2012 AMC 10B Solutions

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1. Each third-grade classroom at Pearl Creek Elementary has 18 students and 2 pet rabbits. How many more students than rabbits are there in all 4 of the third-grade classrooms?

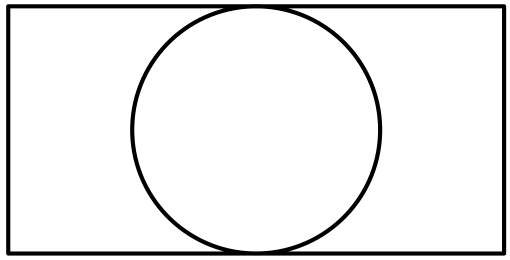
- A 48
- B 56
- C 64
- D 72
- E 80

Solution(s):

There are $18 \cdot 4 = 72$ students. As such, there are $72 \cdot 2 = 144$ rabbits in total. Therefore, the difference is $144 - 72 = 72$.

Thus, the correct answer is **D**.

2. A circle of radius 5 is inscribed in a rectangle as shown. The ratio of the length of the rectangle to its width is $2 : 1$. What is the area of the rectangle?



A 50

B 100

C 125

D 150

E 200

Solution(s):

The smaller side of the rectangle is equal to the diameter, which is $2r = 2 \cdot 5 = 10$.

Due to the $2 : 1$ ratio, the long side is $10 \cdot 2 = 20$.

Therefore, the area is $20 \cdot 10 = 200$.

Thus, the correct answer is **E**.

3. The point in the xy-plane with coordinates $(1000, 2012)$ is reflected across the line $y = 2000$. What are the coordinates of the reflected point?

A $(998, 2012)$

B $(1000, 1988)$

C $(1000, 2024)$

D $(1000, 4012)$

E $(1012, 2012)$

Solution(s):

Notice that the line in question is perfectly horizontal. This means that if we were to construct a perpendicular line segment from the point to the line, to find the reflected coordinates of the point, we simply double the distance along that line segment.

The line segment from $(1000, 2012)$ to $y = 2000$ is of length 12, so the reflected point is along this same line segment, but a distance of 12 on the other side of the horizontal line. This yields $(1000, 1988)$.

Thus, the correct answer is **B**.

4. When Ringo places his marbles into bags with 6 marbles per bag, he has 4 marbles left over. When Paul does the same with his marbles, he has 3 marbles left over. Ringo and Paul pool their marbles and place them into as many bags as possible, with 6 marbles per bag. How many marbles will be left over?

A 1

B 2

C 3

D 4

E 5

Solution(s):

As we know that when Ringo's marbles are divided by 6, we have a remainder of 4, we conclude that he has $6x + 4$ marbles for some x.

Using the same logic, we can also conclude that Paul has $6y + 3$ marbles for some y.

Therefore, the total number of marbles is

$$(6x + 4) + (6y + 3)$$
$$= 6(x + y + 1) + 1$$

Which, when divided by 6, only leaves 1 left over.

Thus, the correct answer is **A**.

5. Anna enjoys dinner at a restaurant in Washington, D.C., where the sales tax on meals is 10%. She leaves a 15% tip on the price of her meal before the sales tax is added, and the tax is calculated on the pre-tip amount. She spends a total of $\$27.50$ for dinner. What is the cost of her dinner without tax or tip in dollars?

- A $\$18$
- B $\$20$
- C $\$21$
- D $\$22$
- E $\$24$

Solution(s):

Suppose the original price is x. Then, the tax is 0.1 and the tip is $0.15x$. This makes the total payment equal to:

$$\begin{aligned} x + 0.1x + 0.15x &= 1.25x \\ &= 27.5 \end{aligned}$$

Therefore, $x = 22$.

Thus, the correct answer is **D**.

6. In order to estimate the value of $x-y$ where x and y are real numbers with $x>y>0$, Xiaoli rounded x up by a small amount, rounded y down by the same amount, and then subtracted her rounded values.

Which of the following statements is necessarily correct?

