2011 AMC 10A Solutions

Typeset by: LIVE, by Po-Shen Loh
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1. A cell phone plan costs $\$20$ each month, plus $5¢$ per text message sent, plus 10 cents for each minute used over 30 hours. In January Michelle sent 100 text messages and talked for 30.5 hours. How much did she have to pay?

A $\$24.00$

B $\$24.50$

C $\$25.50$

D $\$28.00$

E $\$30.00$

Solution(s):

Michelle has to pay $\$20$ for the monthly fee. She also has to pay

$$100 \cdot 5¢ = 500¢ = \$5$$

for the text messages. Finally she talked for $.5$ hours, 30 minutes, over 30 hours. This means she has to pay an extra

$$30 \cdot 10¢ = 300¢ = \$3.$$

Her total cost is

$$\$20 + \$5 + \$3 = \$28.$$

Thus, **D** is the correct answer.

2. A small bottle of shampoo can hold 35 milliliters of shampoo, whereas a large bottle can hold 500 milliliters of shampoo. Jasmine wants to buy the minimum number of small bottles necessary to completely fill a large bottle. How many bottles must she buy?

A 11

B 12

C 13

D 14

E 15

Solution(s):

The desired amount is

$$\frac{500}{35} = \frac{100}{7} = 14\frac{2}{7}.$$

This means that smallest number of small bottles she must be is 15.

Thus, **E** is the correct answer.

3. Suppose $[ab]$ denotes the average of a and b, and $\{a\ b\ c\}$ denotes the average of a, b, and c. What is

$$\{\{1\ 1\ 0\}\ [0\ 1]\ 0\}?$$

A $\dfrac{2}{9}$

B $\dfrac{5}{18}$

C $\dfrac{1}{3}$

D $\dfrac{7}{18}$

E $\dfrac{2}{3}$

Solution(s):

We have that

$$\{1\ 1\ 0\} = \frac{1+1+0}{3} = \frac{2}{3}.$$

We also get that

$$[0\ 1] = \frac{1+0}{2} = \frac{1}{2}.$$

Finally,

$$\left\{\frac{2}{3}\ \frac{1}{2}\ 0\right\} = \frac{\frac{2}{3}+\frac{1}{2}+0}{3} = \frac{7}{18}.$$

Thus, **D** is the correct answer.

4. Let X and Y be the following sums of arithmetic sequences:

$$X = 10 + 12 + 14 + \cdots + 100,$$
$$Y = 12 + 14 + 16 + \cdots + 102.$$

What is the value of $Y - X$?

A 92

B 98

C 100

D 102

E 112

Solution(s):

Note that the terms 10 through 100 are common to both sums. When we subtract, all these terms cancel out.

This means that

$$Y - X = 102 - 10 = 92.$$

Thus, **A** is the correct answer.

5. At an elementary school, the students in third grade, fourth grade, and fifth grade run an average of 12, 15, and 10 minutes per day, respectively. There are twice as many third graders as fourth graders, and twice as many fourth graders as fifth graders. What is the average number of minutes run per day by these students?

A 12

B $\frac{37}{3}$

C $\frac{88}{7}$

D 13

E 14

Solution(s):

WLOG, let there be one fifth grader. This then tells us that there are two fourth graders and four third graders.

We can do this, since we are only interested in the average, which is not impacted by the exact number of students.

The total number of minutes the students spend running is

$$12 \cdot 4 + 15 \cdot 2 + 10 = 88$$

minutes. The total number of students is $1 + 2 + 4 = 7$. The average is then $\frac{88}{7}$.

Thus, **C** is the correct answer.

6. Set A has 20 elements, and set B has 15 elements. What is the smallest possible number of elements in $A \cup B$, the union of A and B?

- A 5
- B 15
- C 20
- D 35
- E 300

Solution(s):

To minimize the number of elements in the union, we want to maximize the overlap between the two sets.

We can then assume that B is contained completely within A, which means that the union is the same as A, which has 20 elements.

Thus, **C** is the correct answer.

