2012 AMC 10A Solutions

Typeset by: LIVE, by Po-Shen Loh
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1. Cagney can frost a cupcake every 20 seconds and Lacey can frost a cupcake every 30 seconds. Working together, how many cupcakes can they frost in 5 minutes?

A 10

B 15

C 20

D 25

E 30

Solution(s):

Cagney can make $60 \div 20 = 3$ cupcakes per minute, and Lacey can make $60 \div 30 = 2$.

In 5 minutes, they can make

$$5(2+3) = 5 \cdot 5 = 25.$$

Thus, **D** is the correct answer.

2. A square with side length 8 is cut in half, creating two congruent rectangles. What are the dimensions of one of these rectangles?

A 2 by 4

B 2 by 6

C 2 by 8

D 4 by 4

E 4 by 8

Solution(s):

Note that the one of the sides remains the same as the original square.

This means that one dimension is 8. The other dimension is the original side cut in half, which is 4.

Thus, **E** is the correct answer.

3. A bug crawls along a number line, starting at -2. It crawls to -6, then turns around and crawls to 5. How many units does the bug crawl altogether?

A 9

B 11

C 13

D 14

E 15

Solution(s):

It crawls a distance of

$$|-2-(-6)| = |-4| = 4$$

when it moves from -2 to -6. It then travels a distance of

$$|-6-5| = |-11| = 11$$

as it moves from -6 to 5.

The total distance is then

$$4+11=15.$$

Thus, **E** is the correct answer.

4. Let $\angle ABC = 24^\circ$ and $\angle ABD = 20^\circ$. What is the smallest possible degree measure for $\angle CBD$?

A 0

B 2

C 4

D 6

E 12

Solution(s):

Note that both of the angles share the ray AB. To minimize the desired degree, we want D to be between A and C.

This would make

$$\angle CBD = \angle ABC - \angle ABD = 4^\circ.$$

Thus, **C** is the correct answer.

5. Last year 100 adult cats, half of whom were female, were brought into the Smallville Animal Shelter. Half of the adult female cats were accompanied by a litter of kittens. The average number of kittens per litter was 4. What was the total number of cats and kittens received by the shelter last year?

A 150

B 200

C 250

D 300

E 400

Solution(s):

We have that there are $100 \div 2 = 50$ female cats. We then have that $50 \div 2 = 25$ cats that have kittens.

Since the average number of kittens per litter is 4, the total number of kittens is

$$4 \cdot 25 = 100.$$

The total number of cats and kittens is then

$$100 + 100 = 200.$$

Thus, **B** is the correct answer.

6. The product of two positive numbers is 9. The reciprocal of one of these numbers is 4 times the reciprocal of the other number. What is the sum of the two numbers?

A $\frac{10}{3}$

B $\frac{20}{3}$

C 7

D $\frac{15}{2}$

E 8

Solution(s):

Let the two numbers be x and y such that

$$xy = 9 \text{ and } \frac{1}{x} = \frac{4}{y}.$$

We get that

$$y = \frac{9}{x}$$

$$\frac{1}{x} = \frac{4x}{9}.$$

$$x = \frac{3}{2},$$

since x is positive.

Then

$$y = 9 \div \frac{3}{2} = 6.$$

The desired sum is then

$$6+\frac{3}{2}=\frac{15}{2}.$$

Thus, **D** is the correct answer.

7. In a bag of marbles, $\frac{3}{5}$ of the marbles are blue and the rest are red. If the number of red marbles is doubled and the number of blue marbles stays the same, what fraction of the marbles will be red?

A $\frac{2}{5}$

B $\frac{3}{7}$

C $\frac{4}{7}$

D $\frac{3}{5}$

E $\frac{4}{5}$

Solution(s):

WLOG, let the number of marbles in the bag be 5. Since we only care about ratios, we can do this. Then there are 3 blue marbles and 2 red marbles.

Doubling the red marbles gives us 4 of them. Then the fraction of red marbles is

$$\frac{4}{4+3}=\frac{4}{7}.$$

Thus, **C** is the correct answer.

8. The sums of three whole numbers taken in pairs are $12, 17,$ and 19. What is the middle number?

- A 4
- B 5
- C 6
- **D** 7
- E 8

Solution(s):

Let the three numbers be a, b, c where $a < b < c$. None of them are equal, since all three sums are different.

