# 2013 AMC 10A Solutions

Typeset by: LIVE, by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc10/2013A/solutions



**1.** A taxi ride costs $\$1.50$ plus $\$0.25$ per mile traveled. How much does a $5$-mile taxi ride cost?

A $2.25

B $2.50

C $2.75

D $3.00

E $3.75

## Solution(s):

The total cost is \begin{align*} $1.5 + 5 \cdot $.25 &= $1.5 + $1.25\\ &= $2.75. \end{align*}

Thus, **C** is the correct answer.

**2.** Alice is making a batch of cookies and needs $2\frac{1}{2}$ cups of sugar. Unfortunately, her measuring cup holds only $\frac{1}{4}$ cup of sugar. How many times must she fill that cup to get the correct amount of sugar?

- A $8$
- B $10$
- C $12$
- D $16$
- E $20$

## Solution(s):

We just need to divide the total amount of sugar needed by the amount of sugar that the cup holds.

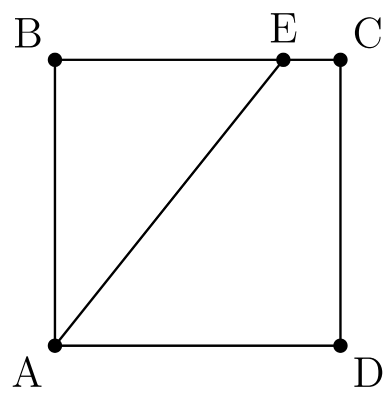
Therefore, She will need to refill the cup

$$\frac{2\frac{1}{2}}{\frac{1}{4}} = \frac{5}{2} \cdot 5 = 10$$

times.

Thus, **B** is the correct answer.

**3.** Square $ABCD$ has side length $10$. Point $E$ is on $\overline{BC}$, and the area of $\triangle ABE$ is $40$. What is $BE$?




A $4$

B $5$

C $6$

D $7$

E $8$

## Solution(s):

We have by the formula for the area of a triangle that

$$40 = \frac{1}{2} \cdot 10 \cdot BE.$$

This gives us

$$40 = 5BE$$

$$BE = 8.$$

Thus, **E** is the correct answer.

**4.** A softball team played ten games, scoring $1, 2, 3, 4, 5, 6, 7, 8, 9$, and $10$ runs. They lost by one run in exactly five games. In each of their other games, they scored twice as many runs as their opponent. How many total runs did their opponents score?

A $35$

B $40$

C $45$

D $50$

E $55$

## Solution(s):

Note that if they scored twice as many runs as their opponents, then they scored an even number of runs.

This means in the games where they scored $2, 4, 6, 8,$ and $10$ runs, their opponents scored $1, 2, 3, 4,$ and $5$ runs respectively.

This sums to

$$1 + 2 + 3 + 4 + 5 = 15.$$

In the other games, their opponents scored $2, 4, 6, 8,$ and $10$ runs.

This sums to

$$2 + 4 + 6 + 8 + 10 = 30.$$

The total number of runs is then

$$15 + 30 = 45.$$

Thus, **C** is the correct answer.

**5.** Tom, Dorothy, and Sammy went on a vacation and agreed to split the costs evenly. During their trip Tom paid \$105, Dorothy paid \$125, and Sammy paid \$175. In order to share costs equally, Tom gave Sammy $t$ dollars, and Dorothy gave Sammy $d$ dollars. What is $t - d$?

A $15$

B $20$

C $25$

D $30$

E $35$

## Solution(s):

The total amount they paid was \$105 + \$125 + \$175 = \$405. This means that each should pay \$405 \div 3 = \$135. This means that Tom has to pay \$30 more, and Dorothy has to pay \$10 more.

Then $t = 30$, $d = 10$, and therefore,

$$t - d = 20.$$

Thus, **B** is the correct answer.

**6.** Joey and his five brothers are ages $3$, $5$, $7$, $9$, $11$, and $13$. One afternoon two of his brothers whose ages sum to $16$ went to the movies, two brothers younger than $10$ went to play baseball, and Joey and the $5$-year-old stayed home. How old is Joey?

- A $3$
- B $7$
- C $9$
- D $11$
- E $13$

## Solution(s):

The two pairs of ages that add to $16$ are $(3, 13)$ and $(7, 9)$.

