2009 AMC 10B Solutions

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1. Each morning of her five-day workweek, Jane bought either a 50-cent muffin or a 75-cent bagel. Her total cost for the week was a whole number of dollars. How many bagels did she buy?

A 1

B 2

C 3

D 4

E 5

Solution:

If Jane buys b bagels, she buys $5-b$ muffins, for a total of

$$50(5-b)+75b=250+25b$$

cents. This is a whole number of dollars when $250+25b$ is a multiple of 100, that is, when $25b \equiv 50 \pmod{100}$, or $b \equiv 2 \pmod{4}$.

The only value with $0 \leq b \leq 5$ is $b=2$.

Thus, the correct answer is **B**.

2. Which of the following is equal to

$$\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{2}-\frac{1}{3}}?$$

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

Solution:

The least common denominator of the small fractions is $12,$ so multiply top and bottom by $12:$

$$\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{2}-\frac{1}{3}}=\frac{4-3}{6-4}=\frac{1}{2}.$$

Thus, the correct answer is **C**.

3. Paula the painter had just enough paint for 30 identically sized rooms. Unfortunately, on the way to work, three cans of paint fell off her truck, so she had only enough paint for 25 rooms. How many cans of paint did she use for the 25 rooms?

A 10

B 12

C 15

D 18

E 25

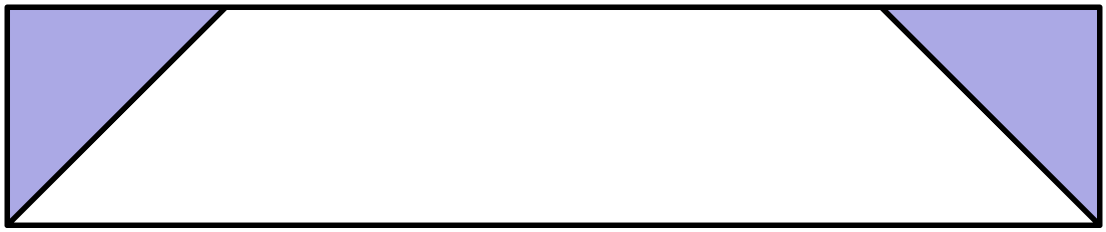
Solution:

The lost 3 cans would have painted $30 - 25 = 5$ rooms, so each room takes $\frac{3}{5}$ of a can.

For 25 rooms she used $\frac{3}{5} \cdot 25 = 15$ cans.

Thus, the correct answer is **C**.

4. A rectangular yard contains two flower beds in the shape of congruent isosceles right triangles. The remainder of the yard has a trapezoidal shape, as shown. The parallel sides of the trapezoid have lengths 15 and 25 meters. What fraction of the yard is occupied by the flower beds?



A $\frac{1}{8}$

B $\frac{1}{6}$

C $\frac{1}{5}$

D $\frac{1}{4}$

E $\frac{1}{3}$

Solution:

The two parallel sides differ by $25 - 15 = 10$, split evenly between the two triangles, so each isosceles right triangle has legs of length 5 and area $\frac{1}{2} \cdot 5^2 = \frac{25}{2}$.

Together the beds cover 25 square meters. The rectangle has length 25 and width 5, so area 125. The fraction is $\frac{25}{125} = \frac{1}{5}$.

Thus, the correct answer is **C**.

5. Twenty percent less than 60 is one-third more than what number?

- A 16
- B 30
- C 32
- D 36
- E 48

Solution:

Twenty percent less than 60 is $\frac{4}{5} \cdot 60 = 48.$

If n is the unknown number, one-third more than n is $\frac{4}{3}n,$ so

$$\frac{4}{3}n = 48$$

$$n = 36.$$

Thus, the correct answer is **D**.

6. Kiana has two older twin brothers. The product of their three ages is 128. What is the sum of their three ages?

- A 10
- B 12
- C 16
- D 18
- E 24

Solution:

Since $128 = 2^7$, every age is a power of 2. Writing the twins' common age as t and Kiana's as k, we need $t^2k = 128$ with $k < t$.

Taking $t = 8$ gives $k = 2$, which works. The sum is $8 + 8 + 2 = 18$.

Thus, the correct answer is **D**.

7. By inserting parentheses, it is possible to give the expression

$$2 \times 3 + 4 \times 5$$

several values. How many different values can be obtained?

A 2

B 3

C 4

D 5

E 6

Solution:

The three operations can be ordered in $3! = 6$ ways, but performing the addition first or last leaves the two multiplications interchangeable, so at most four values arise.

