

2010 AMC 10A

Solutions

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1. Mary's top book shelf holds five books with the following widths, in centimeters: 6, 0.5, 1, 2.5, and 10.

What is the average book width, in centimeters?

- A 1
- B 2
- C 3
- D 4
- E 5

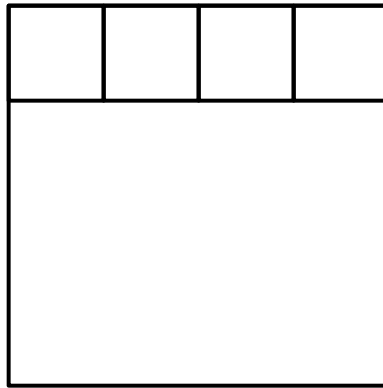
Solution(s):

Converting them all to decimals and adding, we get the average to be

$$\frac{6 + .5 + 1 + 2.5 + 10}{5} = \frac{20}{5} = 4.$$

Thus, **D** is the correct answer.

2. Four identical squares and one rectangle are placed together to form one large square as shown. The length of the rectangle is how many times as large as its width?



A $\frac{5}{4}$

B $\frac{4}{3}$

C $\frac{3}{2}$

D 2

E 3

Solution(s):

WLOG, let the side lengths of the squares be 1.

This means that the length of the rectangle is 4. We also have that the width must be $4 - 1 = 3$.

The desired ratio is then $\frac{4}{3}$.

Thus, **B** is the correct answer.

3. Tyrone had 97 marbles and Eric had 11 marbles. Tyrone then gave some of his marbles to Eric so that Tyrone ended with twice as many marbles as Eric. How many marbles did Tyrone give to Eric?

A 3

B 13

C 18

D 25

E 29

Solution(s):

Let x the number of marbles that Eric ends up with. Then Tyrone ends up with $2x$.

The total number of marbles is $97 + 11 = 108$, so

$$3x = 108$$

$$x = 36.$$

Then, Tyrone ends up with $36 \cdot 2 = 72$ marbles. This means he has to give away $97 - 72 = 25$ marbles.

Thus, **D** is the correct answer.

4. A book that is to be recorded onto compact discs takes 412 minutes to read aloud. Each disc can hold up to 56 minutes of reading. Assume that the smallest possible number of discs is used and that each disc contains the same length of reading. How many minutes of reading will each disc contain?

A 50.2

B 51.5

C 52.4

D 53.8

E 55.2

Solution(s):

Note that $7 \cdot 56 = 392$ and $8 \cdot 56 = 448$, which means that the minimum number of discs needed is 8.

Then the minutes of reading that each disc contains is

$$412 \div 8 = 51.5.$$

Thus, **B** is the correct answer.

5. The area of a circle whose circumference is 24π is $k\pi$. What is the value of k ?

A 6

B 12

C 24

D 36

E 144

Solution(s):

Recall that the formula for the circumference of a circle is $2\pi r$. We then have that

$$24\pi = 2\pi r$$

$$r = 12.$$

The area of a circle is πr^2 , so we have that

$$k\pi = \pi 12^2$$

$$k = 144.$$

Thus, **E** is the correct answer.

6. For positive numbers x and y the operation $\spadesuit(x, y)$ is defined as

$$\spadesuit(x, y) = x - \frac{1}{y}$$

What is $\spadesuit(2, \spadesuit(2, 2))$?

A $\frac{2}{3}$

B 1

C $\frac{4}{3}$

D $\frac{5}{3}$

E 2

Solution(s):

Evaluating the inner expression, we get

$$\spadesuit(2, 2) = 2 - \frac{1}{2} = \frac{3}{2}.$$

Then we have

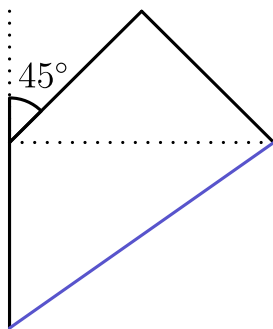
$$\spadesuit\left(2, \frac{3}{2}\right) = 2 - \frac{1}{\frac{3}{2}} = \frac{4}{3}.$$

Thus, **C** is the correct answer.