Then

$$a + b = 12, a + c = 17,$$

and

$$b + c = 19.$$

Adding all three equations together gives us

$$2(a + b + c) = 48.$$

Then

$$a + b + c = 24,$$

from which we can subtract

$$a + c = 17,$$

to get $b = 7$.

Thus, **D** is the correct answer.

9. A pair of six-sided dice are labeled so that one die has only even numbers (two each of $2, 4$, and 6), and the other die has only odd numbers (two of each $1, 3$, and 5). The pair of dice is rolled. What is the probability that the sum of the numbers on the tops of the two dice is 7?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution(s):

The pairs of numbers that sum to 7 are

$$(2,5),(4,3), \text{ and } (6,1).$$

There is a

$$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

chance that we get any of these pairs.

There are 3 pairs, which means that the total probability that the rolls sum to 7 is

$$3 \cdot \frac{1}{9} = \frac{1}{3}.$$

Thus, **D** is the correct answer.

10. Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

A 5

B 6

C 8

D 10

E 12

Solution(s):

Let a be the smallest possible sector angle and d be the difference in the arithmetic sequence.

Then we have that

$$\frac{12}{2}(a+a+11d)=12a+66d$$

is the sum of the arithmetic sequence.

We have that this sums to 360, so

$$12a+66d=360$$

$$2a+11d=60.$$

We want to minimize a, so we maximize d. If $d=5$, then a is not an integer, so $d=4$ and $a=8$.

Thus, **C** is the correct answer.

11. Externally tangent circles with centers at points A and B have radii of lengths 5 and 3, respectively. A line externally tangent to both circles intersects ray AB at point C. What is BC?

A 4

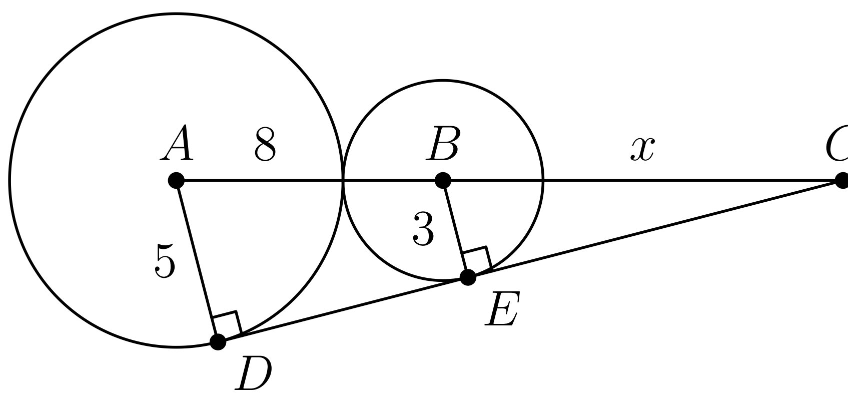
B 4.8

C 10.2

D 12

E 14.4

Solution(s):



Let x be BC. Note that $\triangle CEB$ and $\triangle CDA$ are similar due to angle-angle (tangent lines are perpendicular to radii).

Then

$$\frac{x}{3} = \frac{8+x}{5}.$$

Cross-multiplying gives us

$$5x = 24 + 3x$$
$$x = 12.$$

Thus, **D** is the correct answer.

12. A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not 100 (such as 2012). The 200th anniversary of the birth of novelist Charles Dickens was celebrated on February $7, 2012$, a Tuesday. On what day of the week was Dickens born?

A Friday

B Saturday

C Sunday

D Monday

E Tuesday

Solution(s):

On a typical year with 365 days, moving one year into the future moves the day of the week forward one since

$$365 = 52 \cdot 7 + 1.$$

On a leap year however, the day of the week gets moved forward twice since there is an extra day.

Fifty of the years between 2012 and 1812 are multiples of 4, but we have to discard 1900, which leaves us with 49 leap years.

Therefore, moving back 200 years means we have to go back

$$200 + 49 = 249 = 35 \cdot 7 + 4$$

days, which means we move back 4 days in the week.

This takes us back to Friday, which is the day that corresponds to February $7, 1812$.

Thus, **A** is the correct answer.