Indeed

$$(2 \times 3) + (4 \times 5) = 26, \quad (2 \times 3 + 4) \times 5 = 50,$$
$$2 \times (3 + 4 \times 5) = 46, \quad 2 \times (3 + 4) \times 5 = 70$$

are four distinct values.

Thus, the correct answer is **C**.

8. In a certain year the price of gasoline rose by 20% during January, fell by 20% during February, rose by 25% during March, and fell by $x\%$ during April. The price of gasoline at the end of April was the same as it had been at the beginning of January. To the nearest integer, what is x?

A 12

B 17

C 20

D 25

E 35

Solution:

Let p be the starting price. After March the price is

$$(1.2)(0.8)(1.25)p = 1.2p.$$

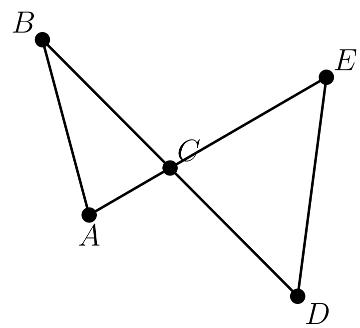
The April drop must return it to p, so it removes $0.2p$, a fraction

$$x = 100 \cdot \frac{0.2p}{1.2p} = \frac{100}{6} \approx 16.7.$$

To the nearest integer, $x = 17.$

Thus, the correct answer is **B**.

9. Segment BD and AE intersect at C, as shown, $AB = BC = CD = CE$, and $\angle A = \frac{5}{2}\angle B$. What is the degree measure of $\angle D$?



A 52.5

B 55

C 57.5

D 60

E 62.5

Solution:

Since $\triangle ABC$ is isosceles with $AB = BC$, we have $\angle A = \angle C$. With $\angle A = \frac{5}{2}\angle B$, the angle sum gives

$$\frac{5}{2}\angle B + \frac{5}{2}\angle B + \angle B = 180^\circ,$$

so $\angle B = 30^\circ$ and $\angle ACB = 75^\circ$.

By vertical angles $\angle DCE = 75^\circ$. Since $CD = CE$, triangle CDE is isosceles, so

$$2\angle D + 75^\circ = 180^\circ,$$

giving $\angle D = 52.5^\circ$.

Thus, the correct answer is **A**.

10. A flagpole is originally 5 meters tall. A hurricane snaps the flagpole at a point x meters above the ground so that the upper part, still attached to the stump, touches the ground 1 meter away from the base. What is x?

- A 2.0
- B 2.1
- C 2.2
- D 2.3
- **E** 2.4

Solution:

The standing stump has height x, and the snapped piece of length $5 - x$ is the hypotenuse of a right triangle with legs x and 1. By the Pythagorean Theorem,

$$x^2 + 1^2 = (5 - x)^2 = x^2 - 10x + 25,$$

so $10x = 24$ and $x = 2.4$.

Thus, the correct answer is **E**.

11. How many 7-digit palindromes (numbers that read the same backward as forward) can be formed using the digits $2, 2, 3, 3, 5, 5, 5$?

- A 6
- B 12
- C 24
- D 36
- E 48

Solution:

A 7-digit palindrome has the form with the middle digit used once and the outer three digits each used twice. Only 5 appears an odd number of times, so 5 must be the middle digit.

The remaining digits $2, 3, 5$ fill the first three positions in some order and mirror to the last three. There are $3! = 6$ such orderings.

Thus, the correct answer is **A**.

12. Distinct points $A, B, C,$ and D lie on a line, with $AB = BC = CD = 1$. Points E and F lie on a second line, parallel to the first, with $EF = 1$. A triangle with positive area has three of the six points as its vertices. How many possible values are there for the area of the triangle?

A 3

B 4

C 5

D 6

E 7

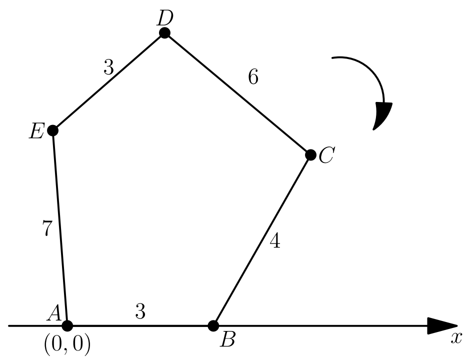
Solution:

A positive-area triangle uses two points on one line as its base and one point on the other line as its apex. The height is always the fixed distance between the lines, so the area depends only on the base length.

Bases on the first line can be $1, 2,$ or 3; a base on the second line is 1. So the distinct base lengths are $1, 2, 3,$ giving three possible areas.

Thus, the correct answer is **A**.