7. Crystal has a running course marked out for her daily run. She starts this run by heading due north for one mile. She then runs northeast for one mile, then southeast for one mile. The last portion of her run takes her on a straight line back to where she started. How far, in miles is this last portion of her run?

- A 1
- B $\sqrt{2}$
- C $\sqrt{3}$**
- D 2
- E $2\sqrt{2}$

Solution(s):



From the diagram, we see that the distance traveled is the hypotenuse of a right triangle.

One of the legs is just 1 from running due north. The other leg is

$$\sqrt{1^2 + 1^2} = \sqrt{2}.$$

The final distance is then

$$\sqrt{\sqrt{2}^2 + 1^2} = \sqrt{3}.$$

Thus, **C** is the correct answer.

8. Tony works 2 hours a day and is paid \$0.50 per hour for each full year of his age. During a six month period Tony worked 50 days and earned \$630. How old was Tony at the end of the six month period?

A 9

B 11

C 12

D 13

E 14

Solution(s):

Since $2 \cdot .5 = 1$, we have that Tony makes a dollar per day per full year of his age.

If he is 12 at the end of the period, then Tony can make a maximum of

$$12 \cdot 50 = 600$$

dollars in the period. If he was 13 at the end, then he could have made

$$13 \cdot 50 = 650.$$

By this, we can see that Tony is 13 the end of the period, since otherwise he would make too much or too little money.

Thus, **D** is the correct answer.

9. A *palindrome*, such as 83438, is a number that remains the same when its digits are reversed. The numbers x and $x + 32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of x ?

A 20

B 21

C 22

D 23

E 24

Solution(s):

Note that x is at most 999. This means that $x + 32$ has a maximum of 1031.

Similarly, we have that the minimum value of $x + 32$ is 1000.

The only palindrome in this range is 1001, so this is what $x + 32$ equals.

Then

$$x + 32 = 1001$$

$$x = 969.$$

The sum of the digits is then

$$9 + 6 + 9 = 24.$$

Thus, **E** is the correct answer.

10. Marvin had a birthday on Tuesday, May 27 in the leap year 2008. In what year will his birthday next fall on a Saturday?

A 2011

B 2012

C 2013

D 2015

E 2017

Solution(s):

Note that on a normal year, we have that

$$365 = 52 \cdot 7 + 1,$$

which means that for a specific day, it moves to the day after the next year.

On a leap year, the day of the week moves forward two since there is an extra day.

Then in 2009, this day falls on a Wednesday. In 2010, it falls on a Thursday.

Similarly, in 2011, it falls on a Friday. In 2012, however, since it is a leap year, it falls on a Sunday.

Now, for the next three years, the day moves forward one. Then in 2016, it moves forward two, landing on a Friday.

Finally, in 2017, the day of the week is a Saturday.

Thus, **E** is the correct answer.

11. The length of the interval of solutions of the inequality

$$a \leq 2x + 3 \leq b$$

is 10. What is $b - a$?

- A 6
- B 10
- C 15
- D 20
- E 30

Solution(s):

Splitting the inequality into two of them and solving gives us

$$a \leq 2x + 3$$

$$x \geq \frac{a - 3}{2}$$

and

$$2x + 3 \leq b$$

$$x \leq \frac{b - 3}{2}.$$

The range of the solutions is then

$$\frac{b - 3}{2} - \frac{a - 3}{2} = 10,$$

which then simplifying gives us

$$(b - 3) - (a - 3) = 20$$

$$b - a = 20.$$

Thus, **D** is the correct answer.

12. Logan is constructing a scaled model of his town. The city's water tower stands 40 meters high, and the top portion is a sphere that holds 100,000 liters of water. Logan's miniature water tower holds 0.1 liters. How tall, in meters, should Logan make his tower?

A 0.04

B $\frac{0.4}{\pi}$

C 0.4

D $\frac{4}{\pi}$

E 4

Solution(s):

The miniature tower holds

$$\frac{100,000}{.1} = 1,000,000$$

times less water than the actual tower. Since this is the ratio for volumes, the ratio of heights is

$$(1,000,000)^{1/3} = 100.$$

This means that the height of the miniature tower is

$$\frac{40}{100} = .4.$$

Thus, **C** is the correct answer.