Note, however, that we need two brothers younger than $10$, but not the $5$-year old, to play baseball.

If the brothers that go to the moves are $7$ and $9$, there are no choices for the brothers that play baseball.

This means that the $3$ and $13$ year olds go to the movies, and the $7$ and $9$ year olds play baseball.

Therefore, Joey is the $11$-year old.

Thus, **D** is the correct answer.

**7.** A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

A $6$

B $8$

C $9$

D $12$

E $16$

## Solution(s):

There are $2$ cases. The first case is the student choose both math classes, the and the second case is they only take one.

**Case 1:** the student takes both math classes

There are $3$ classes left, which means that the student has $3$ choices for their final class.

**Case 2:** the student takes one math class

There are $2$ choices for which math class the student takes. Then, there are $3$ courses left, from which the students must choose $2$.

This is the same as choosing which course the student does not take, which can be done in $3$ ways.

Therefore, this case contributes $2 \cdot 3 = 6$ schedules that the student can take.

The total number of configurations in both cases is $3 + 6 = 9$.

Thus, **C** is the correct answer.

**8.** What is the value of

$$\frac{2^{2014}+2^{2012}}{2^{2014}-2^{2012}}?$$

- A $-1$
- B $1$
- C $\frac{5}{3}$
- D $2013$
- E $2^{4024}$

## Solution(s):

Factoring out a $2^{2012}$, we get:

$$\frac{2^{2012}(2^2+1)}{2^{2012}(2^2-1)}=\frac{5}{3}.$$

Thus, **C** is the correct answer.

**9.** In a recent basketball game, Shenille attempted only three-point shots and two-point shots. She was successful on $20\%$ of her three-point shots and $30\%$ of her two-point shots. Shenille attempted $30$ shots. How many points did she score?

A $12$

B $18$

C $24$

D $30$

E $36$

## Solution(s):

Let $x$ be the number of two-point shots and $y$ be the number of three-point shots.

Then, Shenille scores $.3x$ two-points shots and $.2y$ three-point shots, for a total score of

$$2 \cdot .3x + 3 \cdot .2y = .6x + .6y.$$

We know that

$$x + y = 30$$

$$.6(x + y) = 18.$$

Thus, **B** is the correct answer.

**10.** A flower bouquet contains pink roses, red roses, pink carnations, and red carnations. One third of the pink flowers are roses, three fourths of the red flowers are carnations, and six tenths of the flowers are pink. What percent of the flowers are carnations?

A $15$

B $30$

C $40$

D $60$

E $70$

## Solution(s):

Let the total number of flowers be $x$. There are $.6x$ pink flowers and $.4x$ red flowers.

Then there are

$$\frac{1}{3} \cdot .6x = .2x$$

pink roses, which means there are $.4x$ pink carnations.

There are also

$$\frac{3}{4} \cdot .4x = .3x$$

red carnations. This means there are $.3x + .4x = .7x$ carnations. This is $70\%$ of the total flowers.

Thus, **E** is the correct answer.

**11.** A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly $10$ ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

A $10$

B $12$

C $15$

D $18$

E $25$

## Solution(s):

Let $x$ be the number of students. Then the number of ways to pick a two-person committee is

$$\binom{x}{2} = \frac{x(x-1)}{2}.$$

We know that this equals $10$, so

$$x^2 - x = 20$$

$$x^2 - x - 20 = 0.$$

Factoring yields

$$(x-5)(x+4) = 0$$

$$x = 5,$$

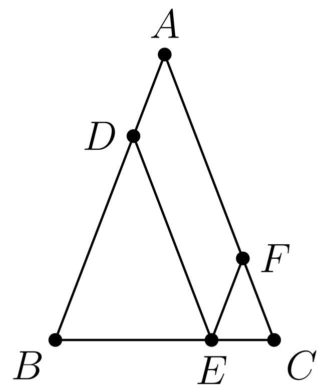
since there cannot be a negative number of students.

Then, the number of ways to pick a $3$-person committee is

$$\binom{5}{3} = \binom{5}{2} = 10.$$

Thus, **A** is the correct answer.

**12.** In $\triangle ABC$, $AB = AC = 28$ and $BC = 20$. Points $D, E$, and $F$ are on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of parallelogram $ADEF$?




