2010 AMC 10B Solutions

Typeset by: LIVE, by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc10/2010B/solutions

1. What is

$$100(100-3)-(100\cdot 100-3)?$$

- A $-20,000$
- B $-10,000$
- C -297
- D -6
- E 0

Solution(s):

We have that

$$\begin{aligned} 100(100-3) &= 100\cdot 97 \\ &= 9700 \end{aligned}$$

and

$$\begin{aligned} 100\cdot 100-3 &= 10000-3 \\ &= 9997. \end{aligned}$$

As such, their difference is:

$$9700-9997=-297.$$

Thus, **C** is the correct answer.

2. Makarla attended two meetings during her 9-hour work day. The first meeting took 45 minutes and the second meeting took twice as long. What percent of her work day was spent attending meetings?

A 15

B 20

C 25

D 30

E 35

Solution(s):

Note that 45 minutes is

$$\frac{45}{60} = \frac{3}{4}$$

hours. The second meeting is then

$$2 \cdot \frac{3}{4} = \frac{3}{2}$$

hours. Both meetings take a total of

$$\frac{3}{4} + \frac{3}{2} = \frac{9}{4}$$

hours then. The percent of Makarla's work day spent attending meetings is

$$100\% \cdot \frac{\frac{9}{4}}{9} = 100\% \cdot \frac{1}{4} = 25\%.$$

Thus, **C** is the correct answer.

3. A drawer contains red, green, blue, and white socks with at least 1 of each color. What is the minimum number of socks that must be pulled from the drawer to guarantee a matching pair?

- A 3
- B 4
- C 5
- D 8
- E 9

Solution(s):

To maximize the number of socks, we want to grab as many single socks as possible before getting a pair.

There are 4 colors, which means that we can draw one sock of each color before drawing a pair.

This means that it takes at least $4+1=5$ socks to be draw before a pair is guaranteed.

Thus, **C** is the correct answer.

4. For a real number x, define $\heartsuit(x)$ to be the average of x and x^2. What is

$$\heartsuit(1) + \heartsuit(2) + \heartsuit(3)?$$

A 3

B 6

C 10

D 12

E 20

Solution(s):

We have

$$\heartsuit(1) = \frac{1+1^2}{2} = 1,$$

$$\heartsuit(2) = \frac{2+2^2}{2} = 3,$$

and

$$\heartsuit(3) = \frac{3+3^2}{2} = 6.$$

Then

$$1 + 3 + 6 = 10.$$

Thus, **C** is the correct answer.

5. A month with 31 days has the same number of Mondays and Wednesdays. How many of the seven days of the week could be the first day of this month?

A 2

B 3

C 4

D 5

E 6

Solution(s):

Note that 31 days leaves a remainder of 3 when divided by 7.

This means that if the month starts on a Saturday, Sunday, Tuesday, or Wednesday, there will be an uneven number of Mondays and Wednesdays.

Then the month can only start on a Monday, Thursday, or Friday.

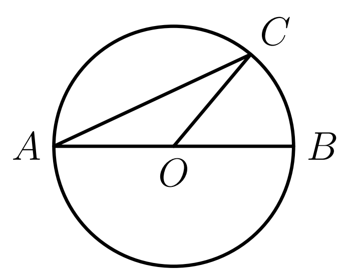
Thus, **B** is the correct answer.

6. A circle is centered at O, $\overline{AB}$ is a diameter and C is a point on the circle with $\angle COB = 50^\circ$. What is the degree measure of $\angle CAB$?

- A 20
- B 25
- C 45
- D 50
- E 65

Solution(s):

Consider the following diagram:



We have that

$$\angle AOC = 180^\circ - 50^\circ = 130^\circ.$$

Since $\triangle AOC$ is isosceles, we have that

$$\angle CAO = \frac{180^\circ - 130^\circ}{2} = 25^\circ.$$

Since $\angle CAB = \angle CAO$, we have that $\angle CAB = 25^\circ$.

Thus, **B** is the correct answer.

7. A triangle has side lengths 10, 10, and 12. A rectangle has width 4 and area equal to the area of the triangle. What is the perimeter of this rectangle?

A 16

B 24

C 28

D 32

E 36

Solution(s):

To find the area of the triangle, we can drop the altitude to the side of length 12.