13. An *iterative average* of the numbers $1, 2, 3, 4,$ and 5 is computed the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

A $\frac{31}{16}$

B 2

C $\frac{17}{8}$

D 3

E $\frac{65}{16}$

Solution(s):

Let the order of the numbers be

$$a, b, c, d, e.$$

Then the iterative average is

$$\frac{\frac{\frac{\frac{a+b}{2}+c}{2}+d}{2}+e}{2}$$

$$=\frac{a+b+2c+4d+8e}{16}.$$

To minimize this, we make the order

$$5, 4, 3, 2, 1,$$

which gives us a sum of

$$\frac{5+4+6+8+8}{16} = \frac{31}{16}.$$

To maximize it, we have to reverse this order to get an average of

$$\frac{1+2+6+16+40}{16} = \frac{65}{16}.$$

The difference between these is

$$\frac{65}{16} - \frac{31}{16} = \frac{34}{16} = \frac{17}{8}.$$

Thus, **C** is the correct answer.

14. Chubby makes nonstandard checkerboards that have 31 squares on each side. The checkerboards have a black square in every corner and alternate red and black squares along every row and column. How many black squares are there on such a checkerboard?

A 480

B 481

C 482

D 483

E 484

Solution(s):

Note that there are 15 rows with 15 black tiles and 16 rows with 16 black tiles.

This can seen by observing that the first row has 16 black tiles, and all the other rows alternate with 15 and 16 tiles.

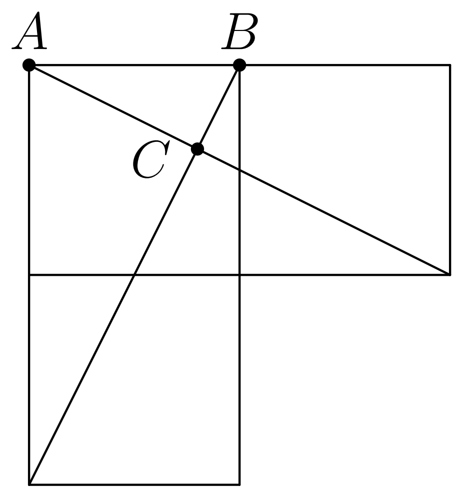
Then, due to the alternating pattern, there will be a total of 16 rows with 16 tiles, and the other rows have 15 tiles.

The total number of black squares is then

$$15^2 + 16^2 = 481.$$

Thus, **B** is the correct answer.

15. Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of $\triangle ABC$?



A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{2}{9}$

D $\frac{1}{3}$

E $\frac{\sqrt{2}}{4}$

Solution(s):

We can use coordinate geometry to figure out where the intersection of the two lines occurs.

Let A be the origin and $B = (1, 0)$. Then the slope of the line through A is $-\frac{1}{2}$, which makes the equation of the line

$$y = -\frac{1}{2}x.$$

The slope of the line through B is 2. The y-intercept is -2. This makes the equation of this line

$$y = 2x - 2.$$

Equating the equations, we get

$$2x - 2 = -\frac{1}{2}x$$

$$x = \frac{4}{5}.$$

This makes the y-coordinate of C

$$-\frac{1}{2} \cdot \frac{4}{5} = -\frac{2}{5}.$$

The area of triangle ABC is then

$$\frac{1}{2} \cdot 1 \cdot \frac{2}{5} = \frac{1}{5}.$$

Thus, **B** is the correct answer.

16. Three runners start running simultaneously from the same point on a 500-meter circular track. They each run clockwise around the course maintaining constant speeds of $4.4, 4.8$, and 5.0 meters per second. The runners stop once they are all together again somewhere on the circular course. How many seconds do the runners run?

A $1,000$

B $1,250$

C $2,500$

D $5,000$

E $10,000$

Solution(s):

Let us find the amount of time that it takes for the runner running at 4.8 meters per second to lap the second fastest person.

We must have that

$$4.8x - 4.4x = 500$$
$$x = 1250,$$

where x is the amount of time it takes for the faster runner to lap the other.

Note that $4.4 \cdot 1250 = 5500$, which means that these two runners always intersect at the starting line.

We now have to find the least time, t, such that t is a multiple of 1250 and the fastest runner ends up at the starting line.

Every 1250 seconds, the fastest runner runs $1250 \cdot 5 = 6250$ meters. Then in 2500 seconds, the fastest runner runs 12500 meters, which is a whole number of laps.

Thus, **C** is the correct answer.

