2011 AMC 10B Solutions

Typeset by: LIVE, by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc10/2011B/solutions

1. What is

$$\frac{2+4+6}{1+3+5}-\frac{1+3+5}{2+4+6}?$$

A -1

B $\frac{5}{36}$

C $\frac{7}{12}$

D $\frac{147}{60}$

E $\frac{43}{3}$

Solution(s):

Simply solving directly:

$$\begin{aligned}
&\frac{2+4+6}{1+3+5}-\frac{1+3+5}{2+4+6}\\
&=\frac{12}{9}-\frac{9}{12}\\
&=\frac{16}{12}-\frac{9}{12}\\
&=\frac{7}{12}.
\end{aligned}$$

Thus, the correct answer is **C**.

2. Josanna's test scores to date are $90, 80, 70, 60$, and 85. Her goal is to raise here test average at least 3 points with her next test. What is the minimum test score she would need to accomplish this goal?

- A 80
- B 82
- C 85
- D 90
- E 95

Solution(s):

Her current average is

$$\frac{90+80+70+60+85}{5}=77,$$

and the sum of her scores is 385. The desired average is then 80, so the sum of scores required is $80 \cdot 6$. Therefore, the answer is

$$80 \cdot 6-385=95.$$

Thus, the correct answer is **E**.

3. At a store, when a length or a width is reported as x inches that means it is at least $x - 0.5$ inches and at most $x + 0.5$ inches. Suppose the dimensions of a rectangular tile are reported as 2 inches by 3 inches. In square inches, what is the minimum area for the rectangle?

A 3.75

B 4.5

C 5

D 6

E 8.75

Solution(s):

The smallest possible dimensions are 1.5×2.5, so the area is

$$1.5 \cdot 2.5 = 3.75.$$

Thus, the correct answer is **A**.

4. LeRoy and Bernardo went on a week-long trip together and agreed to share the costs equally. Over the week, each of them paid for various joint expenses such as gasoline and car rental. At the end of the trip it turned out that LeRoy had paid A dollars and Bernardo had paid B dollars, where $A < B$. How many dollars must LeRoy give to Bernardo so that they share the costs equally?

A $\dfrac{A+B}{2}$

B $\dfrac{A-B}{2}$

C $\dfrac{B-A}{2}$

D $B-A$

E $A+B$

Solution(s):

The amount they each would have to pay is $\dfrac{A+B}{2}$, and LeRoy paid A. Thus, he has to pay

$$\frac{A+B}{2} - A = \frac{B-A}{2}$$

more.

Thus, the correct answer is **C**.

5. In multiplying two positive integers a and b, Ron reversed the digits of the two-digit number a. His erroneous product was 161. What is the correct value of the product of a and b?

A 116

B 161

C 204

D 214

E 224

Solution(s):

The number 161 is equal to $7 \cdot 23$. There are no other pairs of numbers that multiply to 161 besides $1 \cdot 161$, so 23 is the only two digit factor. Thus, 23 is the number reversed, so he mean to get $32 \cdot 7$ which is 224.

Thus, the correct answer is **E**.

6. On Halloween Casper ate $\frac{1}{3}$ of his candies and then gave 2 candies to his brother. The next day he ate $\frac{1}{3}$ of his remaining candies and then gave 4 candies to his sister. On the third day he ate his final 8 candies. How many candies did Casper have at the beginning?

A 30

B 39

C 48

D 57

E 66

Solution(s):

Let c be the total amount of candies.

After day one, he used $\frac{c}{3}+2$ of his candies, so he had $\frac{2c}{3}-2$ left.

After day two, he used

$$\frac{\frac{2c}{3}-2}{3}+4=\frac{2c}{9}+\frac{10}{3}$$

of his candies, so he had $\frac{2c}{9}-\frac{14}{3}$ left.

He had 8 candies after this, so

$$\frac{4c}{9}-\frac{16}{3}=8.$$

This makes $c=30$.