13. As shown below, convex pentagon $ABCDE$ has sides $AB = 3, BC = 4, CD = 6,$ $DE = 3,$ and $EA = 7.$ The pentagon is originally positioned in the plane with vertex A at the origin and vertex B on the positive x-axis. The pentagon is then rolled clockwise to the right along the x-axis. Which side will touch the point $x = 2009$ on the x-axis?



A $\overline{AB}$

B $\overline{BC}$

C $\overline{CD}$

D $\overline{DE}$

E $\overline{EA}$

Solution:

The pentagon has perimeter $3 + 4 + 6 + 3 + 7 = 23.$ One full roll advances the contact point by $23,$ and $2009 = 23 \cdot 87 + 8.$

After 87 rolls, vertex A sits at $x = 23 \cdot 87 = 2001$ and B at $2004.$ Rolling further, C touches at $2004 + 4 = 2008$ and D at $2008 + 6 = 2014.$

Since 2009 lies between 2008 and $2014,$ side $\overline{CD}$ touches that point.

Thus, the correct answer is **C**.

14. On Monday, Millie puts a quart of seeds, 25% of which are millet, into a bird feeder. On each successive day she adds another quart of the same mix of seeds without removing any seeds that are left. Each day the birds eat only 25% of the millet in the feeder, but they eat all of the other seeds. On which day, just after Millie has placed the seeds, will the birds find that more than half the seeds in the feeder are millet?

- A Tuesday
- B Wednesday
- C Thursday
- D Friday
- E Saturday

Solution:

Each quart adds $\frac{1}{4}$ quart of millet, and the birds leave $\frac{3}{4}$ of the standing millet. On day n the millet present is

$$\frac{1}{4}\left(1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{n-1}\right)=1-\left(\frac{3}{4}\right)^{n}.$$

The other seeds always total $\frac{3}{4}$ quart. Millet exceeds half when $1-\left(\frac{3}{4}\right)^{n}>\frac{3}{4}$, i.e. $\left(\frac{3}{4}\right)^{n}<\frac{1}{4}$.

Since $\left(\frac{3}{4}\right)^{4}=\frac{81}{256}>\frac{1}{4}$ but $\left(\frac{3}{4}\right)^{5}=\frac{243}{1024}<\frac{1}{4}$, this first happens on day 5, which is Friday.

Thus, the correct answer is **D**.

15. When a bucket is two-thirds full of water, the bucket and water weigh a kilograms. When the bucket is one-half full of water the total weight is b kilograms. In terms of a and b, what is the total weight in kilograms when the bucket is full of water?

A $\frac{2}{3}a + \frac{1}{3}b$

B $\frac{3}{2}a - \frac{1}{2}b$

C $\frac{3}{2}a + b$

D $\frac{3}{2}a + 2b$

E $3a - 2b$

Solution:

Let x be the bucket's weight and y the weight of a full load of water. Then

$$x + \frac{2}{3}y = a, \qquad x + \frac{1}{2}y = b.$$

Subtracting gives $\frac{1}{6}y = a - b$, so $y = 6a - 6b$, and $x = b - \frac{1}{2}y = 4b - 3a$. The full bucket weighs

$$x + y = (4b - 3a) + (6a - 6b) = 3a - 2b.$$

Thus, the correct answer is **E**.

16. Points A and C lie on a circle centered at O, each of $\overline{BA}$ and $\overline{BC}$ are tangent to the circle, and $\triangle ABC$ is equilateral. The circle intersects $\overline{BO}$ at D. What is $\frac{BD}{BO}$?

A $\frac{\sqrt{2}}{3}$

B $\frac{1}{2}$

C $\frac{\sqrt{3}}{3}$

D $\frac{\sqrt{2}}{2}$

E $\frac{\sqrt{3}}{2}$

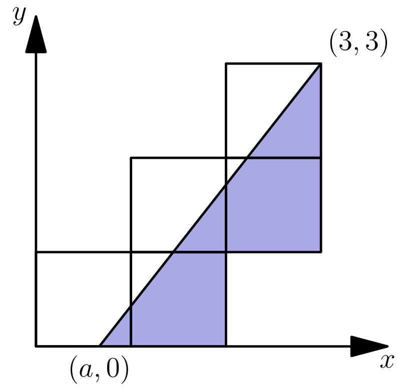
Solution:

Let the radius be r. By symmetry BO bisects the 60° angle ABC, so $\angle OBC = 30^\circ$. Since $OC \perp BC$, triangle BCO is a 30-60-90 triangle with hypotenuse $BO = 2\,OC = 2r$.

Then $BD = BO - OD = 2r - r = r$, so $\frac{BD}{BO} = \frac{r}{2r} = \frac{1}{2}$.