A $48$

B $52$

C $56$

D $60$

E $72$

## Solution(s):

Note that $\triangle DBE \sim \triangle ABC$ and $\triangle FEC \sim \triangle ABC$ due to the parallel lines.

This tells us that $DB = DE$ and $FE = FC$. We have that the perimeter of $ADEF$ is

$$AD + DE + EF + AF$$

$$= AD + DB + FC + AF$$

$$= AB + AC$$

$$= 56.$$

Thus, **C** is the correct answer.

**13.** How many three-digit numbers are not divisible by $5$, have digits that sum to less than $20$, and have the first digit equal to the third digit?

A $52$

B $60$

C $66$

D $68$

E $70$

## Solution(s):

Note that for the number to not be divisible by $5$, the units digits cannot be either $0$ or $5$.

Let $x$ be the hundreds and units digit and $y$ be the tens digit. Then we want

$$2x + y < 20.$$

Casing on the $9$ options of $x$, we get:

If $x$ is $1, 2, 3$, or $4$, then $y$ can be anything since $y < 10$.

If $x = 6$, then $y < 8$, which gives us $8$ solutions.

If $x = 7$, then $y < 6$, which gives us $6$ solutions.

If $x = 8$, then $y < 4$, which gives us $4$ solutions.

If $x = 9$, then $y < 2$, which gives us $2$ solutions.

This gives us a total of

$$4 \cdot 10 + 8 + 6 + 4 + 2 = 60$$

solutions.

Thus, **B** is the correct solution.

**14.** A solid cube of side length $1$ is removed from each corner of a solid cube of side length $3$. How many edges does the remaining solid have?

- A $36$
- B $60$
- C $72$
- D $84$
- E $108$

## Solution(s):

Removing the cubes does not remove any edges from the original cube. It only adds edges.

After removing each cube, we can see that $9$ extra edges are added to the solid.

$8$ cubes are removed, which means $8 \cdot 9 = 72$ edges are added to the original $12$ edges, for a total of $72 + 12 = 84$ edges.

Thus, **D** is the correct answer.

**15.** Two sides of a triangle have lengths $10$ and $15$. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

- A $6$
- B $8$
- C $9$
- D $12$
- E $18$

## Solution(s):

Let $h_1$ be the length of the altitude to the side of length $10$ and similarly define $h_2$ for the other given side.

We have that

$$10h_1 = 15h_2$$

$$h_1 = \frac{3}{2}h_2.$$

The third altitude is the average of the other two, which makes its length

$$\frac{h_2 + \frac{3}{2}h_2}{2} = \frac{5}{4}h_2.$$

Let the third side have length $x$. Then

$$\frac{5}{4}h_2 x = 15h_2$$

$$x = 12.$$

Thus, **D** is the correct answer.

**16.** A triangle with vertices $(6,5)$, $(8,-3)$, and $(9,1)$ is reflected about the line $x=8$ to create a second triangle. What is the area of the union of the two triangles?

A $9$

B $\dfrac{28}{3}$

C $10$

D $\dfrac{31}{3}$

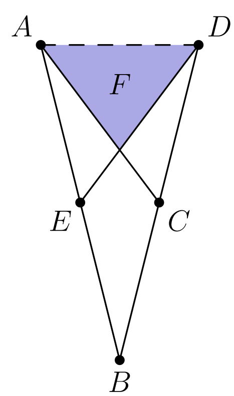
E $\dfrac{32}{3}$

## Solution(s):

Let $A=(6,5)$, $B=(8,-3)$, and $C=(9,1)$. After reflecting, we get the points $E=(7,1)$ and $D=(10,5)$.

Note that the area of the union is the area of $\triangle ABD$ minus the area of region $F$.

Sketching this diagram, we get:




The intersection of $\overline{DE}$ and $\overline{AC}$ occurs at $x=8$ due to symmetry. We also get that $\overline{AC}$ can be represented as the line

$$y=-\frac{4}{3}x+13.$$

Plugging in $x=8$, we get the $y$-coordinate to be: \[\begin{align*} y&=-\dfrac{4}{3} \cdot 8 + 13 \\ &= -\dfrac{32}{3} + 13 \\ &=\dfrac{7}{3}. \end{align*}

The area of region $F$ is then

$$\frac{1}{2} \cdot \left(5 - \frac{7}{3}\right) \cdot 4 = \frac{16}{3}.$$

Finally, the area of $\triangle ABD$ is

$$\frac{1}{2} \cdot 4 \cdot 8 - \frac{16}{3} = \frac{32}{3}.$$

Thus, **E** is the correct answer.