Thus, the correct answer is **A**.

7. The sum of two angles of a triangle is $\frac{6}{5}$ of a right angle, and one of these two angles is 30° larger than the other. What is the degree measure of the largest angle in the triangle?

- A 69
- B 72
- C 90
- D 102
- E 108

Solution(s):

The two angles add to $\frac{6}{5} \cdot 90 = 108$. This makes the other angle $180 - 108 = 72$. Then, if the larger of the two angles is x then the smaller of them is $x - 30$ so their sum is $2x - 30 = 108$, making $x = 69$.

This means no angle is larger than 72, making the largest eqaul to 72.

Thus, the correct answer is **B**.

8. At a certain beach if it is at least $80^{\circ} F$ and sunny, then the beach will be crowded. On June 10 the beach was not crowded. What can be concluded about the weather conditions on June 10?

A The temperature was cooler than 80°F and it was not sunny.

B The temperature was cooler than 80°F or it was not sunny.

C If the temperature was at least 80° F, then it was sunny.

D If the temperature was cooler than 80° F, then it was sunny.

E If the temperature was cooler than 80°F, then it was not sunny.

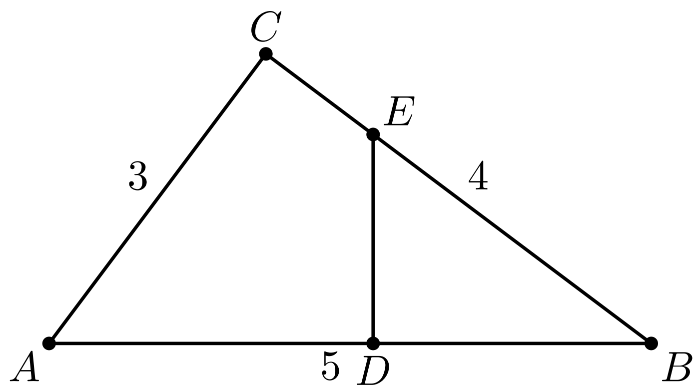
Solution(s):

We know that over 80 degrees and sunny combinded implies crowded, so not crowded implies that the combination of over 80 degrees and sunny is not true. This immeadiately eliminated choice C.

We have no more information, so if it was below 80 degrees, we don't know if it is crowded. Thus, A, D and E are eliminated.

Thus, the correct answer is **B**.

9. The area of $\triangle EBD$ is one third of the area of $\triangle ABC$. Segment DE is perpendicular to segment AB. What is BD?



A $\frac{4}{3}$

B $\sqrt{5}$

C $\frac{9}{4}$

D $\frac{4\sqrt{3}}{3}$

E $\frac{5}{2}$

Solution(s):

By angle angle similarity, we have $BDE \sim BCA$.

Then, since the ratio of the areas is $\frac{1}{3}$, the ratio of the sidelengths is $\frac{1}{\sqrt{3}}$.

As such,

$$\frac{BD}{BC} = \frac{BD}{4} = \frac{1}{\sqrt{3}},$$

making

$$BD = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}.$$

Thus, the correct answer is **D**.

10. Consider the set of numbers $\{1, 10, 10^2, 10^3, \ldots, 10^{10}\}$. The ratio of the largest element of the set to the sum of the other ten elements of the set is closest to which integer?

A 1

B 9

C 10

D 11

E 101

Solution(s):

The largest number is 10^{10}. The rest of the number have a sum of

$$S = \sum_{i=0}^{9} 10^i.$$

Then, $10S = \sum_{i=0}^{9} 10^{i+1}$, making $9S = 10^{10} - 1$. This means that $\dfrac{10^{10}}{9S}$ is close to one, so the ratio between 10^{10} and the sum is close to 9.

Thus, the correct answer is **B**.

11. There are 52 people in a room. what is the largest value of n such that the statement "At least n people in this room have birthdays falling in the same month" is always true?