Then we have a right triangle with one leg $12 \div 2 = 6$ and hypotenuse 10.

The other leg has length

$$\sqrt{10^2 - 6^2} = \sqrt{64} = 8.$$

The area of the triangle is then

$$\frac{8 \cdot 12}{2} = 48.$$

The length of the rectangle is then $48 \div 4 = 12$. Its perimeter is

$$2(4 + 12) = 2 \cdot 16 = 32.$$

Thus, **D** is the correct answer.

8. A ticket to a school play costs x dollars, where x is a whole number. A group of 9 th graders buys tickets costing a total of $\$48$, and a group of 10th graders buys tickets costing a total of $\$64$. How many values for x are possible?

A 1

B 2

C 3

D 4

E 5

Solution(s):

Note that x must divide both 48 and 64, which means that must divide their greatest common divisor.

The greatest common divisor of 48 and 64 is 16, which means x divides 16.

16 has 5 factors, namely all the powers of 2 up to 16.

Thus, **E** is the correct answer.

9. Lucky Larry's teacher asked him to substitute numbers for a, b, c, d, and e in the expression

$$a-(b-(c-(d+e)))$$

and evaluate the result. Larry ignored the parentheses but added and subtracted correctly and obtained the correct result by coincidence. The number Larry substituted for a, b, c, and d were 1, 2, 3, and 4, respectively. What number did Larry substitute for e?

A -5

B -3

C 0

D 3

E 5

Solution(s):

Ignoring the parentheses, Larry would get

$$1-2-3-4+e=e-8.$$

Evaluating with the parentheses, one would get

$$\begin{aligned} &1-(2-(3-(4+e))) \\ &=1-(2-(-1-e)) \\ &=1-(3+e) \\ &=-2-e. \end{aligned}$$

Both of these values are the same, so

$$e-8=-2-e$$

$$e=3.$$

Thus, **D** is the correct answer.

10. Shelby drives her scooter at a speed of 30 miles per hour if it is not raining, and 20 miles per hour if it is raining. Today she drove in the sun in the morning and in the rain in the evening, for a total of 16 miles in 40 minutes. How many minutes did she drive in the rain?

A 18

B 21

C 24

D 27

E 30

Solution(s):

Note that 40 minutes is $\frac{2}{3}$ hours. Let Shelby drive h hours in the sun.

Then she drives $\frac{2}{3} - h$ hours in the rain, which means she travels a total of

$$30h + 20\left(\frac{2}{3} - h\right) = 10h + \frac{40}{3}$$

miles. We have this equals 16, so

$$10h + \frac{40}{3} = 16$$

$$h = \frac{4}{15}.$$

Then Shelby drives

$$60\left(\frac{2}{3} - \frac{4}{15}\right) = 60 \cdot \frac{2}{5} = 24$$

minutes in the rain.

Thus, **C** is the correct answer.

11. A shopper plans to purchase an item that has a listed price greater than $\$100$ and can use any one of the three coupons. Coupon A gives 15% off the listed price, Coupon B gives $\$30$ off the listed price, and Coupon C gives 25% off the amount by which the listed price exceeds $\$100$.

Let x and y be the smallest and largest prices, respectively, for which Coupon A saves at least as many dollars as Coupon B or C. What is $y - x$?

A 50

B 60

C 75

D 80

E 100

Solution(s):

Let p be the price of the item. Then coupon A saves $.15p$. Coupon B saves $\$30$.

Coupon C will save

$$.25(p - 100) = .25p - 25.$$

We must have that

$$.15p > 30$$

$$p \geq 200$$

and

$$.15p > .25p - 25$$

$$250 \geq p.$$

This shows that $x = 200$ and $y = 250$. Therefore $y - x = 50$.

Thus, **A** is the correct answer.

12. At the beginning of the school year, 50% of all students in Mr. Well's class answered "Yes" to the question "Do you love math", and 50% answered "No." At the end of the school year, 70% answered "Yes" and 30% answered "No." Altogether, $x\%$ of the students gave a different answer at the beginning and end of the school year. What is the difference between the maximum and the minimum possible values of x?

- A 0
- B 20
- C 40
- D 60
- E 80

Solution(s):

To minimize x, we want to have as many kids as possible maintain their answer.