17. Let a and b be relatively prime positive integers with $a > b > 0$ and

$$\frac{a^3 - b^3}{(a-b)^3} = \frac{73}{3}.$$

What is $a - b$?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution(s):

Recall that we can factor

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

Canceling out this factor gives us that

$$\frac{a^2 + ab + b^2}{a^2 - 2ab + b^2} = \frac{73}{3}.$$

Cross-multiplying and rearranging gives us

$$70a^2 - 149ab + 70b^2 = 0.$$

Since $b \neq 0$, we can divide through by b^2 to get

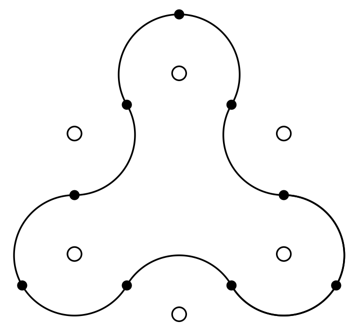
$$70\left(\frac{a}{b}\right)^2 - 149\frac{a}{b} + 70 = 0.$$

Applying the quadratic formula and noting that $a > b$ gives us that $\dfrac{a}{b} = \dfrac{10}{7}$.

Since a and b are relatively prime, we have that $a = 10$ and $b = 7$. Their difference is $10 - 7 = 3$.

Thus, **C** is the correct answer.

18. The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?



A $2\pi + 6$

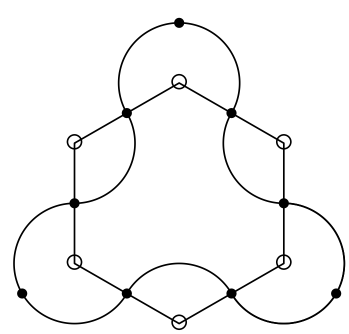
B $2\pi + 4\sqrt{3}$

C $3\pi + 4$

D $2\pi + 3\sqrt{3} + 2$

E $\pi + 6\sqrt{3}$

Solution(s):



Note that the region enclosed by the curve but outside the hexagon consists of 3 sectors with angle 240°.

This means that together they form 2 whole circles with radius 1. Now to find the area of the region inside both the hexagon and the curve.

This area can be found by finding the area of the hexagon and subtracting out the areas of the sectors outside the curve.

There are three $\frac{1}{3}$ circles that together form a whole circle. The area of the hexagon can be given by

$$\frac{3}{2} \cdot 2^2 \cdot \sqrt{3} = 6\sqrt{3}.$$

The desired area is then

$$2\pi + (6\sqrt{3} - \pi) = \pi + 6\sqrt{3}.$$

Thus, **E** is the correct answer.

19. Paula the painter and her two helpers each paint at constant, but different, rates. They always start at $8:00$ AM, and all three always take the same amount of time to eat lunch.

On Monday the three of them painted 50% of a house, quitting at $4:00$ PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at $2:12$ PM. On Wednesday Paula worked by herself and finished the house by working until $7:12$ P.M.

How long, in minutes, was each day's lunch break?

- A 30
- B 36
- C 42
- D 48
- E 60

Solution(s):

Let Paula work at a rate of $x\%$ per hour and the helpers combined work at a rate of y. Let l be the duration of the lunch break.

Then we have the following equations.

$$(8-l)(x+y)=50$$
$$(6.2-l)y=24$$
$$(11.2-l)x=36.$$

Adding the second and third equations together gives us

$$6.2y+11.2x-L(x+y)=50.$$

We can then subtract the first equation from this to get

$$-1.8y+3.2x=0$$

$$h=\frac{16}{9}p.$$

We can now substitute this into the second equation, which gives us

$$(6.2 - l)\frac{16}{9}x = 24$$

$$(6.2 - l)p = \frac{27}{2}.$$

Finally, subtracting this from the third equation gets us

$$5p = 26 - \frac{27}{2}$$

$$p = \frac{5}{2}.$$

Plugging in this value gives us $l = \frac{4}{5}$, which is the same as 48 minutes.

Thus, **D** is the correct answer.

20. A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random.

The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability the grid is now entirely black?

A $\dfrac{49}{512}$

B $\dfrac{7}{64}$

C $\dfrac{121}{1024}$

D $\dfrac{81}{512}$

E $\dfrac{9}{32}$

Solution(s):

Since the center square does not get affected by the rotation, we have that it must be black.

Now, let us analyze the corners. Clearly, if they are all black, that works. If only one is white, that also works.

If two are white, they must diagonal from each other, since otherwise a white will move onto a white, keeping it white.

There is no way for three or all four of them to be white, since that ensures a white square will move onto a white square.

This gives us

$$1 + 4 + \binom{4}{2} = 7$$

possible colorings for the corners. Similarly, we have that there are also 7 working colorings for the edges.