13. Angelina drove at an average rate of 80 kmh and then stopped 20 minutes for gas. After the stop, she drove at an average rate of 100 kmh. Altogether she drove 250 km in a total trip time of 3 hours including the stop. Which equation could be used to solve for the time t in hours that she drove before her stop?

A $80t + 100 \left(\frac{8}{3} - t \right) = 250$

B $80t = 250$

C $100t = 250$

D $90t = 250$

E $80 \left(\frac{8}{3} - t \right) + 100t = 250$

Solution(s):

Before the stop, Angelina drove for $80t$ km using the distance formula.

The stop takes $\frac{1}{3}$ of an hour, which means that Angelina travels for

$$2 - \frac{1}{3} = \frac{8}{3}$$

hours after the stop. Then after the stop, Angelina drives for

$$100 \left(\frac{8}{3} - t \right)$$

km. Since the total distance driven is 250 km, which makes the final equation

$$\frac{8}{3}t + 100 \left(\frac{8}{3} - t \right) = 250.$$

Thus, **A** is the correct answer.

14. Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on \overline{AB} and \overline{BC} , respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD , and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

A 60°

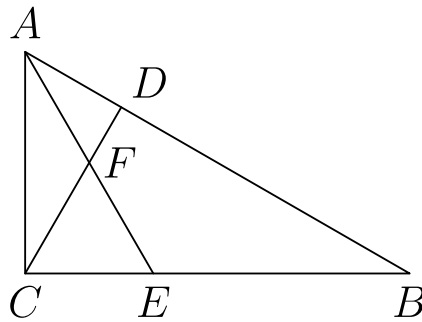
B 75°

C 90°

D 105°

E 120°

Solution(s):



Let $\angle BAE = \angle ACD = x$. Note that $\angle CFE = 60^\circ$ since $\triangle CFE$ is equilateral.

We then have that

$$\angle AFC = 180^\circ - \angle CFE = 120^\circ.$$

Then:

$$\begin{aligned} \angle FAC &= 180^\circ - 120^\circ - x \\ &= 60^\circ - x \\ &= \angle EAC. \end{aligned}$$

We then get that

$$\begin{aligned} \angle BAC &= \angle BAE + \angle EAC \\ &= x + 60^\circ - x \\ &= 60^\circ. \end{aligned}$$

Since $AB = 2 \cdot AC$ and $\angle BAC = 60^\circ$, we have that $\triangle ABC$ is a $30 - 60 - 90$ triangle.

Thus, **C** is the correct answer.

15. In a magical swamp there are two species of talking amphibians: toads, whose statements are always true, and frogs, whose statements are always false. Four amphibians, Brian, Chris, LeRoy, and Mike live together in this swamp, and they make the following statements.

Brian: "Mike and I are different species."

Chris: "LeRoy is a frog."

LeRoy: "Chris is a frog."

Mike: "Of the four of us, at least two are toads."

How many of these amphibians are frogs?

- A 0
- B 1
- C 2
- D 3
- E 4

Solution(s):

If Brian is a frog, then he must be lying, which means that Mike must be a frog.

If Brian is a toad, then he must be telling the truth, which also means that Mike is a frog.

Therefore, Mike is a frog, which means that Mike is lying. This means that there is at most one toad.

Then, at least one of LeRoy and Chris is a frog. This means the other is telling the truth, which makes them a toad.

This means there is one toad, which makes there be 3 frogs.

Thus, **D** is the correct answer.

16. Nondegenerate $\triangle ABC$ has integer side lengths, \overline{BD} is an angle bisector, $AD = 3$, and $DC = 8$. What is the smallest possible value of the perimeter?

A 30

B 33

C 35

D 36

E 37

Solution(s):

Using the Angle Bisector Theorem, we have that

$$\frac{AB}{3} = \frac{BC}{8}$$

$$AB = \frac{3}{8}BC.$$

For AB and BC to be integers, we must have that BC is a multiple of 8.