**17.** Daphne is visited periodically by her three best friends: Alice, Beatrix, and Claire. Alice visits every third day, Beatrix visits every fourth day, and Claire visits every fifth day.

All three friends visited Daphne yesterday. How many days of the next $365$-day period will exactly two friends visit her?

A $48$

B $54$

C $60$

D $66$

E $72$

## Solution(s):

Note that the least common multiple of $3, 4$, and $5$ is $60$.

We can split up the year into $6$ $60$-day periods, and count the number of times exactly two friends visit.

We have that Alice and Beatrix visit together

$$\frac{60}{3 \cdot 4} - 1 = 5 - 1 = 4$$

times. We subtract $1$ since on the $60$ th day, all $3$ friends visit, which we don't want to count.

Similarly, Alice and Claire visit

$$\frac{60}{3 \cdot 5} - 1 = 4 - 1 = 3$$

times and Beatrix and Claire visit

$$\frac{60}{4 \cdot 5} - 1 = 3 - 1 = 2$$

times. This means that in any $60$-day period, exactly $2$ friends visit

$$4 + 3 + 2 = 9$$

times. There are $6$ periods, which means that on $6 \cdot 9 = 54$ days, exactly $2$ friends visit.

Thus, **B** is the correct answer.

**18.** Let points

$$A=(0,0),\ B=(1,2),$$
$$C=(3,3),\ D=(4,0).$$

Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through $A$. This line intersects $\overline{CD}$ at point $\left(\frac{p}{q},\frac{r}{s}\right)$, where these fractions are in lowest terms. What is

$$p+q+r+s?$$

A $54$

B $58$

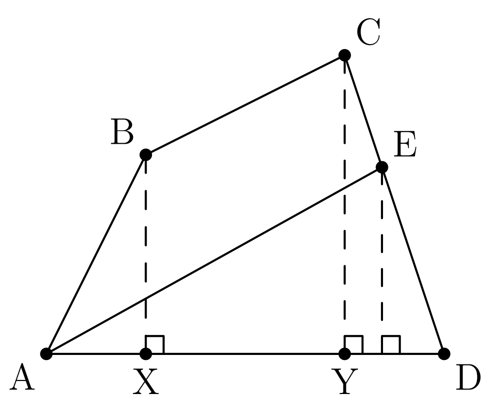
C $62$

D $70$

E $75$

## Solution(s):

We begin by finding the are of $ABCD$. To do this, label the point $E=(\frac{p}{q},\frac{r}{s})$ and drop altitudes from $B$ and $C$ to the $x$-axis as follows:




Looking at the triangle $\triangle ABX$, observe that the area is equal to:

$$[\triangle ABX]=\frac{1}{2}\cdot 1\cdot 2=1$$

Similarly, looking at the trapezoid $XBCY$, observe that the area is equal to:

$$[XBCY] = \frac{1}{2}(3-1)(2+3) = 5$$

And lastly, looking at the triangle $\triangle CYD$, observe that the area is equal to:

$$[\triangle CYD] = \frac{1}{2}(1)(3) = \frac{3}{2}$$

Therefore the area of $ABCD$ is equal to:

$$[ABCD] = 1 + 5 + \frac{3}{2} = \frac{15}{2}$$

The area of $\triangle ADE$ is then $\dfrac{15}{4}$. This means that the height of this triangle is

$$\frac{1}{2} \cdot 4h = \frac{15}{4}$$

$$h = \frac{15}{8}.$$

We have that the slope of $\overline{CD}$ is $-3$, which means that this line can be expressed with

$$y = -3x + 12.$$

Plugging in the value $y = h$, we get

$$\frac{15}{8} = -3x + 12$$

$$x = \frac{27}{8}.$$

We now have both the $x$ and $y$-coordinates of the intersection point. The desired sum is then

$$15 + 27 + 8 + 8 = 58.$$

Thus, **B** is the correct answer.