This gives us a total probability of

$$\frac{7 \cdot 7}{2^9} = \frac{49}{512}.$$

Thus, **A** is the correct answer.

21. Let points

$$A = (0,0,0),$$
$$B = (1,0,0),$$
$$C = (0,2,0),$$
$$D = (0,0,3).$$

Points E, F, G, and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of $EFGH$?

A $\sqrt{2}$

B $\dfrac{2\sqrt{5}}{3}$

C $\dfrac{3\sqrt{5}}{4}$

D $\sqrt{3}$

E $\dfrac{2\sqrt{7}}{3}$

Solution(s):

Note that $EF = \dfrac{1}{2}AD$ since it is a midsegment of $\triangle ABD$. Similarly, $HG = \dfrac{1}{2}AD$ and $FG = \dfrac{1}{2}BC$.

We also have that $\overline{EF}$ and $\overline{HG}$ are perpendicular to the xy-plane, which means that they are perpendicular to $\overline{FG}$ and $\overline{EH}$.

This tells us that $EFGH$ is rectangle since $EF = HG$. We have $EF = \dfrac{1}{2} \cdot 3 = \dfrac{3}{2}$.

We also have that

$$FG = \frac{1}{2}\sqrt{1^2 + 2^2} = \frac{\sqrt{5}}{2}.$$

The area of $EFGH$ is then

$$EF \cdot FG = \frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \frac{3\sqrt{5}}{4}.$$

Thus, **C** is the correct answer.

22. The sum of the first m positive odd integers is 212 more than the sum of the first n positive even integers. What is the sum of all possible values of n?

A 255

B 256

C 257

D 258

E 259

Solution(s):

Recall that the sum of the first m odd numbers is m^2 and the first n even numbers is $n^2 + n$.

We have that

$$m^2 = n^2 + n + 212.$$

We can view this equation as a quadratic in terms of n as follows.

$$n^2 + n + (212 - m^2) = 0.$$

We can apply the quadratic formula to get

$$n = \frac{-1 + \sqrt{1 - 4(212 - m^2)}}{2}.$$

We have that m and n are integers, so

$$1 - 4(212 - m^2) = 4m^2 - 847$$

must be a square number. Also, since we add -1 and divide by 2, this must be odd.

Let

$$x = \sqrt{4m^2 - 947}.$$

Squaring and rearranging gives us

$$4m^2 - x^2 = 847$$

$$(2m + x)(2m - x)$$

$$= 847 = 7 \cdot 11^2.$$

Note that

$$n = \frac{-1 + x}{2}.$$

Given the values for $2m + x$ and $2m - x$, we have that the difference between the two is $2x$.

The possible pairs of values are

$$847 \cdot 1, 121 \cdot 7, 77 \cdot 11.$$

These pairs contribute the following values for x respectively: $423, 57, 33$. Then the possible values for n are $211, 28$, and 16.

Thus, **A** is the correct answer.

23. Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

A 60

B 170

C 290

D 320

E 660

Solution(s):

We case on the value of friends that each person has. This value ranges from 1 to 4 since all of them are not friends.

Note that the cases for 1 and 2 friends correspond with the case for 4 and 3 friends, since choosing who are friends determines who are not friends.

Case 1: everyone has 1 friend

This means that the 6 people must split up into 3 pairs where the people in each pair are friends.

There are 5 choices for the friend for the first person. This leaves 4 people remaining.

There are then 3 choices for the friend of the next person. The remaining 2 people are then forced to be friends.

Therefore, there are $3 \cdot 5 = 15$ possibilities for this case.

Case 2: everyone has 2 friends

There are two possibilities for this case. There could be two triples where everyone in a triple is friends with each other.

For this possibility, there are $\binom{6}{3} = 20$ ways to choose the people in the first pair. We have to divide by 2 since we can swap the pairs. This gives us $20 \div 2 = 10$ configurations.

The second possibility is if the friends form a hexagon where the person at each vertex is friends with the adjacent people.

The first person can be placed anywhere on the hexagon. There are $\binom{5}{2} = 10$ ways to choose the people adjacent to this person.

The final 3 people can placed in $3! = 6$ ways in the remaining spots. This case then has a total number of

$$10 + 10 \cdot 6 = 70$$

configurations.

The total number of arrangements is then have

$$2(15 + 70) = 170.$$

Thus, **B** is the correct answer.