7. Which of the following equations does not have a solution?

A $(x+7)^2 = 0$

B $|-3x| + 5 = 0$

C $\sqrt{-x} - 2 = 0$

D $\sqrt{x} - 8 = 0$

E $|-3x| - 4 = 0$

Solution(s):

A simplifies to

$$x + 7 = 0$$
$$x = -7,$$

so it has a solution.

B simplifies to

$$|-3x| = -5,$$

which has no solution since absolute value makes everything positive.

Let us make sure that all the other choices have solutions.

C simplifies to

$$\sqrt{-x} = 2$$
$$-x = 4$$
$$x = -4,$$

which is fine.

D simplifies to

$$\sqrt{x} = 8$$
$$x = 64,$$

which works.

Finally, **E** simplifies to

$$|-3x| = 4$$

$$-3x = \pm 4$$

$$x = \pm\frac{4}{3},$$

which has a solution as well.

Thus, **B** is the correct answer.

8. Last summer 30% of the birds living on Town Lake were geese, 25% were swans, 10% were herons, and 35% were ducks. What percent of the birds that were not swans were geese?

- A 20
- B 30
- C 40
- D 50
- E 60

Solution(s):

WLOG, let there be a 100 birds. Then 75 birds are not swans. The desired percentage is then

$$\frac{30}{75} \cdot 100 = 40\%.$$

Thus, **C** is the correct answer.

9. A rectangular region is bounded by the graphs of the equations $y = a, y = -b, x = -c$, and $x = d$, where a, b, c, and d are all positive numbers. Which of the following represents the area of this region?

A $ac + ad + bc + bd$

B $ac - ad + bc - bd$

C $ac + ad - bc - bd$

D $-ac - ad + bc + bd$

E $ac - ad - bc + bd$

Solution(s):

Note that the region is a rectangle with side lengths

$$a - (-b) = a + b$$

and

$$d - (-c) = c + d.$$

The area is then

$$(a + b)(c + d) =$$

$$ac + ad + bc + bd.$$

Thus, **A** is the correct answer.

10. A majority of the 30 students in Ms. Demeanor's class bought pencils at the school bookstore. Each of these students bought the same number of pencils, and this number was greater than 1. The cost of a pencil in cents was greater than the number of pencils each student bought, and the total cost of all the pencils was $\$17.71$. What was the cost of a pencil in cents?

A 7

B 11

C 17

D 23

E 77

Solution(s):

Let p be the number of pencils that each student bought, s be the number of students that bought pencils, and c be the cost of a pencil.

We have that

$$psc = 1771 = 7 \cdot 11 \cdot 23.$$

We also have the following restrictions:

$$30 \geq s > 15, p > 1, c > p.$$

From the above prime factorization, we have that $s = 23$ is the only value that satisfies the conditions.

Finally, we get that $p = 7$ and $c = 11$ are the only remaining values that satisfy the other conditions.

Thus, **B** is the correct answer.

11. Square $EFGH$ has one vertex on each side of square $ABCD$. Point E is on $\overline{AB}$ with $AE = 7 \cdot EB$. What is the ratio of the area of $EFGH$ to the area of $ABCD$?

A $\frac{49}{64}$

B $\frac{25}{32}$

C $\frac{7}{8}$

D $\frac{5\sqrt{2}}{8}$

E $\frac{\sqrt{14}}{4}$

Solution(s):

Let $x = EB$. Then $AB = 8x$. Applying the Pythagorean Theorem to a side of $EFGH$, we get

$$\sqrt{(7x)^2 + x^2} = \sqrt{50x^2}.$$

The desired ratio is then

$$\frac{\sqrt{50x^2}^2}{(8x)^2} = \frac{50}{64} = \frac{25}{32}.$$

Thus, **B** is the correct answer.

12. The players on a basketball team made some three-point shots, some two-point shots, and some one-point free throws. They scored as many points with two-point shots as with three-point shots. Their number of successful free throws was one more than their number of successful two-point shots. The team's total score was 61 points. How many free throws did they make?