To minimize the perimeter, we can set $BC = 8$ and $AB = 3$. This, however, makes the triangle degenerate.

BC must then be 16 and $AB = 6$. Since $DC = 11$, the perimeter is

$$16 + 6 + 11 = 33.$$

Thus, **B** is the correct answer.

17. A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid?

A 7

B 8

C 10

D 12

E 15

Solution(s):

Note that all the cut out solids intersect in the middle of the cube.

This region of intersection is a cube with side length 2. Then the area of the cutout region is

$$\begin{aligned} 3 \cdot 2 \cdot 2 \cdot 3 - 2 \cdot 2^3 &= 36 - 16 \\ &= 20. \end{aligned}$$

We have to subtract out the center region twice since it is included in all 3 regions.

The remaining volume is then

$$3^3 - 20 = 27 - 20 = 7.$$

Thus, **A** is the correct answer.

18. Bernardo randomly picks 3 distinct numbers from the set

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set

$$\{1, 2, 3, 4, 5, 6, 7, 8\}$$

and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

A $\frac{47}{72}$

B $\frac{37}{56}$

C $\frac{2}{3}$

D $\frac{49}{72}$

E $\frac{39}{56}$

Solution(s):

There are two cases: Bernardo picks a 9 or he doesn't.

Case 1: Bernardo picks a 9

Since a number is fixed, there are $\binom{8}{2} = 28$ ways to choose the other two numbers.

There are a total of $\binom{9}{3} = 84$ ways to pick all three numbers. The probability is then

$$\frac{28}{84} = \frac{1}{3}.$$

Note that if Bernardo picks a 9, he automatically has a greater number than Silvia.

This means that Bernardo always wins in this case.

Case 2: Bernardo doesn't pick a 9

There is a $1 - \frac{1}{3} = \frac{2}{3}$ chance of this happening. Since both people are choosing from the same numbers, they have an equal chance of winning.

We still need to find the probability that the numbers are the same. There is a

$$\frac{1}{\binom{8}{3}} = \frac{1}{56}$$

chance that Silvia chooses the same numbers as Bernardo. The probability that Bernardo gets a higher number is then

$$\frac{1 - \frac{1}{56}}{2} = \frac{55}{112}.$$

The total probability of Bernardo getting a higher number is then

$$\frac{1}{3} + \frac{2}{3} \cdot \frac{55}{112} = \frac{37}{56}.$$

Thus, **B** is the correct answer.

19. Equiangular hexagon $ABCDEF$ has side lengths

$$AB = CD = EF = 1$$

and

$$BC = DE = FA = r.$$

The area of $\triangle ACE$ is 70% of the area of the hexagon. What is the sum of all possible values of r ?

A $\frac{4\sqrt{3}}{3}$

B $\frac{10}{3}$

C 4

D $\frac{17}{4}$

E 6

Solution(s):

Note that $\triangle ACE$ is equilateral. Using the law of cosines, we get that

$$AC^2 = r^2 + 1^2 - 2r \cos \frac{2\pi}{3}.$$

The area of $\triangle ACE$ is then

$$\frac{\sqrt{3}}{4}(r^2 + r + 1).$$

Recall that the formula for the area of a triangle can be given by

$$\frac{1}{2}AB \sin \theta.$$

Using this formula on $\triangle ABC$, we get its area to be

$$\frac{1}{2} \cdot 1 \cdot r \cdot \sin 120^\circ = \frac{r\sqrt{3}}{4}.$$

The area of all three triangles is then

$$3 \cdot \frac{r\sqrt{3}}{4} = \frac{3r\sqrt{3}}{4}.$$

The area of the entire hexagon is then

$$\frac{\sqrt{3}}{4}(r^2 + 4r + 1).$$

The problem conditions tells us that

$$\begin{aligned} \frac{\sqrt{3}}{4}(r^2 + r + 1) &= \\ \frac{7}{10} \cdot \frac{\sqrt{3}}{4}(r^2 + 4r + 1). \end{aligned}$$

Simplifying this gives us

$$r^2 - 6r + 1 = 0,$$

which we can then use Vieta's formulas on to get the sum of values is 6. Intuitively, this means that if our solutions to this equation are r_0 and r_1 , we have that:

$$\begin{aligned} 0 &= r^2 - 6r + 1 \\ &= (r - r_0)(r - r_1) \\ &= r^2 - (r_0 + r_1)r + r_0r_1 \end{aligned}$$

Which implies that the sum of possible solutions is equal to $r_0 + r_1 = 6$.