We then need at least $70 - 50 = 20$ percent of the students to change their answer.

To maximize x, we can have everybody that answered no change their answer, but some who answered yes must stay yes.

We need $70 - 50 = 20$ percent of the people who said yes to stay yes, which means only $50 - 20 = 30$ percent can switch.

This makes the maximum percent of students that can switch $50 + 30 = 80$ percent. The difference is then $80 - 20 = 60$.

Thus, **D** is the correct answer.

13. What is the sum of all the solutions of

$$x = |2x - |60 - 2x||?$$

A 32

B 60

C 92

D 120

E 124

Solution(s):

We first take care of the outer absolute value. We have either

$$x = 2x - |60 - 2x|$$

or

$$x = -2x + |60 - 2x|.$$

These simplify to

$$x = |60 - 2x| \text{ and } 3x = |60 - 2x|.$$

We have 2 cases for each equation. For the first one, we have

$$x = 60 - 2x \text{ and } x = 2x - 60.$$

Solving both gives us

$$3x = 60$$

$$x = 20$$

and

$$-x = -60$$

$$x = 60.$$

For the other equation, we have

$$3x = 60 - 2x \text{ and } 3x = 2x - 60.$$

Again solving both, we ahve

$$5x = 60$$

$$x = 12$$

and $x = -60$. Note that x cannot be negative since that means the original absolute value is negative, which is not possible.

Adding up all the solutions gives us

$$20 + 60 + 12 = 92.$$

Thus, **C** is the correct answer.

14. The average of the numbers $1, 2, 3, \cdots, 98, 99$, and x is $100x$. What is x?

A $\frac{49}{101}$

B $\frac{50}{101}$

C $\frac{1}{2}$

D $\frac{51}{101}$

E $\frac{50}{99}$

Solution(s):

Recall that the sum of the first n integers is $\frac{n(n+1)}{2}$.

Then, we have that

$$\frac{\frac{99 \cdot 100}{2} + x}{100} = 100x,$$

which simplifies to

$$\begin{aligned} 99 \cdot 50 &= (100^2 - 1)x \\ &= 101 \cdot 99x, \end{aligned}$$

by difference of squares. Dividing gives us $x = \frac{50}{101}$.

Thus, **B** is the correct answer.

15. On a 50-question multiple choice math contest, students receive 4 points for a correct answer, 0 points for an answer left blank, and -1 point for an incorrect answer. Jesse's total score on the contest was 99. What is the maximum number of questions that Jesse could have answered correctly?

A 25

B 27

C 29

D 31

E 33

Solution(s):

Let x be the number of questions Jesse answered correctly and y be the number he answered incorrectly.

Then

$$4x - y = 99 \text{ and } x + y \leq 50.$$

We have that

$$y = 4x - 99$$

$$5x - 99 \leq 50.$$

Rearranging and simplifying tells us that $x \leq 29.8$. Since x is an integer, its maximum value is 29.

Thus, **C** is the correct answer.

16. A square of side length 1 and a circle of radius $\dfrac{\sqrt{3}}{3}$ share the same center. What is the area inside the circle, but outside the square?

A $\dfrac{\pi}{3} - 1$

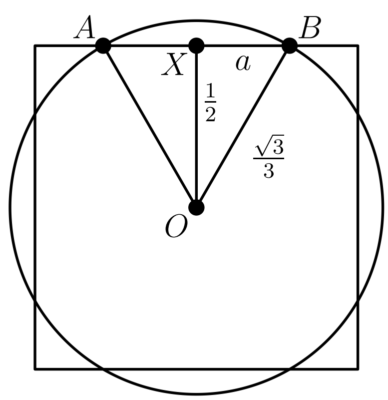
B $\dfrac{2\pi}{9} - \dfrac{\sqrt{3}}{3}$

C $\dfrac{\pi}{18}$

D $\dfrac{1}{4}$

E $\dfrac{2\pi}{9}$

Solution(s):



Drop the altitude from O to $\overline{AB}$ at X.

Looking at the side lengths, we see that $\triangle OBX$ is a $30-60-90$ triangle.