- A Her estimate is larger than $x-y$
- B Her estimate is smaller than $x-y$
- C Her estimate equals $x-y$
- D Her estimate equals $y-x$
- E Her estimate is 0

Solution(s):

The value when x is rounded up is greater than x.

The value when $-y$ is rounded up is greater than $-y$.

Therefore, we add two numbers which are greater than their corresponding parts in $x-y$, making it greater.

Thus, the correct answer is **A**.

7. For a science project, Sammy observed a chipmunk and a squirrel stashing acorns in holes. The chipmunk hid 3 acorns in each of the holes it dug. The squirrel hid 4 acorns in each of the holes it dug. They each hid the same number of acorns, although the squirrel needed 4 fewer holes. How many acorns did the chipmunk hide?

A 30

B 36

C 42

D 48

E 54

Solution(s):

Let the number of acorns they each hid be x. Then, the number of holes from the chipmunk is $\frac{x}{3}$ and the number of holes from the squirrel is $\frac{x}{4}$.

This means

$$\frac{x}{3} - \frac{x}{4} = \frac{x}{12} = 4$$

Therefore, $x = 48$.

Thus, the correct answer is **D**.

8. What is the sum of all integer solutions to

$$1 < (x-2)^2 < 25?$$

- A 10
- B 12
- C 15
- D 19
- E 25

Solution(s):

Suppose we have $x = 2 + k$ as a solution. Then, $x = 2 - k$ would also be a solution as

$$((2+k)-2)^2 = ((2-k)-2)^2$$

The sum of these two solutions would be 4. Thus, the sum of all integer solutions to the above equation is four times the number of positive k's that work.

To find the number of k's, we need to find the number of positive solutions to:

$$1 < k^2 < 25,$$

which would be 3, as $k = 2, 3, 4$.

Therefore, there are a total of $4 \cdot 3 = 12$ solutions.

Thus, the correct answer is **B**.

9. Two integers have a sum of 26. When two more integers are added to the first two integers the sum is 41. Finally when two more integers are added to the sum of the previous four integers the sum is 57. What is the minimum number of odd integers among the 6 integers?

A 1

B 2

C 3

D 4

E 5

Solution(s):

The sum of the first two numbers is even, so we can make them both even.

The sum of the next two numbers is odd, so one must be even and one must be odd.

The sum of the last two numbers is even, so we can make them both even.

Therefore, we can have 1 odd number, making it the minimum.

Thus, the correct answer is **A**.

10. How many ordered pairs of positive integers (M, N) satisfy the equation

$$\frac{M}{6} = \frac{6}{N}?$$

- A 6
- B 7
- C 8
- D 9
- E 10

Solution(s):

By cross multiplying, we can see that $MN = 36$. Thus, we can make M any factor of 36 and then determine N from it.

Since $36 = 2^2 \cdot 3^2$, we have $(2+1)(2+1) = 9$ possible choices for M, each of which also determine a unique N.

Thus, the correct answer is **D**.

11. A dessert chef prepares the dessert for every day of a week starting with Sunday. The dessert each day is either cake, pie, ice cream, or pudding. The same dessert may not be served two days in a row. There must be cake on Friday because of a birthday. How many different dessert menus for the week are possible?

A 729

B 972

C 1024

D 2187

E 2304

Solution(s):

We know that there must be cake served on Friday, and as such, on Saturday, we cannot have cake. Therefore, we have 3 choices for Saturday's menu.

Furthermore, for each of the 5 previous days, we could go backwards and have 3 choices on each day, making the total number of choices $3^5 \cdot 3 = 729$.

Thus, the correct answer is **A**.

12. Point B is due east of point A. Point C is due north of point B. The distance between points A and C is $10\sqrt{2}$, and $\angle BAC = 45^\circ$. Point D is 20 meters due north of point C. The distance AD is between which two integers?

- A 30 and 31
- B 31 and 32
- C 32 and 33
- D 33 and 34
- E 34 and 35

Solution(s):

We know AB and BC are perpendicular, so

$$AB^2 + BC^2 = (10\sqrt{2})^2 = 200.$$

Also, as $\angle BAC = 45^\circ$, we know that $\triangle ABC$ is an isoceles right triangle, so $AB = BC$, making $2AB^2 = 200$. Thus, $AB = BC = 10$.