A 13

B 14

C 15

D 16

E 17

Solution(s):

Let x be the number of successful two-point shots. Then we have that

$$2x + 2x + (x + 1) = 61,$$

which simplifies to

$$5x + 1 = 61$$
$$x = 12.$$

The number of successful free throws is then $12 + 1 = 13$.

Thus, **A** is the correct answer.

13. How many even integers are there between 200 and 700 whose digits are all different and come from the set

$$\{1, 2, 5, 7, 8, 9\}?$$

A 12

B 20

C 72

D 120

E 200

Solution(s):

Since the hundreds digit can only be a 2 or 5, we can case on this value.

Case 1: hundreds digit is 2

The only option for the units digit is 8, since the number must be even. This leaves 4 options for the tens digit.

This gives us $1 \cdot 4 = 4$ numbers for this case.

Case 2: hundreds digit is 5

Similarly to above, 2 and 8 are the only options for the units digit, leaving 4 options for the tens digit.

This gives us $2 \cdot 4 = 8$ numbers for this case.

The total number of integers is then $4 + 8 = 12$.

Thus, **A** is the correct answer.

14. A pair of standard 6-sided dice is rolled once. The sum of the numbers rolled determines the diameter of a circle. What is the probability that the numerical value of the area of the circle is less than the numerical value of the circle's circumference?

A $\frac{1}{36}$

B $\frac{1}{12}$

C $\frac{1}{6}$

D $\frac{1}{4}$

E $\frac{5}{18}$

Solution(s):

For the area to be less than the circumference, we must have

$$\pi r^2 < 2\pi r$$
$$r < 2.$$

This means the diameter must be less than 4. There are three possible rolls that satisfy this:

$$(1,1),(1,2),(2,1).$$

The probability is then

$$\frac{3}{36} = \frac{1}{12}.$$

Thus, **B** is the correct answer.

15. Roy bought a new battery-gasoline hybrid car. On a trip the car ran exclusively on its battery for the first 40 miles, then ran exclusively on gasoline for the rest of the trip, using gasoline at a rate of 0.02 gallons per mile. On the whole trip he averaged 55 miles per gallon. How long was the trip in miles?

- A 140
- B 240
- C 440
- D 640
- E 840

Solution(s):

Let d be the distance the car drove solely on gasoline. We have that

$$\frac{40+x}{0.02x} = 55.$$

Cross-multiplying and simplifying gives

$$40 = .1x$$

$$x = 400.$$

The total length of the trip is then $400 + 40 = 440.$

Thus, **C** is the correct answer.

16. Which of the following is equal to $\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}$?

A $3\sqrt{2}$

B $2\sqrt{6}$

C $\frac{7\sqrt{2}}{2}$

D $3\sqrt{3}$

E 6

Solution(s):

Since we have square roots, we can try to change the inside of each radical to be a perfect square.

Note that we can rewrite the expression as

$$\sqrt{6-6\sqrt{2}+3}+\sqrt{6+6\sqrt{2}+3}.$$

Factoring and simplifying gives us

$$\sqrt{(\sqrt{6}-\sqrt{3})^2}+\sqrt{(\sqrt{6}+\sqrt{3})^2}$$

$$=\sqrt{6}-\sqrt{3}+\sqrt{6}+\sqrt{3}=2\sqrt{6}.$$

Thus, **B** is the correct answer.

17. In the eight term sequence A, B, C, D, E, F, G, H, the value of C is 5 and the sum of any three consecutive terms is 30. What is (A+H?\)

A 17

B 18

C 25

D 26

E 43

Solution(s):

From the condition about the sequence, we get that

$$A + B + C = 30$$

$$B = 25 - A.$$

Similarly, we get

$$B + C + D = 30$$

$$D = A.$$

Propagating these values through the sequence and repeating the condition for every consecutive triple, we get that

$$E = 25 - A, F = 5, G = A,$$

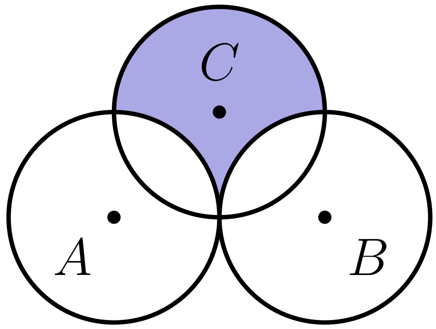
and finally, $H = 25 - A.$

The desired sum is then

$$A + 25 - A = 25.$$

Thus, **C** is the correct answer.