This means that the area of sector AOB is

$$\frac{1}{6} \cdot \pi \cdot \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{\pi}{18}.$$

We also have that the area of $\triangle AOB$ is

$$\frac{1}{2} \cdot 2 \cdot \frac{\sqrt{3}}{6} \cdot \frac{1}{2} = \frac{\sqrt{3}}{12.}.$$

The area of the sector outside the square and inside the circle is then

$$\frac{\pi}{18} - \frac{\sqrt{3}}{12}.$$

There are 4 of these regions, which gives us a total area of

$$4\left(\frac{\pi}{18} - \frac{\sqrt{3}}{12}\right) = \frac{2\pi}{9} - \frac{\sqrt{3}}{3}.$$

Thus, **B** is the correct answer.

17. Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37^{th} and 64^{th}, respectively. How many schools are in the city?

A 22

B 23

C 24

D 25

E 26

Solution(s):

Let there be x schools. Then there are $3x$ participants in the contest.

Since everyone has a unique score, there is also a unique median. This means x is odd.

Note there are at least 23 teams, since otherwise a 64^{th} placed wouldn't exist.

We also have at most 23 teams, since otherwise we have Andrea's place as being greater than Beth's.

This means that there are 23 teams.

Thus, **B** is the correct answer.

18. Positive integers a, b, and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \ldots, 2010\}$.

What is the probability that $abc + ab + a$ is divisible by 3?

A $\frac{1}{3}$

B $\frac{29}{81}$

C $\frac{31}{81}$

D $\frac{11}{27}$

E $\frac{13}{27}$

Solution(s):

Note that

$$abc + ab + a = a(bc + b + 1).$$

This means that if a is divisible by 3, the whole expression is as well.

Since 2010 is divisible by 3, we have that a is divisible by 3 with probability $\frac{1}{3}$.

Now consider a not divisible by 3. For the expression to be divisible by 3, we must have that $bc + b + 1$ is divisible by 3.

This means that

$$bc + b = b(c + 1) \equiv 2 \pmod{3}.$$

The only possibility for this is that one of the factors is 2 mod 3 and the other is 1 mod 3.

For each of the two cases, there is a $\frac{1}{3}$ chance that each of the factors is the desired modulus, for a probability of

$$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

There are two cases, which means that this happens with a $\frac{2}{9}$ probability.

The total probability is then

$$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{2}{9} = \frac{13}{27}.$$

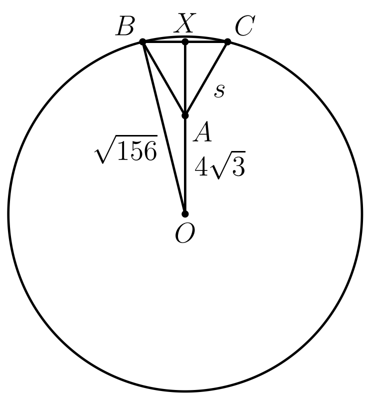
Thus, **E** is the correct answer.

19. A circle with center O has area 156π. Triangle ABC is equilateral, $\overline{BC}$ is a chord on the circle, $OA = 4\sqrt{3}$, and point O is outside $\triangle ABC$. What is the side length of $\triangle ABC$?

- A $2\sqrt{3}$
- B 6
- C $4\sqrt{3}$
- D 12
- E 18

Solution(s):

Consider the following diagram:



Using the formula for the area of a circle, we have that $BO = \sqrt{156}$ since it is a radius.

Extend $\overline{AO}$ to intersect $\overline{BC}$ at X. Let s be the side length of $\triangle ABC$.

Then we have that

$$BX = \frac{s}{2}$$

and

$$AX = \frac{s\sqrt{3}}{2}.$$

We have that $\triangle OXB$ is right, which means that we can apply the Pythagorean Theorem. This gives us

$$(\sqrt{156})^2 = \left(\frac{s}{2}\right)^2$$

$$+\left(\frac{s\sqrt{3}}{2} + 4\sqrt{3}\right)^2.$$

Simplifying, we get

$$156 = s^2 + 12s + 48$$

$$s^2 + 12s - 108 = 0$$

$$(s-6)(s+18) = 0.$$

Since s is positive, we must have that $s = 6$.