A 2

B 3

C 4

D 5

E 12

Solution(s):

It isn't nessicarily true for $n \geq 6$ as we could have 5 people born in the first 4 months and 4 people born in the subsequent months.

However, one month must be greater than or equal to 5 as the average of the number of people born in each month is $\frac{52}{12}$ which is greater than 4, and some month must be above average.

Thus, the correct answer is **D**.

12. Keiko walks once around a track at exactly the same constant speed every day. The sides of the track are straight, and the ends are semicircles. The track has a width of 6 meters, and it takes her 36 seconds longer to walk around the outside edge of the track than around the inside edge. What is Keiko's speed in meters per second?

A $\frac{\pi}{3}$

B $\frac{2\pi}{3}$

C π

D $\frac{4\pi}{3}$

E $\frac{5\pi}{3}$

Solution(s):

Let the radius of the inside loop be r and let the straights have length s. Then, the distance he walks is on the inside is then $2\pi r + 2s$. Then, the radius of the outside is $r + 6$, so the distance he walks is on the inside is then

$$2\pi(r+6) + 2s = 2\pi r + 12\pi + 2s.$$

Therefore, he walks 12π more meters in 36 seconds.

Since $d = vt$ where v is speed, we have $12\pi = 36v$. Thus, $v = \frac{\pi}{3}$.

Thus, the correct answer is **A**.

13. Two real numbers are selected independently at random from the interval $[-20, 10]$. What is the probability that the product of those numbers is greater than zero?

A $\frac{1}{9}$

B $\frac{1}{3}$

C $\frac{4}{9}$

D $\frac{5}{9}$

E $\frac{2}{3}$

Solution(s):

There is 0 probability that our number is 0, so we need to just find the probability that the product isn't less than 0. The number product is less than zero if one of the numbers is less than 0 and one of them is greater than 0.

First, there are 2 ways to choose the designated lower number. Then, the probability that the designated lower number is less than 0 is $\frac{2}{3}$ and the probability that the designated higher number is greater than 0 is $\frac{1}{3}$.

This makes the probability that the product is less than 0 equal to

$$2 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9}.$$

As such, the probability that the product is greater than 0 equal to $\frac{5}{9}$.

Thus, the correct answer is **D**.

14. A rectangular parking lot has a diagonal of 25 meters and an area of 168 square meters. In meters, what is the perimeter of the parking lot?

- A 52
- B 58
- C 62
- D 68
- E 70

Solution(s):

Let the side lengths be l, w. We wish to find $2(l+w)$. From the Pythagorean Theorem, we get

$$25 = \sqrt{l^2 + w^2}$$

$$l^2 + w^2 = 625$$

We also know $lw = 168$.

As such

$$\begin{aligned} l^2 + 2lw + w^2 &= (l+w)^2 \\ &= 961 \\ &= 31^2. \end{aligned}$$

This makes $l + w = 31$, and as such, our answer is $31 \cdot 2 = 62$.

Thus, the correct answer is **C**.

15. Let @ denote the "averaged with" operation: $a@b = \frac{a+b}{2}$. Which of the following distributive laws hold for all numbers x, y, and z?

I. $x@(y+z) = (x@y) + (x@z)$

II. $x + (y@z) = (x+y)@(x+z)$

III. $x@(y@z) = (x@y)@(x@z)$

A I only

B II only

C III only

D I and III only

E II and III only

Solution(s):

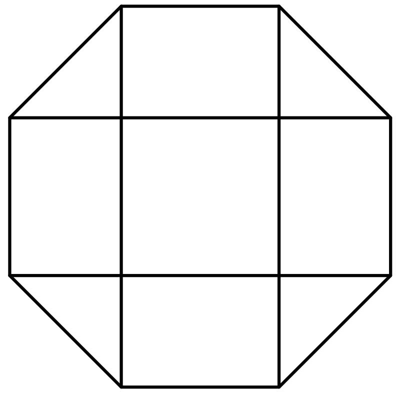
In text 1, the left hand side equals $\dfrac{x+y+z}{2}$ and the right hand side equals $x + \dfrac{y+z}{2}$, so they aren't equal.