As such, we know that

$$BD = BC + CD = 30.$$

Thus, by the Pythagorean Theorem, we have that

$$\begin{aligned} AD^2 &= AB^2 + BD^2 \\ &= 10^2 + 30^2 \\ &= 1000 \end{aligned}$$

Thus, since

$$31^2 < AD^2 < 32^2,$$

we have

$$31 < AD < 32$$

Thus, the correct answer is **B**.

13. It takes Clea 60 seconds to walk down an escalator when it is not operating, and only 24 seconds to walk down the escalator when it is operating. How many seconds does it take Clea to ride down the operating escalator when she just stands on it?

A 36

B 40

C 42

D 48

E 52

Solution(s):

Let c represent Clea's speed walking down the non-operational escalator. Similarly, let e represent Clea's speed standing still on the operational escalator. Then, as speed is equal to distance over time, we know that $c = \frac{d}{60}$ and

$$c + e = \frac{d}{24}.$$

Therefore,

$$e = \frac{d}{24} - \frac{d}{60} = \frac{d}{40},$$

meaning that Clea takes 40 to descend the escalator by simply standing still.

Thus, the correct answer is **B**.

14. Two equilateral triangles are contained in square whose side length is $2\sqrt{3}$. The bases of these triangles are the opposite side of the square, and their intersection is a rhombus. What is the area of the rhombus?

A $\frac{3}{2}$

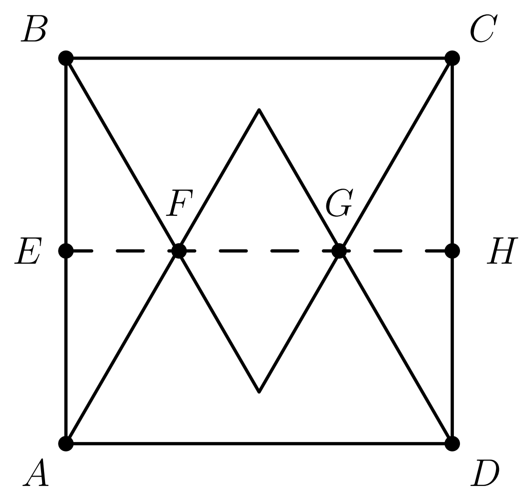
B $\sqrt{3}$

C $2\sqrt{3}-1$

D $8\sqrt{3}-12$

E $\frac{4\sqrt{3}}{3}$

Solution(s):



This rhombus is created by placing two congruent equilateral triangles. Let the side length of it be s. Then, the area of one of them is $\frac{s^2\sqrt{3}}{4}$, making the total area $\frac{s^2\sqrt{3}}{2}$.

The side length of the larger equilateral triangle is $2\sqrt{3}$. The height of it is 3 since the height is equal to $2\sqrt{3}\sin(60^\circ)$.

Half of the sqaure is $\sqrt{3}$, so the height of the smaller triangle is $3-\sqrt{3}$. Thus, the ratio between s and $2\sqrt{3}$ is $\frac{3-\sqrt{3}}{3}$.

As such,

$$s = \frac{2\sqrt{3}(3-\sqrt{3})}{3} = 2\sqrt{3} - 2.$$

Therefore, the combined area is

$$\begin{aligned}\frac{s^2\sqrt{3}}{2} &= \frac{(2\sqrt{3}-2)^2\sqrt{3}}{2} \\ &= \frac{(16-8\sqrt{3})\sqrt{3}}{2} \\ &= 8\sqrt{3} - 12\end{aligned}$$

Thus, the correct answer is **D**.

15. In a round-robin tournament with 6 teams, each team plays one game against each other team, and each game results in one team winning and one team losing. At the end of the tournament, the teams are ranked by the number of games won. What is the maximum number of teams that could be tied for the most wins at the end of the tournament?

A 2

B 3

C 4

D 5

E 6

Solution(s):

They would have to share $\binom{6}{2} = 15$ wins.