Thus, the correct answer is **B**.

17. Five unit squares are arranged in the coordinate plane as shown, with the lower left corner at the origin. The slanted line, extending from $(a, 0)$ to $(3, 3)$, divides the entire region into two regions of equal area. What is a?



A $\frac{1}{2}$

B $\frac{3}{5}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{4}{5}$

Solution:

The five unit squares have total area 5, so each region must have area $\frac{5}{2}$.

The region to the lower right of the line is a right triangle with legs $3 - a$ and 3, minus the one unit square it does not cover. Setting its area to $\frac{5}{2}$ gives

$$\frac{3(3-a)}{2} - 1 = \frac{5}{2},$$

so $3(3 - a) = 7$ and $a = \frac{2}{3}$.

Thus, the correct answer is **C**.

18. Rectangle $ABCD$ has $AB = 8$ and $BC = 6$. Point M is the midpoint of diagonal $\overline{AC}$, and E is on $\overline{AB}$ with $\overline{ME} \perp \overline{AC}$. What is the area of $\triangle AME$?

A $\frac{65}{8}$

B $\frac{25}{3}$

C 9

D $\frac{75}{8}$

E $\frac{85}{8}$

Solution:

By the Pythagorean Theorem, $AC = \sqrt{8^2 + 6^2} = 10$, so $AM = 5$. Right triangles AME and ABC share angle A, so they are similar with

$$\frac{ME}{AM} = \frac{BC}{AB} = \frac{6}{8},$$

giving $ME = \frac{15}{4}$.

Then $\text{area}(\triangle AME) = \frac{1}{2} \cdot AM \cdot ME = \frac{1}{2} \cdot 5 \cdot \frac{15}{4} = \frac{75}{8}$.

Thus, the correct answer is **D**.

19. A particular 12-hour digital clock displays the hour and minute of a day. Unfortunately, whenever it is supposed to display a $1,$ it mistakenly displays a $9.$ For example, when it is 1:16 PM the clock incorrectly shows 9:96 PM. What fraction of the day will the clock show the correct time?

A $\frac{1}{2}$

B $\frac{5}{8}$

C $\frac{3}{4}$

D $\frac{5}{6}$

E $\frac{9}{10}$

Solution:

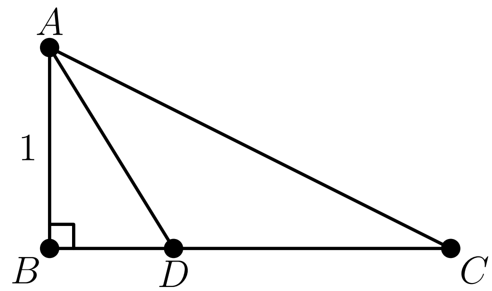
Among the hours 1 through $12,$ exactly $1, 10, 11, 12$ contain a $1,$ so the hour is correct $\frac{8}{12} = \frac{2}{3}$ of the time.

A minute is displayed wrong when its tens digit is 1 (minutes 10–19) or its units digit is 1 $(01, 11, \ldots, 51)$, which is 15 of the 60 minutes. So the minute is correct $\frac{45}{60} = \frac{3}{4}$ of the time.

The clock is correct $\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$ of the day.

Thus, the correct answer is **A**.

20. Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The bisector of $\angle BAC$ meets $\overline{BC}$ at D. What is BD?



A $\dfrac{\sqrt{3}-1}{2}$

B $\dfrac{\sqrt{5}-1}{2}$

C $\dfrac{\sqrt{5}+1}{2}$

D $\dfrac{\sqrt{6}+\sqrt{2}}{2}$

E $2\sqrt{3}-1$

Solution:

By the Pythagorean Theorem, $AC = \sqrt{1^2 + 2^2} = \sqrt{5}$. The Angle Bisector Theorem gives

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{1}{\sqrt{5}},$$

so $DC = \sqrt{5}\,BD$.

Since $BD + DC = 2$, we have $BD(1+\sqrt{5}) = 2$, so

$$BD = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}.$$

Thus, the correct answer is **B**.

21. What is the remainder when

$$3^0 + 3^1 + 3^2 + \cdots + 3^{2009}$$

is divided by 8?

- A 0
- B 1
- C 2
- D 4
- E 6

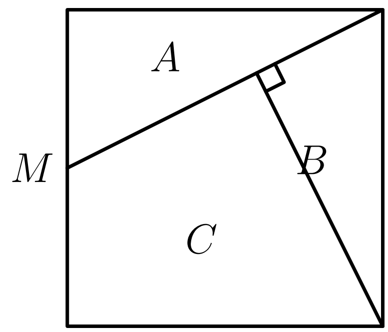
Solution:

Any four consecutive powers of 3 sum to a multiple of $3^0 + 3^1 + 3^2 + 3^3 = 40$, which is divisible by 8.