**19.** In base $10$, the number $2013$ ends in the digit $3$. In base $9$, on the other hand, the same number is written as $(2676)_9$ and ends in the digit $6$. For how many positive integers $b$ does the base-$b$-representation of $2013$ end in the digit $3$?

A $6$

B $9$

C $13$

D $16$

E $18$

## Solution(s):

Note that the units digit of represents the remainder when the number is divided by the base.

The question then boils down to finding all numbers, $b$, such that $2013$ leaves a remainder of $3$ when divided by $b$.

This means that $b$ must divide $2010$. Also note that $b \geq 4$, since otherwise the remainder cannot be $3$.

The prime factorization of $2010$ is

$$2010 = 2 \cdot 3 \cdot 5 \cdot 67.$$

Then, $2010$ has

$$(1+1)^4 = 2^4 = 16$$

factors. It has $3$ factors less than $4$, namely $1, 2,$ and $3$. This means there are $16 - 3 = 13$ valid values for $b$.

Thus, **C** is the correct answer.

**20.** A unit square is rotated $45^\circ$ about its center. What is the area of the region swept out by the interior of the square?

A $1-\dfrac{\sqrt{2}}{2}+\dfrac{\pi}{4}$

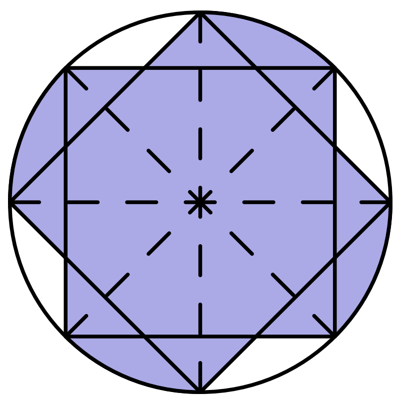
B $\dfrac{1}{2}+\dfrac{\pi}{4}$

C $2-\sqrt{2}+\dfrac{\pi}{4}$

D $\dfrac{\sqrt{2}}{2}+\dfrac{\pi}{4}$

E $1+\dfrac{\sqrt{2}}{4}+\dfrac{\pi}{8}$

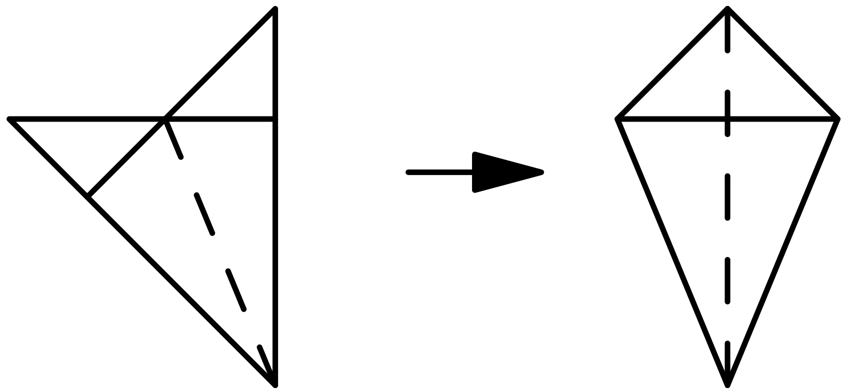
## Solution(s):



The above figure illustrates the region that the square sweeps out. We can find the area of the shaded region by splitting it up.

We have four sectors and eight triangles that we can divide the shaded region into.

Note that two triangle regions can be combined to form a kite, which we can easily find the area of.



The area of the four sectors is half the area of the circle. The radius of the circle can be calculating by realizing that the radius is one half the diameter of the square.

Therefore,

$$r = \frac{\sqrt{2}}{2}$$

$$\frac{1}{2}\pi r^2 = \frac{\pi}{4}.$$

One of the diagonals of the kite is $r$. Let the other be $x$.

Then we get the equation

$$x + x\sqrt{2} = 1$$

$$x = \sqrt{2} - 1.$$

This equation comes from splitting up a side of the square into $x$, and the legs of two isosceles right triangles whose hypotenuse is $x$.