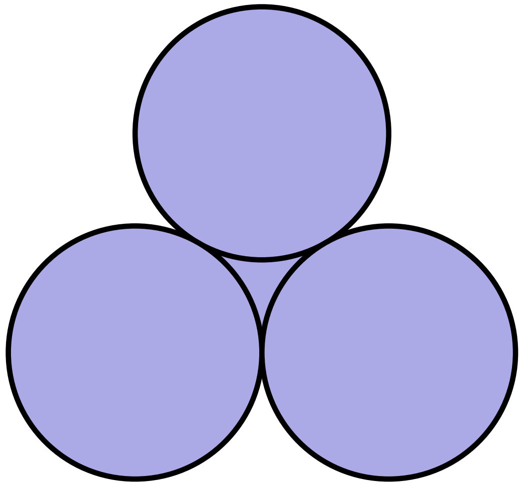
This means, we can't have a 6 way tie as that would be 2.5 wins per team.

If we had a 5 way tie, each team could have 3 wins, which is possible if one team losses all of its games, and out of the 5 winning teams, they each split their games.

If we label the teams from 1 to 5, and designate team 6 to lose all their games. To get a 5 way tie, we could have each team $1 \leq x \leq 5$ beat team $x+1 \mod 5$ and team $x+2 \mod 5$, as well as team 6.

Thus, the correct answer is **D**.

16. Three circles with radius 2 are mutually tangent. What is the total area of the circles and the region bounded by them, as shown in the figure?



A $10\pi + 4\sqrt{3}$

B $13\pi - \sqrt{3}$

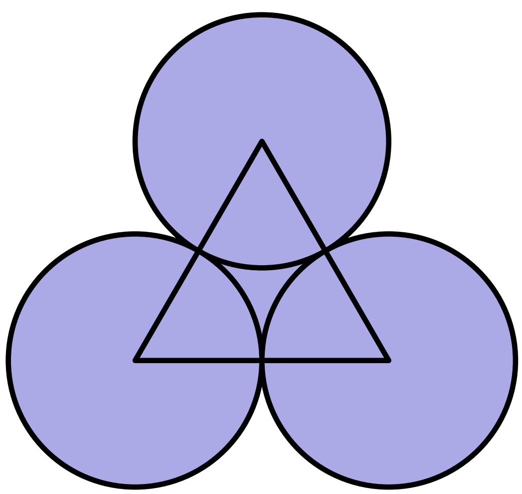
C $12\pi + \sqrt{3}$

D $10\pi + 9$

E 13π

Solution(s):

Consider the following diagram:



We first construct 3 circles, and take $\frac{1}{6}$ of it out for each of them, leaving 2.5 times the area of a circle.

Together, they have an area of

$$2.5r^2\pi = 2.5 \cdot 4\pi - 10\pi.$$

Then, add an equilateral with a side length of 4. The area is then

$$\frac{s^2\sqrt{3}}{4} = \frac{16\sqrt{3}}{4} = 4\sqrt{3}.$$

Therefore, the total is

$$10\pi + 4\sqrt{3}.$$

Thus, the correct answer is **A**.

17. Jesse cuts a circular paper disk of radius 12 along two radii to form two sectors, the smaller having a central angle of 120 degrees. He makes two circular cones, using each sector to form the lateral surface of a cone. What is the ratio of the volume of the smaller cone to that of the larger?

A $\frac{1}{8}$

B $\frac{1}{4}$

C $\frac{\sqrt{10}}{10}$

D $\frac{\sqrt{5}}{6}$

E $\frac{\sqrt{10}}{5}$

Solution(s):

The diagonal length is $x = 12$ for both cones since it is the radius. Let the cones each have radii of r_1, r_2 and heights of h_1, h_2, with the smaller cone being cone 1.

Then, the ratio between the cones is

$$\frac{\frac{r_1^2 h_1 \pi}{3}}{\frac{r_2^2 h_2 \pi}{3}} = \frac{r_1^2 h_1}{r_2^2 h_2}.$$

Then, for each cone, $h^2 + r^2 = x^2$ ad their diagonal length is x, so $h = \sqrt{x^2 - r^2}$.