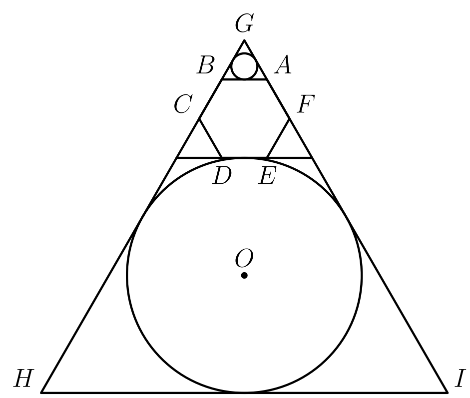
Thus, **B** is the correct answer.

20. Two circles lie outside regular hexagon $ABCDEF$. The first is tangent to $\overline{AB}$, and the second is tangent to $\overline{DE}$. Both are tangent to lines BC and FA. What is the ratio of the area of the second circle to that of the first circle?

- A 18
- B 27
- C 36
- D 81
- E 108

Solution(s):

Consider the following diagram:



We can see that the smaller circle is inscribed within an equilateral triangle of side length 1.

The inradius of this equilateral triangle is $\dfrac{\sqrt{3}}{6}$. The area of the circle is then

$$\pi \cdot \left(\frac{\sqrt{3}}{6}\right)^2 = \frac{\pi}{12}.$$

Let O be the center of the larger circle. Drop the perpendicular from O to $\overline{GH}$ at J. Draw $\overline{OG}$.

We have that $\triangle OJG$ is right. Since $\angle HGI = 60^\circ$, we also have that $\triangle OJG$ is a $30-60-90$ triangle.

Let $OJ = r$. Then $OG = 2r$. We also have that OG is the sum of the height of the hexagon, equilateral triangle, and radius of the circle.

Then

$$OG = \frac{\sqrt{3}}{2} + \sqrt{3} + r.$$

Substituting in OG, we get

$$2r = \frac{\sqrt{3}}{2} + \sqrt{3} + r.$$

Simplifying gives us

$$r = \frac{3\sqrt{3}}{2}.$$

The area of the larger circle is then

$$\pi \cdot \left(\frac{3\sqrt{3}}{2}\right)^2 = \frac{27}{4}\pi.$$

The desired ratio is then

$$\frac{\frac{27\pi}{4}}{\frac{\pi}{12}} = 81.$$

Thus, **D** is the correct answer.

21. A palindrome between 1000 and $10,000$ is chosen at random. What is the probability that it is divisible by 7?

A $\frac{1}{10}$

B $\frac{1}{9}$

C $\frac{1}{7}$

D $\frac{1}{6}$

E $\frac{1}{5}$

Solution(s):

Note that we can express any 4 digit number as $abcd$. This can be expressed in long form as

$$10^3a + 10^2b + 10c + d.$$

Since in a palindrome, we have that $a = d$ and $b = c$. We can simplify this to get

$$1001a + 110b.$$

Note that 1001 is divisible by 7. This means that $110b$ must also be divisible by 7.

The only way for this to happen is if b is 0 or 7 since 110 is not divisible by 7.

There are 9 options for a and 2 options for b, for a total of $9 \cdot 2 = 18$ palindromes.

The total number of palindromes is $9 \cdot 10$ since there are 9 options for the thousands digit and 10 options for the hundreds digit.

The desired probability is then

$$\frac{18}{9 \cdot 10} = \frac{1}{5}.$$

Thus, **E** is the correct answer.

22. Seven distinct pieces of candy are to be distributed among three bags. The red bag and the blue bag must each receive at least one piece of candy; the white bag may remain empty. How many arrangements are possible?

- A 1930
- B 1931
- C 1932
- D 1933
- E 1934

Solution(s):

We can count this with complementary counting. The total number of ways to distribute the candies with no restrictions is

$$3^7 = 2187.$$

To find the number of invalid arrangements, we have to count the number of ways where either the red or blue bag is empty.

For the case where the red bag is empty, each candy has 2 options for the bag that goes into. There are then

$$2^7 = 128$$

arrangements for this case. Similarly, there are 128 arrangements for the case where the blue bag is empty.

There is an overlap of one case where both bags are empty. The final answer is then

$$2187 - (128 + 128 - 1) = 1932.$$

Thus, **C** is the correct answer.