In text 2, the left hand side equals $x + \dfrac{y+z}{2}$ and the right hand side equals $x + \dfrac{y+z}{2}$, so they are equal.

In text 3, the left hand side equals $\dfrac{2x+y+z}{4}$ and the right hand side equals $\dfrac{2x+y+z}{4}$, so they are equal.

Thus, the correct answer is **E**.

16. A dart board is a regular octagon divided into regions as shown. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands within the center square?



A $\frac{\sqrt{2}-1}{2}$

B $\frac{1}{4}$

C $\frac{2-\sqrt{2}}{2}$

D $\frac{\sqrt{2}}{4}$

E $2-\sqrt{2}$

Solution(s):

Let the side length be 1. Then, the area of the center is 1.

Then, we must find the area of the octagon. It can be found as a square with 4 isoceles right triangles taken out. The side length of this square is

$$1+2\cdot\frac{1}{\sqrt{2}}=1+\sqrt{2}.$$

It has an area of

$$(1+\sqrt{2})^2=3+2\sqrt{2}.$$

Then, the side length of the right triangles is $\frac{1}{\sqrt{2}}$, making the area of one equal to

$$\frac{\frac{1}{\sqrt{2}}^{2}}{2} = \frac{1}{4}.$$

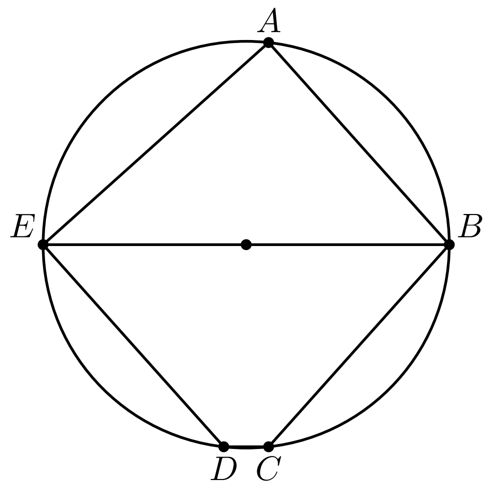
This makes them have a total combined area of 1, so the area of the octagon is $2 + 2\sqrt{2}$.

Thus, the ratio is

$$\begin{aligned}
&\frac{1}{2 + 2\sqrt{2}} \\
&= \frac{-2 + 2\sqrt{2}}{(2 + 2\sqrt{2})(-2 + 2\sqrt{2})} \\
&= \frac{2(\sqrt{2} - 1)}{4} \\
&= \frac{\sqrt{2} - 1}{2}.
\end{aligned}$$

Thus, the correct answer is **A**.

17. In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?



A 120

B 125

C 130

D 135

E 140

Solution(s):

The angle is equal to the angle of the major arc

$$\frac{\overparen{BAD}}{2} = 180^\circ - \frac{\overparen{BCD}}{2}.$$

Then, since $AB||ED$, AE and BD are equal making their arcs equal, making our answer equal to $180^\circ - \dfrac{\overparen{AE}}{2}$.

Then, by the angle ratio, we have $\overparen{AE} : \overparen{ED} = 5:4$. Since the sum of the arcs is 180°, the arc $\overparen{AE}$ is eqaul to 100°. Therefore, the answer is

$$180^\circ - \frac{\overset{\frown}{100^\circ}}{2} = 130^\circ.$$

Thus, the correct answer is **C**.

18. Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

A 15

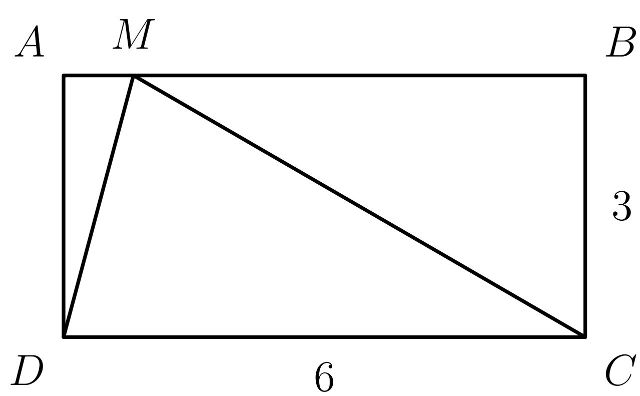
B 30

C 45

D 60

E 75

Solution(s):



The angles $\angle AMD$ and $\angle MDC$ are equal since $AB \mid\mid DC$.

As such, $\angle MDC = \angle DMC$, making MDC isoceles and $MC = DC = 6$.

As we can see, $\sin(CMB) = \frac{1}{2}$, making $\angle CMB = 30^\circ$.

Therefore, $\angle AMC = 150^\circ$. Since $\angle AMD$ is half of that, $\angle AMD = 75^\circ$.

Thus, the correct answer is **E**.

19. What is the product of all the roots of the equation

$$\sqrt{5|x|+8} = \sqrt{x^2-16}?$$

A -64

B -24

C -9

D 24

E 576

Solution(s):

The equation is equal to

$$\sqrt{5|x|+8} = \sqrt{|x|^2-16}.$$

Solving, we get that:

$$\begin{aligned} 5|x|+8 &= |x|^2-16 \\ |x|^2-5|x|-24 &= 0 \\ (|x|-8)(|x|+3) &= 0 \end{aligned}$$

This makes $|x| = -3, 8$, making $|x| = 8$ the only possible value. Thus, $x = 8, -8$ with a product of -64.

Thus, the correct answer is **A**.

20. Rhombus $ABCD$ has side length 2 and $\angle B = 120°$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R?

A $\dfrac{\sqrt{3}}{3}$

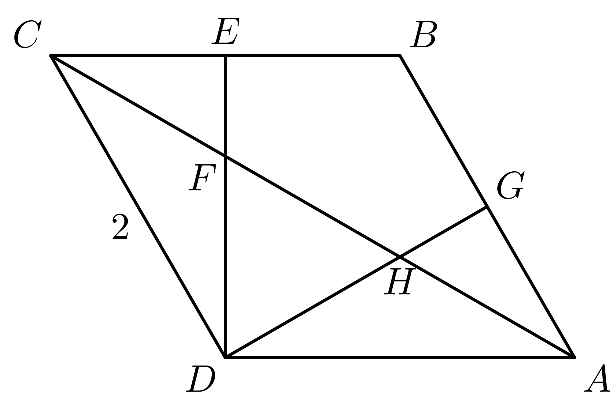
B $\dfrac{\sqrt{3}}{2}$

C $\dfrac{2\sqrt{3}}{3}$

D $1+\dfrac{\sqrt{3}}{3}$

E 2

Solution(s):



To find the points closest to B, we must find the points closer to B when taking the perpendicular bisector between B and any other point.

If we take perpendicular bisector of B and D, the amount on the side closer to B is equal to the area of ABC which is equal to

$$\frac{2 \cdot 2 \cdot \sin(120^\circ)}{2} = \sqrt{3}.$$

If we take perpendicular bisector of B and A, the amount on the side closer to C is equal to the area of CEF which has $CE = 1$ and $EF = \tan(30^\circ) = \dfrac{1}{\sqrt{3}}$. Therefore, the amount subtracted here is the area of CEF which is equal to

$$\frac{1 \cdot \frac{1}{\sqrt{3}}}{2} = \frac{\sqrt{3}}{6}.$$

Doing the same with the perpendicular bisector of B and C has the same area subtracted as above, so the area is equal to

$$\begin{aligned}\sqrt{3} - 2 \cdot \frac{\sqrt{3}}{6} &= \sqrt{3} - \frac{\sqrt{3}}{3} \\ &= \frac{2\sqrt{3}}{3}.\end{aligned}$$

Thus, the correct answer is **C**.