This makes the ratio

$$\frac{r_1^2 \sqrt{x^2 - r_1^2}}{r_2^2 \sqrt{x^2 - r_r^2}}.$$

Then, with the smaller section, the curcumference of the base is $\frac{1}{3}$ of the circumference of the larger circle, so its radius is $\frac{1}{3}$ of the larger radius, making it $\frac{1}{3}x$.

Also, with the larger section, the curcumference of the base is $\frac{2}{3}$ of the circumference of the larger circle, so its radius is $\frac{2}{3}$ of the larger radius, making it $\frac{2}{3}x$.

This makes the ratio

$$\begin{aligned}\frac{(\frac{x}{3})^2\sqrt{x^2-(\frac{x}{3})^2}}{(\frac{2x}{3})^2\sqrt{x^2-(\frac{2x}{3})^2}} &= \frac{\sqrt{\frac{8}{9}}}{2^2\sqrt{\frac{5}{9}}} \\ &= \frac{\sqrt{10}}{10}\end{aligned}$$

Thus, the correct answer is **C**.

18. Suppose that one of every 500 people in a certain population has a particular disease, which displays no symptoms. A blood test is available for screening for this disease. For a person who has this disease, the test always turns out positive.

For a person who does not have the disease, however, there is a 2% false positive rate. In other words, for such people, 98% of the time the test will turn out negative, but 2% of the time the test will turn out positive and will incorrectly indicate that the person has the disease.

Let p be the probability that a person who is chosen at random from this population and gets a positive test result actually has the disease. Which of the following is closest to p?

A $\frac{1}{98}$

B $\frac{1}{9}$

C $\frac{1}{11}$

D $\frac{49}{99}$

E $\frac{98}{99}$

Solution(s):

The probability that someone has a positive test is equal to the probability someone is positive times the probability the test is positive plus the probability the test is negative times the probability the test is negative. This would be

$$\frac{1}{500} \cdot 1 + \frac{499}{500} \cdot 0.02$$
$$= \frac{499 \cdot 0.02 + 1}{500}$$

The probability that someone is positive given that they had a positive test is equal to the probability someone is positive times the probability the test is positive divided by the probability the test is positive. This would be

$$\frac{\frac{1}{500}\cdot 1}{\frac{499\cdot 0.02+1}{500}} = \frac{1}{499\cdot 0.02+1}$$

This is very close to

$$\frac{1}{500\cdot 0.02+1} = \frac{1}{11}.$$

Thus, the correct answer is **C**.

19. In rectangle $ABCD$, $AB = 6$, $AD = 30$, and G is the midpoint of $\overline{AD}$. Segment AB is extended 2 units beyond B to point E, and F is the intersection of $\overline{ED}$ and $\overline{BC}$. What is the area of quadrilateral $BFDG$?

A $\frac{133}{2}$

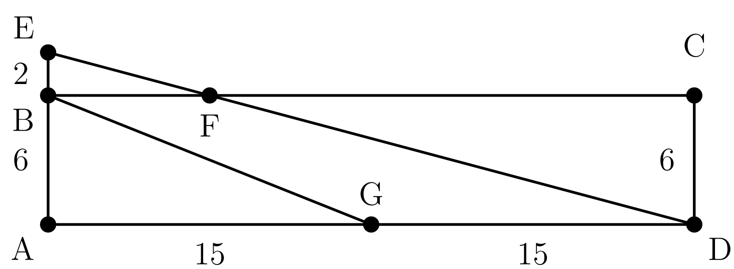
B 67

C $\frac{135}{2}$

D 68

E $\frac{137}{2}$

Solution(s):



The polygon $BFDG$ is a trapezoid with bases DG and BF and height 6. Also, since G is the midpoint between A and D, we have $GD = 15$.

We can see that $EBF \sim EAD$, so

$$\frac{BF}{AD} = \frac{EB}{EA}$$

$$\frac{BF}{30} = \frac{2}{8}$$

$$BF = 7.5$$

This makes the area of $BFDG$ equal to $\frac{6(15 + 7.5)}{2} = \frac{135}{2}$.

Thus, the correct answer is **C**.