The terms from 3^2 to 3^{2009} split into such blocks and contribute remainder 0. What remains is $3^0 + 3^1 = 4$.

Thus, the correct answer is **D**.

22. A cubical cake with edge length 2 inches is iced on the sides and the top. It is cut vertically into three pieces as shown in this top view, where M is the midpoint of a top edge. The piece whose top is triangle B contains c cubic inches of cake and s square inches of icing. What is $c + s$?



A $\frac{24}{5}$

B $\frac{32}{5}$

C $8 + \sqrt{5}$

D $5 + \frac{16\sqrt{5}}{5}$

E $10 + 5\sqrt{5}$

Solution:

Set the top face as a 2×2 square. The cut from M toward the far corner creates the top triangle A with legs 1 and 2, so area 1 and hypotenuse $\sqrt{5}$.

Triangle B is similar to A but with hypotenuse 2, so its area is $\left(\frac{2}{\sqrt{5}}\right)^2 \cdot 1 = \frac{4}{5}$. Since the cake has height 2, the volume is $c = \frac{4}{5} \cdot 2 = \frac{8}{5}$.

The icing on this piece is its top ($\frac{4}{5}$) plus the full cube side face it borders ($2 \times 2 = 4$), so $s = \frac{4}{5} + 4 = \frac{24}{5}$. Therefore $c + s = \frac{8}{5} + \frac{24}{5} = \frac{32}{5}$.

Thus, the correct answer is **B**.

23. Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the start line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture?

A $\frac{1}{16}$

B $\frac{1}{8}$

C $\frac{3}{16}$

D $\frac{1}{4}$

E $\frac{5}{16}$

Solution:

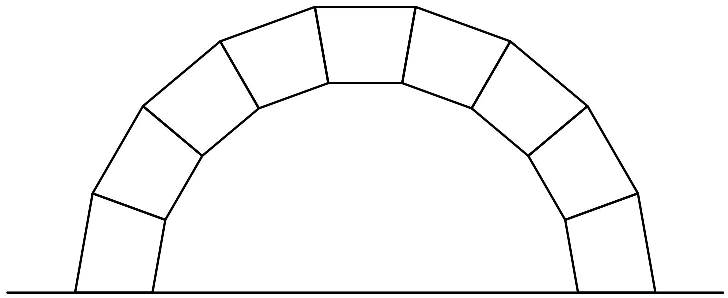
The picture spans $\frac{1}{8}$ lap on each side of the start. After 600 seconds Rachel is 30 seconds short of the line; running $\frac{1}{4}$ lap in 22.5 seconds, she is in view between $30 - 11.25 = 18.75$ and $30 + 11.25 = 41.25$ seconds of the 10th minute.

After 600 seconds Robert is 40 seconds from the line; running $\frac{1}{4}$ lap in 20 seconds, he is in view between 30 and 50 seconds.

Both appear between 30 and 41.25 seconds, a window of length 11.25 out of 60, so the probability is $\frac{11.25}{60} = \frac{3}{16}$.

Thus, the correct answer is **C**.

24. The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid. What is x?



A 100

B 102

C 104

D 106

E 108

Solution:

Adding a mirror image completes the arch into a symmetric closed loop of 18 trapezoids. Their inner edges form a regular 18-gon, each interior angle of which is

$$\frac{(18-2)\cdot 180^\circ}{18} = 160^\circ.$$

At each inner vertex, two of the trapezoids' larger angles x meet the 160° angle around a full turn: $x + x + 160^\circ = 360^\circ$, so $x = 100$.

Thus, the correct answer is **A**.

25. Each face of a cube is given a single narrow stripe painted from the center of one edge to the center of its opposite edge. The choice of the edge pairing is made at random and independently for each face. What is the probability that there is a continuous stripe encircling the cube?

A $\frac{1}{8}$

B $\frac{3}{16}$

C $\frac{1}{4}$

D $\frac{3}{8}$

E $\frac{1}{2}$

Solution:

Each face's stripe has 2 orientations, giving $2^6 = 64$ equally likely configurations.

An encircling stripe runs around one of the 3 pairs of opposite faces. For a given band, the 4 faces it crosses must each be oriented to continue it, a probability of $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

The three bands are mutually exclusive, so the probability is $3 \cdot \frac{1}{16} = \frac{3}{16}$.

Thus, the correct answer is **B**.

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