21. Brian writes down four integers $w > x > y > z$ whose sum is 44. The pairwise positive differences of these numbers are $1, 3, 4, 5, 6,$ and 9. What is the sum of the possible values for w?

A 16

B 31

C 48

D 62

E 93

Solution(s):

The largest difference must be 9 so $w - z = 9$. Then, $(w - x) + (x - z) = 9$, so they must both be two different pariwise differences that add to 9. Thus,

$$w - x = 6, x - z = 3$$

or

$$w - x = 5, x - z = 4.$$

or the other way around.

Similarly,

$$w - y = 6, y - z = 3$$

or

$$w - y = 5, y - z = 4$$

or the other way around.

Then, we must have $x - y = 1$ since it is the only place for it to be. Thus, we could have

$$y - z = 3, x - z = 4$$

or

$$y - z = 5, x - z = 6.$$

Thus, the cases are

$$w - z = 9,$$

$$w - y = 6,$$

$$w - y = 5$$

or

$$w - z = 9,$$

$$w - y = 5,$$

$$w - y = 3.$$

Then, for the first case, we have:

$$w + x + y + z = 44$$

$$w + w - 5 + w - 6 + w - 9 = 44$$

$$4w = 64$$

$$w = 16.$$

Also, for the second case, we have:

$$w + x + y + z = 44$$

$$w + w - 3 + w - 4 + w - 9 = 44$$

$$4w = 60$$

$$w = 15.$$

The sum over all cases is then 31.

Thus, the correct answer is **B**.

22. A pyramid has a square base with sides of length 1 and has lateral faces that are equilateral triangles. A cube is placed within the pyramid so that one face is on the base of the pyramid and its opposite face has all its edges on the lateral faces of the pyramid. What is the volume of this cube?

A $5\sqrt{2}-7$

B $7-4\sqrt{3}$

C $\frac{2\sqrt{2}}{27}$

D $\frac{\sqrt{2}}{9}$

E $\frac{\sqrt{3}}{9}$

Solution(s):

Let the side length of the cube be s. Then, we take the diagonal cross section of the cube.

This would have a $1-1\sqrt{2}$ right triangle. Then, the base has $s\sqrt{2}$ and two legs of right isoceles triangles. The legs of the isoceles triangle is s, so the side equal to $\sqrt{2}$ is also equal to $(\sqrt{2}+1)s$.

Therefore,

$$(\sqrt{2}+2)s=\sqrt{2}$$

$$s=\frac{\sqrt{2}}{\sqrt{2}+2}$$

$$s=\frac{(\sqrt{2})(2-\sqrt{2})}{(\sqrt{2}+2)(2-\sqrt{2})}$$

$$s=\sqrt{2}-1.$$

Then the volume is

$$s^3=(\sqrt{2}-1)^3=5\sqrt{2}-7.$$

Thus, the correct answer is **A**.

23. What is the hundreds digit of 2011^{2011}?

A 1

B 4

C 5

D 6

E 9

Solution(s):

We must find $2011^{2011} \mod 1000$.

This is equivalent to $11^{2011} \mod 1000$.

By the binomial theorem, we get that this is equal to

$$(10+1)^{2011} = \sum_{i=0}^{2011} 10^i \binom{2011}{i}.$$

Then, if $i \geq 3$, it is a multiple of 1000, so

$$\begin{aligned}
&\sum_{i=0}^{2} 10^i \binom{2011}{i} \\
&\equiv 100 \cdot \frac{2011 \cdot 2010}{2} \\
&+ 100 \cdot 2011 \cdot 1005 \\
&+ 2011 \cdot 10 + 1 \\
&\equiv 100 \cdot 11 \cdot 5 + 11 \cdot 10 + 1 \\
&\equiv 5651 \\
&\equiv 651
\end{aligned}$$

This has a hundreds digit of 6.

Thus, the correct answer is **D**.