20. Bernardo and Silvia play the following game. An integer between 0 and 999 inclusive is selected and given to Bernardo. Whenever Bernardo receives a number, he doubles it and passes the result to Silvia. Whenever Silvia receives a number, she adds 50 to it and passes the result to Bernardo. The winner is the last person who produces a number less than 1000.

Let N be the smallest initial number that results in a win for Bernardo. What is the sum of the digits of N?

A 7

B 8

C 9

D 10

E 11

Solution(s):

Suppose x was our initial number. Then, it becomes $2x$ when given to Silva, and $2x + 50$ when given to Bernardo. Repeatedly doing this can yield that it eventually becomes $16x + 700$ when given to Silva and $16x + 750$ when given to Bernardo. Any more iterations makes the number greater than 1000.

The number given to Silva must be below 1000 and the number Silva makes is greater than 1000, so

$$16x + 750 > 1000 > 16x + 700.$$

Therefore, $300 > 16x > 250$. This makes $18 \geq x \geq 16$, so $N = 16$. As such, the sum of the digits of N is $1 + 6 = 7$.

Thus, the correct answer is **A**.

21. Four distinct points are arranged on a plane so that the segments connecting them have lengths a, a, a, a, $2a$, and b. What is the ratio of b to a?

A $\sqrt{3}$

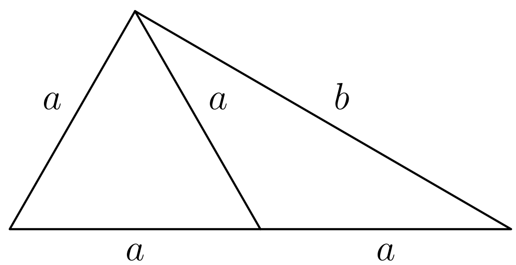
B 2

C $\sqrt{5}$

D 3

E π

Solution(s):



We must have some equilateral triangle of length a as otherwise, we would have 2 of the points being the same which contradicts the distinctness. Then, the other point is a away from some point on the equilateral triangle, and $2a$ away from another point in the triangle.

As such, those three points must be colinear. This makes us have an isoceles triangle with lengths $a - a - b$, with the opposite angle to b being 120°. Thus, b can be split into 2 right triangles, making

$$\begin{aligned} b &= 2a\cos(60^\circ) \\ &= a(2 \cdot \frac{\sqrt{3}}{2}) \\ &= a\sqrt{3}. \end{aligned}$$

Therefore, their ratio is $\sqrt{3}$.

Thus, the correct answer is **A**.

22. Let $(a_1, a_2, \ldots a_{10})$ be a list of the first 10 positive integers such that for each $2 \leq i \leq 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. How many such lists are there?

A 120

B 512

C 1024

D $181,440$

E $362,880$

Solution(s):

Suppose we have a_1. Then we can either add $a_1 - 1$ or $a_1 + 1$. Then, when every we add some number, we must have it in a connected interval to the numbers before.

Thus, our last interval would be $[1, 10]$. If we construct a list backwards, we need to take either the lowest or highest numbers in the list. We can do this 9 times to get a_{10}, then a_9 and continually until we get a_2. Then, a_1 is given. For each index, we choose an upper or lower, so there are 2 choices. Thus, the total is $2^9 = 512$.

Thus, the correct answer is **B**.

23. A solid tetrahedron is sliced off a solid wooden unit cube by a plane passing through two nonadjacent vertices on one face and one vertex on the opposite face not adjacent to either of the first two vertices. The tetrahedron is discarded and the remaining portion of the cube is placed on a table with the cut surface face down. What is the height of this object?

A $\frac{\sqrt{3}}{3}$

B $\frac{2\sqrt{2}}{3}$

C 1

D $\frac{2\sqrt{3}}{3}$

E $\sqrt{2}$

Solution(s):

Suppose we have the unit cube where our points are $(\pm1, \pm1, \pm1)$.

Then, we choose our non adjacent points to be $(1,1,0), (1,0,1), (0,1,1)$. If we put it on a table, the top point would be the current point $(0,0,0)$, so we must find the distance from the origin to the plane containing $(1,1,0), (1,0,1)$, and $(0,1,1)$.