Finally, the area of all $4$ kites is

$$4 \cdot \frac{1}{2} \cdot (\sqrt{2} - 1) \cdot \frac{\sqrt{2}}{2} = 2 - \sqrt{2}.$$

The total area is then

$$2 - \sqrt{2} + \frac{\pi}{4}.$$

Thus, **C** is the correct answer.

**21.** A group of $12$ pirates agree to divide a treasure chest of gold coins among themselves as follows. The $k^{\text{th}}$ pirate to take a share takes $\frac{k}{12}$ of the coins that remain in the chest. The number of coins initially in the chest is the smallest number for which this arrangement will allow each pirate to receive a positive whole number of coins. How many coins does the $12^{\text{th}}$ pirate receive?

A $720$

B $1296$

C $1728$

D $1925$

E $3850$

## Solution(s):

Let $x$ be the number of coins initially in the treasure chest. Note that after the $k^{\text{th}}$ pirate, $\frac{12-k}{k}$ of the coins are left.

This means that

$$x \cdot \frac{11}{12} \cdot \frac{10}{12} \cdot \frac{9}{12} \cdot \frac{8}{12} \cdot \frac{7}{12} \cdot \frac{6}{12} \cdot \frac{5}{12} \cdot \frac{4}{12} \cdot \frac{3}{12} \cdot \frac{2}{12} \cdot \frac{1}{12}$$

is an integer.

We want to minimize $x$ while keeping the above expression an integer. Cancelling out the common factors of the numerator and denominator will tell us what $x$ needs to be.

After doing this, we get

$$11 \cdot 5 \cdot 7 \cdot 5 = 1925$$

is left in the numerator. If we set $x$ equal to the resulting denominator, we have that the $12$th pirate gets $1925$ coins.

Thus, **D** is the correct answer.

**22.** Six spheres of radius $1$ are positioned so that their centers are at the vertices of a regular hexagon of side length $2$. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

A $\sqrt{2}$

B $\frac{3}{2}$

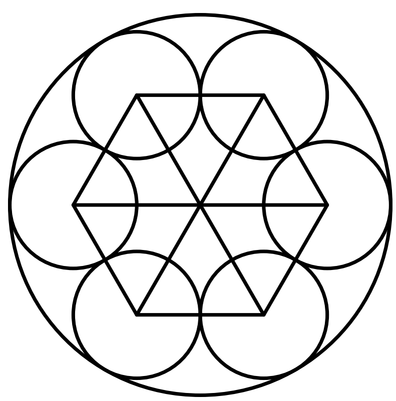
C $\frac{5}{3}$

D $\sqrt{3}$

E $2$

## Solution(s):

We can consider the cross-section of the largest sphere that contains all the six smaller spheres.



From this we can see that the radius of the largest sphere is $2 + 1 = 3$.

Now to find the radius, $r$, of the eight sphere, we can construct a right triangle connecting the centers of a small sphere, the seventh sphere, and the eight sphere.

The distance between the small sphere and the seventh sphere is $2$. The other leg is $3 - r$, the hypotenuse is $1 + r$.

We can apply the Pythagorean Theorem to get

$$2^2 + (3 - r)^2 = (1 + r)^2.$$

Simplifying yields

$$4 + 9 - 6r = 1 + 2r$$

$$12 = 8r$$

$$r = \frac{3}{2}.$$

Thus, **B** is the correct answer.

**23.** In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center $A$ and radius $AB$ intersects $\overline{BC}$ at points $B$ and $X$. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is $BC$?

A $11$

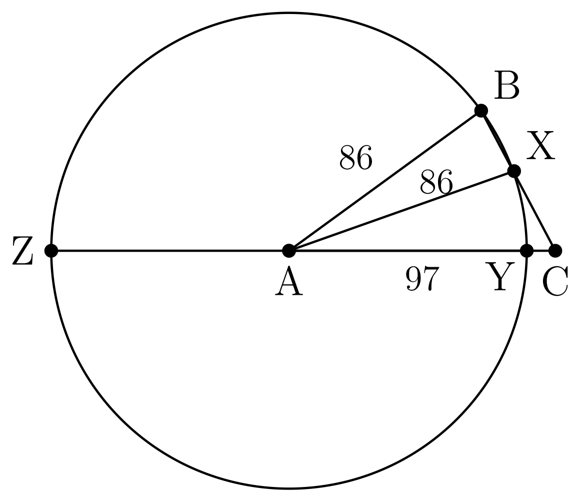
B $28$

C $33$

D $61$

E $72$

## Solution(s):




Let $Z$ and $Y$ be the intersections of $\overline{AC}$ and the circle as shown in the diagram. We have that