24. A lattice point in an xy-coordinate system is any point (x, y) where both x and y are integers. The graph of $y = mx + 2$ passes through no lattice point with $0 < x \leq 100$ for all m such that $\frac{1}{2} < m < a$. What is the maximum possible value of a?

A $\dfrac{51}{101}$

B $\dfrac{50}{99}$

C $\dfrac{51}{100}$

D $\dfrac{52}{101}$

E $\dfrac{13}{25}$

Solution(s):

The lattice point x, y is a coordingate intesected by $y = mx + 2$, if and only if $y = mx$ intersects the lattice point $(x, y - 2)$, so it suffices to look at $y = mx$ instead.

Thus, we must find the smallest $m >$ such that it intersects a lattice point. We will inspect each x and find the smallest m that intersects that lattice point and take the maximum.

If x is even, then the number would be

$$\frac{x+2}{2x} = \frac{1}{2} + \frac{1}{x}.$$

The minimum of this would be $x = 100$ which is

$$\frac{1}{2} + \frac{1}{100} = \frac{51}{100}.$$

If x is even, then the number would be

$$\frac{x+1}{2x} = \frac{1}{2} + \frac{1}{2x}.$$

The minimum of this would be $x = 99$ which is

$$\frac{1}{2} + \frac{1}{2 \cdot 99} = \frac{50}{99}.$$

The minimum of the possible m is then $\dfrac{50}{99}$ since it is less than $\dfrac{51}{100}$.

Thus, the correct answer is **B**.

25. Let T_1 be a triangle with side lengths $2011, 2012$, and 2013. For $n \geq 1$, if $T_n = \triangle ABC$ and D, E, and F are the points of tangency of the incircle of $\triangle ABC$ to the sides AB, BC, and AC, respectively, then T_{n+1} is a triangle with side lengths AD, BE, and CF, if it exists. What is the perimeter of the last triangle in the sequence (T_n)?

A $\dfrac{1509}{8}$

B $\dfrac{1509}{32}$

C $\dfrac{1509}{64}$

D $\dfrac{1509}{128}$

E $\dfrac{1509}{256}$

Solution(s):

Suppose we have the side lengths of $a = BC, b = AC, c = AB$ and the side lengths of the next triangle is x, y, z.

Then, we know that $x = AD = AF$ by the congruence of ADI and AFI since they are right triangles with an equal hypotenuse and an equal length.

Similarly, $y = BD = BE$ and $z = CF = CE$.

Thus,

$$x + y = AB = c,$$

$$x + z = AC = b,$$

and

$$y + z = BC = a.$$

Then, we have

$$x + y + z = \frac{a + b + c}{2}$$

so

$$z=\frac{a+b-c}{2},$$

$$y=\frac{a-b+c}{2},$$

$$x=\frac{-a+b+c}{2}.$$

Suppose $a=b-1, c=b+1$. Then,

$$z=\frac{b-1+b-(b-1)}{2}$$

$$=\frac{b}{2}+1,$$

$$y=\frac{b-1-b+b+1}{2}$$

$$=\frac{b}{2},$$

$$x=\frac{-(b+1)+b+b+1}{2}$$

$$=\frac{b}{2}-1.$$

Thus, $y=\frac{1}{2}y$, $x=y+1$, and $z=y-1$. This means that the triangle would always be in the form $s-1, s, s+1$ for all T_i.

This would satisfy the triangle inequality if $s-1+s>s+1$ making $s>2$. Also, the perimeter is equal to $3s$.

Thus, the middle term of T_i is equal to $\dfrac{2012}{2^{i-1}}$ since it always halves, so the first time it is less than 2 is if $i=11$. This makes the last $i=10$, making the middle term equal to

$$\frac{2012}{2^{10}}=\frac{2012}{2^{9}}=\frac{503}{128}.$$

Then, the perimeter equals

$$3 \cdot \frac{503}{128} = \frac{1509}{128}.$$

Thus, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc10/2011B