Since the tetrahedron is rotationally symmetric, the point $(0,0,0)$ is closest to is the center of mass which would be $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. Therefore, the distance is

$$\sqrt{3\left(\frac{2}{3}\right)^2} = \frac{2\sqrt{3}}{3}.$$

Thus, the correct answer is **D**.

24. Amy, Beth, and Jo listen to four different songs and discuss which ones they like. No song is liked by all three. Furthermore, for each of the three pairs of the girls, there is at least one song liked by those two girls but disliked by the third. In how many different ways is this possible?

A 108

B 132

C 671

D 846

E 1105

Solution(s):

There are two cases: Each pair has exactly one liked song in common, or some pair has 2 liked songs in common. This is because each pair must have at least 1 liked song in common, and any more pairs than in the cases would result in 5 songs.

Case 1: Each pair has exactly one liked song in common

There are 4 ways to choose the song that one pair likes, 3 ways to choose the song that the second pair likes, and 2 ways to choose the song the third pair likes if we choose some order for them. Then, for the last song, one of them could like it which has 3 cases or none of them likes it which is another case. Thus, the number of solutions in this case is

$$4 \cdot 3 \cdot 2 \cdot (3+1) = 96.$$

Case 2: Some pair has 2 liked songs in common

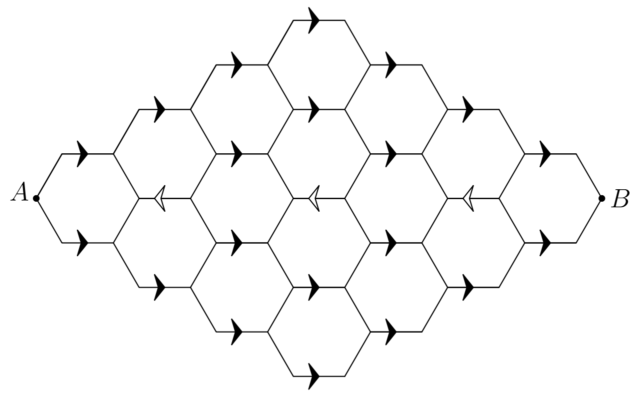
There are 3 ways to choose the pair that has 2 liked songs in common. Then, there are $\binom{4}{3} = 6$ ways to choose which songs they like. Finally, there are $\binom{2}{1} = 2$ ways to figure out who likes the last song. Thus, the number of solutions in this case is

$$3 \cdot 6 \cdot 2 = 36$$

The total amount is then $96 + 36 = 132.$

Thus, the correct answer is **B**.

25. A bug travels from A to B along the segments in the hexagonal lattice pictured below. The segments marked with an arrow can be traveled only in the direction of the arrow, and the bug never travels the same segment more than once. How many different paths are there?



A 2112

B 2304

C 2368

D 2384

E 2400

Solution(s):

For each of the arrows in the first column, we have one way to get to them. Then, we could get to any of the second column forward arrows, and directly to any of the third column arrows, making 4 ways to get there. Furthermore, we have 1 more way to get there by going forward on the opposite side twice, (top/bottom) go to the backwards arrow, and go to the specified arrow on the 3rd column.

It is symmetric, so the number of paths from any of the fifth column arrows to B is 5. Thus, for every path from a third column arrow to a fifth column arrow, there are $5 \cdot 5 = 25$ paths.

Note that with the answer choices, one can deduce that the answer is E since it is the only multiple of 5. However, we will solve the rest anyways.

If we don't take any of the back arrows, we can choose a forwards arrow in the third, fourth, and fifth column, making $4^3 = 64$ paths from the third to fifth column.

If we do take one of the back arrows, we must first choose a blue arrow, a forward arrow to go to on its side, then go to the back arrow, choose a forward arrow on the other side, and then choose a forward arrow in the fifth row in the opposite side as the arrow chosen in the third row.

This has $4 \cdot 2 \cdot 2 \cdot 2 = 32$ cases. Thus, we have 96 ways to go from the third to fifth column, making our answer $96 \cdot 5 \cdot 5 = 2400$.

Thus, the correct answer is **E**.

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