Thus, **E** is the correct answer.

20. A fly trapped inside a cubical box with side length 1 meter decides to relieve its boredom by visiting each corner of the box. It will begin and end in the same corner and visit each of the other corners exactly once. To get from a corner to any other corner, it will either fly or crawl in a straight line. What is the maximum possible length, in meters, of its path?

A $4 + 4\sqrt{2}$

B $2 + 4\sqrt{2} + 2\sqrt{3}$

C $2 + 3\sqrt{2} + 3\sqrt{3}$

D $4\sqrt{2} + 4\sqrt{3}$

E $3\sqrt{2} + 5\sqrt{3}$

Solution(s):

Note that all the paths the fly can take have lengths of 1, $\sqrt{2}$, or $\sqrt{3}$.

We want to maximize the number of longer length paths. We cannot travel any interior diagonal twice, since that would make the fly visit the same vertex twice.

It also possible to visit all the vertices by traveling along diagonals, so we will never have to travel a path of length 1.

This means that we can maximize the distance by traveling along 4 interior diagonals and 4 diagonals on the faces.

This path is possible by traveling along a face and then an interior diagonal, repeating this in a way that avoids visiting the same vertex twice.

The path has length

$$4\sqrt{2} + 4\sqrt{3}.$$

Thus, **D** is the correct answer.

21. The polynomial

$$x^3 - ax^2 + bx - 2010$$

has three positive integer roots. What is the smallest possible value of a ?

A 78

B 88

C 98

D 108

E 118

Solution(s):

If r, s, t are the roots of the polynomial, we know that:

$$\begin{aligned} 0 &= x^3 - ax^2 + bx - 2010 \\ &= (x - r)(x - s)(x - t) \\ &= x^3 - (r + s + t)x^2 \\ &\quad + (rs + st + rt)x - (rst) \end{aligned}$$

As such, we know that $rst = 2010$, or in English, the product of the three roots is 2010.

As

$$2010 = 2 \cdot 3 \cdot 5 \cdot 67,$$

we have that one of the three roots must have two of these prime numbers as factors.

Again using the first fact, we have that $r + s + t = a$ is the sum of all the roots. To minimize this, we should have 2 and 3 multiplied together.

Then, we can let the roots be 6, 5, and 67, which makes

$$a = 6 + 5 + 67 = 78.$$

Thus, **A** is the correct answer.

22. Eight points are chosen on a circle, and chords are drawn connecting every pair of points. No three chords intersect in a single point inside the circle. How many triangles with all three vertices in the interior of the circle are created?

A 28

B 56

C 70

D 84

E 140

Solution(s):

We need 3 chords to form the side lengths of the triangles. Each of these chords requires 2 points on the circle.

This means that we need to choose $3 \cdot 2 = 6$ points from the 8. Also note that any 6 points determine a triangle.

This is because we don't want to create chords that don't intersect in the circle, which leaves only one way to form the triangle.

The number of ways to choose the six points is then

$$\binom{8}{6} = \binom{8}{2} = 28.$$

Thus, **A** is the correct answer.

23. Each of 2010 boxes in a line contains a single red marble, and for $1 \leq k \leq 2010$, the box in the k th position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

- A 45
- B 63
- C 64
- D 201
- E 1005

Solution(s):

Since there are $k + 1$ marbles in the k th box, there is a $\frac{k}{k + 1}$ chance Isabella draws a white marble from it.

The probability of drawing a red marble is then $\frac{1}{k + 1}$. To stop after drawing the n th marble, the first $n - 1$ marbles must have been white.