18. Circles $A, B,$ and C each have radius 1. Circles A and B share one point of tangency. Circle C has a point of tangency with the midpoint of $\overline{AB}$. What is the area inside circle C but outside circle A and circle B?



A $3-\frac{\pi}{2}$

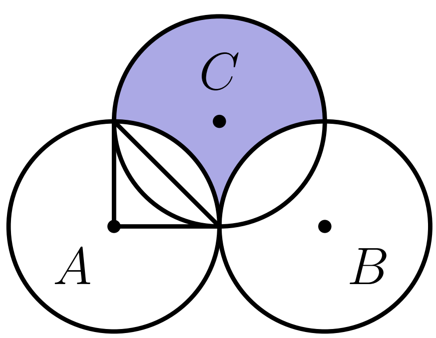
B $\frac{\pi}{2}$

C 2

D $\frac{3\pi}{4}$

E $1+\frac{\pi}{2}$

Solution(s):



The area of this region is the area of circle C minus the area of the overlapping region in B and C.

From the diagram, we can find the area of half of one of the overlapping regions by finding the area of the sector and subtracting the area of the triangle.

This area is then

$$\frac{1}{4}\pi \cdot 1^2 - \frac{1}{2} \cdot 1 \cdot 1 = \frac{\pi}{4} - \frac{1}{2}.$$

There are four of these that we must subtract, which leaves us with a final answer of

$$\pi \cdot 1^2 - 4(\frac{\pi}{4} - \frac{1}{2}) = 2.$$

Thus, **C** is the correct answer.

19. In 1991 the population of a town was a perfect square. Ten years later, after an increase of 150 people, the population was 9 more than a perfect square. Now, in $2011,$ with an increase of another 150 people, the population is once again a perfect square. Which of the following is closest to the percent growth of the town's population during this twenty-year period?

- A 42
- B 47
- C 52
- D 57
- E 62

Solution(s):

Let the population in 1991 be p^2. Then let the population in 2001 be q^2+9.

Using these values, we have

$$p^2+150=q^2+9$$

$$q^2-p^2=141.$$

Factoring, we get

$$(q-p)(q+p)=141.$$

As p and q are integers, we have that the only possible values for $q-p$ and $q+p$ are $(1,141)$ and $(3,47)$.

Trying the first pair, we have

$$q-p=1 \text{ and } q+p=141,$$

which adding together and dividing gives us $q=71$ and $p=70$.

We have that p^2+300 is not a square number, which means that this pair is the wrong one.

Trying the other pair and using the same strategy gives us $q=25$ and $p=22$.

Now, $p^2 + 300 = 784$, which is a perfect square. The percent increase in population is then

$$\frac{300}{24^2} \cdot 100\% \approx 62\%.$$

Thus, **E** is the correct answer.

20. Two points on the circumference of a circle of radius r are selected independently and at random. From each point a chord of length r is drawn in a clockwise direction. What is the probability that the two chords intersect?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution(s):

Fix one of the points on the circumference and its chord. Then consider the regular hexagon inscribed in the circle with this point at a vertex.

From this, we can see that the only way for the other point's chord to intersect the current one is if it is within an adjacent arc to the point.

Otherwise, the chord will not reach far enough to intersect the fixed chord, which is why the point must lie on an adjacent arc.

The desired probability is then

$$\frac{2}{6} = \frac{1}{3}.$$

Thus, **D** is the correct answer.

21. Two counterfeit coins of equal weight are mixed with 8 identical genuine coins. The weight of each of the counterfeit coins is different from the weight of each of the genuine coins. A pair of coins is selected at random without replacement from the 10 coins. A second pair is selected at random without replacement from the remaining 8 coins. The combined weight of the first pair is equal to the combined weight of the second pair. What is the probability that all 4 selected coins are genuine?