23. The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

A 18

B 24

C 36

D 42

E 60

Solution(s):

Note that 1 and 9 must be in the top left and bottom right corners respectively. We must also have that 2 and 8 are next to these squares.

We can then case on the center square. Note that the only possible values are 4, 5, or 6.

Case 1: the center is 4

The 3 is necessarily next to the 1, since there is no other option that is less than 4.

Any number can be in the square next to the 8, but the other two squares are then fixed. There are

$$2 \cdot 2 \cdot 3 = 12$$

cases (two places for the 2, two places for the 8, and three choices for the square adjacent to 8).

Case 2: the center is 5

We can case on the position of the 3. If the 3 is in the top right square, the 4 is necessarily next to the 1.

If the 8 is above the 9, then the other two squares are fixed. If it is to the left of the 9, the other two squares can be filled arbitrarily.

Now consider when the 3 is below the 1. There are two spots for the 8, and the square next to the 8 can be any number.

The other two squares are then fixed. This means that this case has a total of

$$2(1+2+2\cdot 3)=18.$$

We multiply by two since the 2 can be either to the right of or below the 1.

Case 3: the center is 6

This is similar to case 1 since the 7 is fixed instead of the 3.

The total number of arrangements is then

$$12+18+12=42.$$

Thus, **D** is the correct answer.

24. A high school basketball game between the Raiders and Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

A 30

B 31

C 32

D 33

E 34

Solution(s):

Let a be the number of points scored in the first quarter. Let r be the common ratio for the Raiders and d the difference for the Wildcats.

First, we can establish that r is an integer. Assume for the sake of contradiction that it is not and $r = \frac{m}{n}$ where $\gcd(m, n) = 1$.

For ar, ar^2, and ar^3 to all be integers, we must have that n, n^2, and n^3 divide a. We can then let $a = n^3k$ for some integer k.

Then we have that

$$k\left(n^3 + n^2m + nm^2 + m^3\right) < 100$$

Minimizing n and k, we can assume that $n = 2$ and $k = 1$. This gives us $m < 4$, which means that $r = \dfrac{3}{2}$.

Testing out this value shows that it does not work. Since this value does not work, no other non-integer rational ratio will work. This means r is integral.

Then the sum of the quarter scores for the Wildcats is

$$a + (a + d) + (a + 2d) + (a + 3d)$$
$$= 4a + 6d.$$

We must have that

$$a(1 + r + r^2 + r^3) =$$

$$4a + 6d + 1.$$

Trying out values, let $r = 2$. Then we have

$$15a = 4a + 6d + 1,$$

which simplifies to

$$11a = 6d + 1.$$

Looking for the smallest multiple of 11 that leaves a remainder of 1 when divided by 6, we get 55.

With this value, we get $a = 5$ and $d = 9$. The sum of the scores of the first two quarters for both teams is

$$5 + 10 + 5 + 14 = 34.$$

Thus, **E** is the correct answer.

25. Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$P(1) = P(3) = P(5) = P(7) = a,$$

and

$$P(2) = P(4) = P(6) = P(8)$$
$$= -a.$$

What is the smallest possible value of a?

A 105

B 315

C 945

D $7!$

E $8!$

Solution(s):

We can define a new function

$$Q(x) = P(x) - a$$

so that Q has roots at $1, 3, 5,$ and 7. Then we can factor Q to get

$$Q(x) = (x-1)(x-3)(x-5)$$
$$(x-7)(x).$$

We can plug in the values of $2, 4, 6,$ and 8 to get

$$R(2) = (2-1)(2-3)(2-5)$$
$$(2-7)Q(2) = -15Q(2) = -2a$$

And

$$R(4) = (4-1)(4-3)(4-5)$$
$$(4-7)Q(4) = 9Q(4) = -2a$$

And

$$R(6) = (6-1)(6-3)(6-5)$$

$$(6-7)Q(6) = -15Q(6) = -2a$$

And

$$R(8) = (8-1)(8-3)(8-5)$$

$$(8-7)Q(8) = 105Q(8) = -2a$$

To minimize a, we can set $a = \mathrm{lcm}(15, 9, 15, 105)$. Then $a = 315$.

Thus, **B** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2010B