This happens with a probability of

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \dots \cdot \frac{n-1}{n} \cdot \frac{1}{n+1}.$$

Note that all the numerators cancel with the adjacent denominator, which means that this expression reduces to $\frac{1}{n(n+1)}$.

We have to find the smallest n such that

$$\frac{1}{n(n+1)} < \frac{1}{2010}$$

$$n(n+1) > 2010.$$

Guessing and checking gives us that the smallest n that works is 45.

Thus, **A** is the correct answer.

24. The number obtained from the last two nonzero digits of $90!$ is equal to n . What is n ?

A 12

B 32

C 48

D 52

E 68

Solution(s):

We first find the number of zeros in $90!$. The number of zeros is determined by the number of factors of 10.

The number of factors of 10 is given by the number of factors of 2 and 5.

There are clearly more factors of 5, which means that $90!$ has

$$\left\lfloor \frac{90}{5} \right\rfloor + \left\lfloor \frac{90}{25} \right\rfloor = 18 + 3 = 21$$

factors of 10.

Now, we need to find the two rightmost digits of $N = \frac{90!}{10^{21}}$. We can find the value mod 100 by finding the values mod 4 and mod 25.

Since there are more factors of 2 than 5 in $90!$, the value of $N \pmod{4}$ is just 0.

Now, we just need to consider $M = \frac{90!}{5^{21}} \pmod{25}$. M is the product of all the non-multiples of 5 less than 90.

Consider

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdots 24 \pmod{25}.$$

We can rewrite it is

$$(1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdots 12)^2$$

using the fact that $-x \equiv 25 - x \pmod{25}$ and that there are even number of numbers in the square.

Then multiplying out some terms, we have that

$$\begin{aligned}1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 11 \cdot 12 &\equiv \\ -1 \cdot 7 \cdot 9 \cdot 11 &\equiv 7 \pmod{25}\end{aligned}$$

using the fact that

$$3 \cdot 8 = 2 \cdot 12 = 4 \cdot 6 \equiv -1.$$

Then

$$7^2 = 49 \equiv -1 \pmod{25}.$$

This set of numbers appears 3 times in M , with the final being a partial expression of

$$1 \cdot 2 \cdot 3 \cdots 14 \equiv -1 \pmod{25},$$

using similar tricks to above.

Then

$$M \equiv (-1)^4 \equiv 1 \pmod{25}.$$

We also have that

$$N = \frac{M}{2^{21}} \equiv \frac{1}{2^{21}} \pmod{25}.$$

Then using Euler's theorem, we get that

$$\begin{aligned}\frac{1}{2^{21}} &\equiv 12^{21} \equiv 12^{\phi(25)} \cdot 12 \equiv \\ &12 \pmod{25}.\end{aligned}$$

Since 12 is divisible by 4, we have that $N \equiv 12 \pmod{100}$.

Thus, **A** is the correct answer.

25. Jim starts with a positive integer n and creates a sequence of numbers. Each successive number is obtained by subtracting the largest possible integer square less than or equal to the current number until zero is reached. For example, if Jim starts with $n = 55$, then his sequence contains 5 numbers:

$$\begin{array}{r} 55 \\ 55 - 7^2 = 6 \\ 6 - 2^2 = 2 \\ 2 - 1^2 = 1 \\ 1 - 1^2 = 0 \end{array}$$

Let N be the smallest number for which Jim's sequence has 8 numbers. What is the units digit of N ?

- A 1
- B 3
- C 5
- D 7
- E 9

Solution(s):

We can just work backwards starting with 0. From this, we can add on 1^1 to get 1.

We can again add on 1^1 to get 2. Again, adding on 1^1 gives us 3.

If we add on 1^1 now, we get 4, but then 1^1 is not the greatest square less than 4.

Then adding on 2^2 gives us 7. We repeat this process for 8 steps to get 7223.

$$\begin{array}{r} 7223 \\ 7223 - 84^2 = 167 \\ 167 - 12^2 = 23 \\ 23 - 4^2 = 7 \\ 7 - 2^2 = 3 \\ 3 - 1^2 = 2 \\ 2 - 1^2 = 1 \\ 1 - 1^2 = 0 \end{array}$$

Thus, **B** is the correct answer.

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