A $\frac{7}{11}$

B $\frac{9}{13}$

C $\frac{11}{15}$

D $\frac{15}{19}$

E $\frac{15}{16}$

Solution(s):

There are two cases: either all the coins are not counterfeit or each pair has a counterfeit.

For the first case, there are $\binom{8}{2} = 28$ ways to choose the coins for the first pair and $\binom{6}{2} = 15$ choices for the second pair.

We also have to divide by 2 since we can swap the pairs. This gives us

$$28 \cdot 15 \div 2 = 210$$

configurations for this case.

For the second case, there are $\binom{8}{2} = 28$ ways to choose the non-counterfeit coins. There is only one choice for the counterfeit coins.

There are two ways to create the two pairs, two choices for which counterfeit coin goes with a genuine coin.

This means that there are $28 \cdot 2 = 56$ configurations for this case.

The desired probability is then

$$\frac{210}{210+56}=\frac{210}{266}=\frac{15}{19}.$$

Thus, **D** is the correct answer.

22. Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

- A 2520
- B 2880
- C 3120
- D 3250
- E 3750

Solution(s):

Note that there are only 3 cases: all the vertices are different, there is one pair of adjacent vertices with the same colors, or there are 2 pairs (each pair has a different color).

Case 1 : all vertices have different colors

This case just gives us $6! = 720$ different coloring's.

Case 2 : one pair of adjacent vertices has the same color

There are

$$\frac{6!}{2} = 360$$

ways to choose the colors for this case. There are then 5 options for the pair of vertices.

This gives us a total of

$$360 \cdot 5 = 1800$$

colorings for this case.

Case 3 : two pairs of adjacent vertices have the same color

There are 5 choices for the vertex that is not in a pair. There are then

$$6 \cdot 5 \cdot 4 = 120$$

choices for the colors. There are then a total of

$$120 \cdot 5 = 600$$

colorings for this case.

There are a total of

$$720 + 1800 + 600 = 3120$$

colorings for all the cases.

Thus, **C** is the correct answer.

23. Seven students count from 1 to 1000 as follows:

• Alice says all the numbers, except she skips the middle number in each consecutive group of three numbers. That is, Alice says 1, 3, 4, 6, 7, $9, \ldots,$ 997, 999, 1000.

• Barbara says all of the numbers that Alice doesn't say, except she also skips the middle number in each consecutive group of three numbers.

• Candice says all of the numbers that neither Alice nor Barbara says, except she also skips the middle number in each consecutive group of three numbers.

• Debbie, Eliza, and Fatima say all of the numbers that none of the students with the first names beginning before theirs in the alphabet say, except each also skips the middle number in each of her consecutive groups of three numbers.

• Finally, George says the only number that no one else says.

What number does George say?

A 37

B 242

C 365

D 728

E 998

Solution(s):

We can walk through all the iterations to find what is left.

Alice does not say the numbers

$$2, 5, 8, 11, 14, 17, \ldots, 998.$$

After Barbara says her numbers, the remaining ones are

$$5, 14, 23, 32, 41, \ldots, 995.$$

Note that both of these are arithmetic sequences where the common difference is increased by a multiple of 3.

This pattern continues as the numbers remaining after Candace says hers are

$$14, 41, 68, 95, \ldots, 986.$$

Then after Debbie, they are

$$41, 122, 203, \ldots, 959$$

and after Eliza, they are

$$122, 365, 608, 878.$$

Finally, the only number left after Fatima goes is 365, which is the number that George will have to say.

Thus, **C** is the correct answer.

24. Two distinct regular tetrahedra have all their vertices among the vertices of the same unit cube. What is the volume of the region formed by the intersection of the tetrahedra?

A $\frac{1}{12}$

B $\frac{\sqrt{2}}{12}$

C $\frac{\sqrt{3}}{12}$

D $\frac{1}{6}$

E $\frac{\sqrt{2}}{6}$

Solution(s):

Note that the sides of the tetrahedron intersect each other at the midpoints.

If we split up each tetrahedron into five congruent smaller tetrahedra, we can see that only the middle tetrahedron overlaps between the two larger ones.