$$CY = 97 - 86 = 11$$

and

$$CZ = 86 + 97 = 183.$$

We can then apply Power of a Point to get

$$CX \cdot CB = CY \cdot CZ$$

$$CX \cdot CB = 11 \cdot 183$$

$$= 11 \cdot 3 \cdot 61.$$

Since $BX$ and $CX$ are integers, we also have that $CB$ is an integer. Also, by the Triangle Inequality,

$$BC < 86 + 97 = 183.$$

Using these two facts combined with $CX < CB$, we have that the only pair of values that work is $CX = 31$ and $BC = 61$.

Thus, **D** is the correct answer.

**24.** Central High School is competing against Northern High School in a backgammon match. Each school has three players, and the contest rules require that each player play two games against each of the other school's players. The match takes place in six rounds, with three games played simultaneously in each round. In how many different ways can the match be scheduled?

A $540$

B $600$

C $720$

D $810$

E $900$

## Solution(s):

Label the players from Central High School $A, B$, and $C$, and the players from Northern High School $X, Y$, and $Z$.

There are

$$\frac{6!}{2!2!2!} = 90$$

ways to figure out the order in which $A$ plays his matches.

We just need to figure out the order in which $B$ plays their matches. Then $C$'s matches are fixed.

WLOG, let the order of matches $A$ plays be

$$XXYYZZ.$$

To figure out the possible arrangements for $B$, we can case on the positions of $X$ 's and $Y$'s.

If $X$ goes in the middle two spots, then $Y$ has to go in the last two spots, Otherwise, the $Z$'s will overlap.

Similarly, if the $X$'s go in the last two spots, then there is only spot to put the $Y$'s.

If one $X$ goes in the middle $2$, and the other $X$ goes in the last two, then there are $2$ options for the $Y$'s go.

The $Y'$s have $2$ options in the first $2$ spots and the other is forced to be in the remaining spot in the last $2$.

There are $2 \cdot 2$ ways to place the $X'$s in the above configuration. There are then a total of

$$1 + 1 + 2 \cdot 2 \cdot 2 = 10$$

ways to determine the schedule for $B'$s matches.

We then have to multiply by $90$ for the number of configurations for $A'$s matches.

The total number of ways to order the games is then

$$90 \cdot 10 = 900.$$

Thus, **E** is the correct answer.

**25.** All $20$ diagonals are drawn in a regular octagon. At how many distinct points in the interior of the octagon (not on the boundary) do two or more diagonals intersect?

A $49$

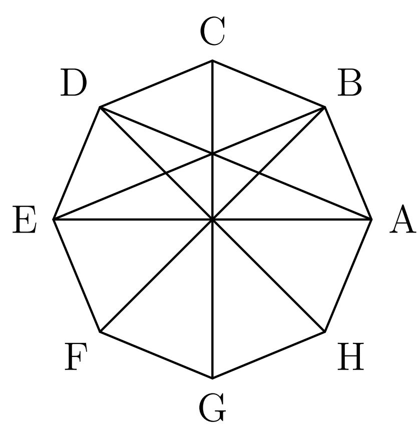
B $65$

C $70$

D $96$

E $128$

## Solution(s):




We can over count the number of intersections and then subtract out the ones we double counted.

Assuming that every set of $4$ points contribute one intersection, we have

$$\binom{8}{4} = 70$$

intersections.

Four diagonals intersect in the center, which means that we need to subtract out

$$\binom{4}{2} - 1 = 5$$

because of this. We subtract one since we need to still keep one of them.

As shown in the diagram, multiple lines can also intersect in places such as $\overline{AD}, \overline{CG}$, and $\overline{BE}$.

There are $8$ of these intersections, which means that we have to subtract out

$$8\left(\binom{3}{2}-1\right)=16$$

more intersections.

This means that the total number of intersections is

$$70-5-16=49.$$

Thus, **A** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2013A