The two -values are then

The distance between the two is which makes the perimeter

Thus, **D** is the correct answer.

18. Hexadecimal (base-16) numbers are written using numeric digits through as well as the letters through to represent through Among the first positive integers, there are whose hexadecimal representation contains only numeric digits. What is the sum of the digits of

A

B

C

D

E

Solution(s):

Note that converted to hexadecimal is Now we need to count the number of numbers that have only numerical digits in their hexadecimal.

The first digit can be or The second and third digits can be any number from This gives us

numbers. This, however, includes which is not a positive integer so we have to subtract one.

The sum of the digits in is

Thus, **E** is the correct answer.

19. The isosceles right triangle has right angle at and area The rays trisecting intersect at and What is the area of

A

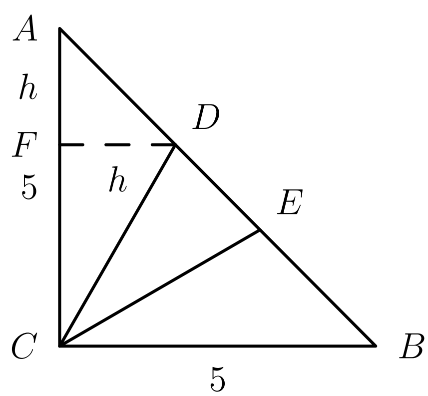
B

C

D

E

Solution(s):



Note that since it trisects a right angle. This means that we can split into a right triangle and a right triangle.

We have that since is isosceles. Using this, we have that

Cross-multiplying gives us

Rationalizing the denominator gives us

The area of is then

Note that the area of is We also have that

Finally, we get that

Thus, **D** is the correct answer.

20. A rectangle with positive integer side lengths in has area and perimeter Which of the following numbers cannot equal

A

B

C

D

E

Solution(s):

Let and be the side lengths of the rectangle. Then we have that

Adding, we get

This means that must be the product of two numbers which are both greater than

The only answer choice that cannot be expressed as such is

Thus, **B** is the correct answer.

21. Tetrahedron has and What is the volume of the tetrahedron?

A

B

C

D

E

Solution(s):

We claim that triangles and are perpendicular to each other.

We can show this be dropping the altitudes from to and from to in each triangle.

Since and we have that the feet of these altitudes will coincide at point

Then we have that

We then have that which shows that is an isosceles right triangle.

This proves the above claim. Finally, the volume of the tetrahedron is

Thus, **C** is the correct answer.

22. Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

A

B

C

D

E

Solution(s):

We can case on the number of people standing.

Case people standing

There is only way for this to happen: everyone sits.

Case person stands

There are choices for which person stands.

Case people stand

There are ways to choose the pair of people that stand. There are, however, pairs of adjacent people, so we have to subtract those out for a total of configurations.

Case people stand

There are choices for the first person. This rules out the person standing and their two neighbors, leaving people to choose from.

There are ways to choose the other people, but of these pairs involve adjacent people standing.

Therefore, there are a total of arrangements for this case. Note, however, that we have divide by since we overcounted which person we choose first.

This case only has configurations then.

Case people stand

There are only choices depending on which half of the people stand up.

These cases together contribute

configurations. There are possibilities for the coin flips, which makes the desired probability

Thus, **A** is the correct answer.

23. The zeroes of the function

are integers. What is the sum of the possible values of

A

B

C

D

E

Solution(s):

Let the zeroes be and Using Vieta's formulas, we have that and

Then we get that

which rearranges to

The only possible pairs that work are

For any of these pairs, we have that

We want all the such unique values of We get that they are

The sum of these values is

Thus, **C** is the correct answer.

24. For some positive integers there is a quadrilateral with positive integer side lengths, perimeter right angles at and and How many different values of are possible?

A

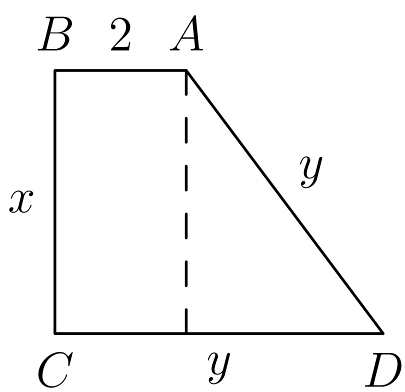
B

C

D

E

Solution(s):



Drop the altitude from down to Then we have that

which simplifies to

Since and are both integers, we have that must be one more than a perfect square.

We know that

We have that Guessing and checking tells us that is the max value, whereas is the minimum value.

This means that there are values of for which all the problem constraints are satisfied.

Thus, **B** is the correct answer.

25. Let be a square of side length Two points are chosen independently at random on the sides of The probability that the straight-line distance between the points is at least — is ——— where and are positive integers with What is

A

B

C

D

E

Solution(s):

Fix one of the points. Then the probability the other point is on the same side is —

The probability that the other point is on an adjacent side is — and — for the opposite side.

Case the other point is one the same side

Let the two points be and If we view them as points in the unit square, then we see that the area such that

—

forms a triangle with area

— —

Case the points are on adjacent sides

WLOG, let the two points be on the bottom and left sides. Then the two points are and

The distance between the two points is ——— We want this to be greater than —

When graphed as above, we get that the points fill out the unit square except for a quarter circle of radius —

This means that the probability that the distance is greater than — is

— — —

Case the points are on opposite sides

The length will always be greater than — which means that the probability is

The total probability is therefore

— — — — —

———

The desired sum is

Thus, **A** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2015A