This means we just need to find the volume of the larger tetrahedron, and divide it by 8 to get the volume of a smaller one.

We divide by 8 since the ratio of the side lengths of the larger tetrahedron to the smaller one is 2, which we then cube to find the ratio of volumes.

Since the side length of the large tetrahedron is the diagonal of a face, we know that it is equal to $\sqrt{2}$.

Recall that the formula for the volume of a tetrahedron is

$$\frac{1}{3}Bh.$$

Using the formula for the area of an equilateral triangle, we have that

$$B = \frac{1}{4}(\sqrt{2})^2\sqrt{3} = \frac{\sqrt{3}}{2}.$$

We then have to apply the Pythagorean Theorem to find the height. Drop the altitude from the top vertex of the tetrahedron to the center of the base.

Note that the center of an equilateral triangle is its centroid, which means that the distance from a vertex to a centroid is $\frac{2}{3}$ the altitude of the equilateral triangle.

The altitude is just

$$\frac{\sqrt{2}}{2} \cdot \sqrt{3} = \frac{\sqrt{6}}{2}.$$

Then this side length of the right triangle is

$$\frac{2}{3} \cdot \frac{\sqrt{6}}{2} = \frac{\sqrt{6}}{3}.$$

The hypotenuse of the right triangle is just the side length of the tetrahedron, which is $\sqrt{2}$.

The height is then

$$\sqrt{(\sqrt{2})^2 - \left(\frac{\sqrt{6}}{3}\right)^2} = \sqrt{2 - \frac{2}{3}}$$

$$= \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}.$$

Finally, the volume of the large tetrahedron is

$$\frac{1}{3} \cdot \frac{\sqrt{3}}{2} \cdot \frac{2\sqrt{3}}{3} = \frac{4}{3}.$$

The area of the smaller tetrahedron is then

$$\frac{4}{3} \div 8 = \frac{1}{6}.$$

Thus, **D** is the correct answer.

25. Let R be a unit square region and $n \geq 4$ an integer. A point X in the interior of R is called *n-ray partitional* if there are n rays emanating from X that divide R into n triangles of equal area. How many points are 100-ray partitional but not 60-ray partitional?

A 1500

B 1560

C 2320

D 2480

E 2500

Solution(s):

Let us first find all the points that are 100-ray and 60-ray partitional.

First, consider an extreme 100-ray partitional point. Let this be the point in the top-left corner.

Note that we must draw rays through the vertices of the square, since otherwise we will end up with non-triangular regions.

Since this is the topmost and leftmost point, we have that the areas of the top triangular region and the left triangle region must be minimized.

This means that they do not have any rays going through them, which also means that their areas must be the same.

Then we have that the distance from the point to the top and left side must be the same. We know have 96 rays to split among the other 2 regions.

Since the other two regions also have the same area, we will have to have 48 rays in each region. This means that those two regions are split into 49 equal regions.

Let x be the distance between the point and the top of the square. We then have that

$$\frac{1 \cdot x}{2} = \frac{(1-x) \cdot 1}{2} \cdot \frac{1}{49}.$$

Simplifying gives

$$49x = 1 - x$$

$$x = \frac{1}{50}.$$

Now, if we move the point right to the next 100-partitional point, we have that a ray from the right region gets moved to the left region.

Doing the same analysis again would tell us that the point is $\frac{2}{50}$ away from the left side and $\frac{1}{50}$ from the top side.

Repeating this process, moving the point right and down, gets us that all the 100-partitional points form a 49×49 grid, with each point $\frac{1}{50}$ away from adjacent points.

Similarly, we can find the 60-partitional points form a 29×29 grid where the points are $\frac{1}{30}$ apart.

We now have to find the overlap between these two grids. Note that the gcd of 60 and 100 is 10. This means that all the points that are on both grids themselves form another grid that is 9×9 and $\frac{1}{10}$ apart.

This means that there are $9^2 = 81$ points that are on both grids. Then there are

$$49^2 - 81 = 2401 - 81 = 2320$$

points that are 100-partitional and not 60-partional.

Thus, **C** is the correct answer.

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