24. Let a, b, and c be positive integers with $a \geq b \geq c$ such that

$$a^2 - b^2 - c^2 + ab = 2011$$

and

$$a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc$$
$$= -1997.$$

What is a?

A 249

B 250

C 251

D 252

E 253

Solution(s):

Adding together the equations gives us

$$2(a^2 + b^2 + c^2) - 2(ab + ac + bc)$$
$$= 14.$$

We can group terms and factor this to get

$$(a-b)^2 + (a-c)^2 + (b-c)^2$$
$$= 14.$$

Note that every term on the left hand side is a positive square integer. The only triple of squares that add to 14 is $9, 4,$ and 1.

We have that $a - c$ is the biggest difference among the three pairs. Therefore, $a - c = 3$.

We cannot discern which of the other terms we can match with the other squares. Let us try $a - b = 1$ and $b - c = 2$.

Plugging in these values into the first equation gives us

$$a^2-(a-1)^2-(a-3)^2$$

$$+a(a-1)=2011.$$

Simplifying yields $7a=2021$. Since 2021 is not divisible by 7, we have that $a-b=2$ and $b-c=1$.

Plugging these in again and solving gives us $a=253$.

Thus, **E** is the correct answer.

25. Real numbers x, y, and z are chosen independently and at random from the interval $[0, n]$ for some positive integer n. The probability that no two of x, y, and z are within 1 unit of each other is greater than $\frac{1}{2}$. What is the smallest possible value of n?

A 7

B 8

C 9

D 10

E 11

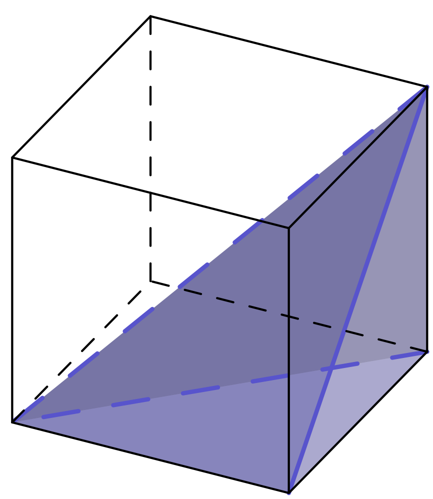
Solution(s):

This problem lends itself to geometric probability since we can view the interval as a range on an axis.

WLOG, let

$$n \geq x \geq y \geq z \geq 0.$$

Then we have that the points (x, y, z) which satisfy this restriction form a tetrahedron.



The height of this tetrahedron is n, and the base has an area of $\frac{n^2}{2}$. This makes the volume

$$\frac{1}{3} \cdot \frac{n^2}{2} \cdot n = \frac{n^3}{6}.$$

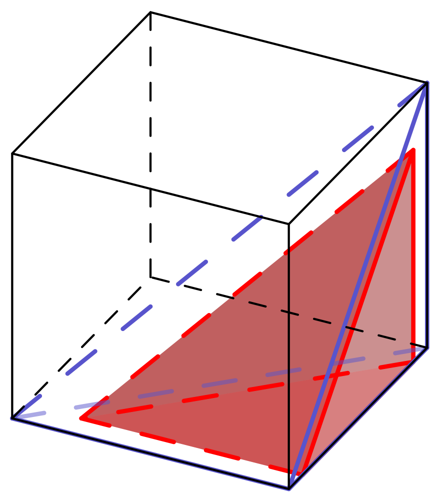
Now we have to apply the restrictions from the problem statement. We need to find the region where

$$|x-y|, |x-z|, |y-z| \geq 1.$$

From our ordering condition that we imposed, these inequalities reduce to

$$x - y \geq 1 \text{ and } y - z \geq 1.$$

These two restrictions form another tetrahedron as shown below.



Note that in the new tetrahedron, all the dimensions have been reduced by 2. This makes the height $n-2$ and the base $\dfrac{(n-2)^2}{2}$.

The area is then

$$\frac{1}{2} \cdot \frac{(n-2)^2}{2} \cdot (n-2)$$

$$= \frac{(n-2)^3}{6}.$$

The desired probability is then

$$\frac{(n-2)^3}{6} \div \frac{n^3}{6} = \frac{(n-2)^3}{n^3}.$$

Plugging in all the answer choices, we get that the smallest value such that this fraction is greater than $\dfrac{1}{2}$ is 10.

Thus, **D** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2012A