2009 AMC 10A Solutions

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1. One can holds 12 ounces of soda. What is the minimum number of cans needed to provide a gallon $(128$ ounces$)$ of soda?

A 7

B 8

C 9

D 10

E 11

Solution:

Since $\dfrac{128}{12} = 10\dfrac{2}{3}$, ten cans hold only 120 ounces, which is not enough.

Therefore 11 cans are needed.

Thus, the correct answer is **E**.

2. Four coins are picked out of a piggy bank that contains a collection of pennies, nickels, dimes, and quarters. Which of the following could *not* be the total value of the four coins, in cents?

- A 15
- B 25
- C 35
- D 45
- E 55

Solution:

To get a multiple of 5 cents, the number of pennies must be a multiple of 5. With only four coins, that means using no pennies, but then the four coins are each worth at least 5 cents, for a total of at least 20 cents.

So 15 cents cannot be made. The others can: $25 = 10 + 3(5)$, $35 = 3(10) + 5$, $45 = 25 + 10 + 2(5)$, and $55 = 25 + 3(10)$.

Thus, the correct answer is **A**.

3. Which of the following is equal to

$$1+\frac{1}{1+\frac{1}{1+1}}?$$

A $\frac{5}{4}$

B $\frac{3}{2}$

C $\frac{5}{3}$

D 2

E 3

Solution:

Working outward,

$$1+\frac{1}{1+\frac{1}{1+1}}=1+\frac{1}{1+\frac{1}{2}}=1+\frac{1}{\frac{3}{2}}=1+\frac{2}{3}=\frac{5}{3}.$$

Thus, the correct answer is **C**.

4. Eric plans to compete in a triathlon. He can average 2 miles per hour in the $\frac{1}{4}$-mile swim and 6 miles per hour in the 3-mile run. His goal is to finish the triathlon in 2 hours. To accomplish his goal what must his average speed, in miles per hour, be for the 15-mile bicycle ride?

A $\frac{120}{11}$

B 11

C $\frac{56}{5}$

D $\frac{45}{4}$

E 12

Solution:

The swim takes $\frac{1/4}{2} = \frac{1}{8}$ hour and the run takes $\frac{3}{6} = \frac{1}{2}$ hour. This leaves

$$2 - \frac{1}{8} - \frac{1}{2} = \frac{11}{8}$$

hours for the bicycle ride.

His average speed must be

$$\frac{15}{11/8} = \frac{120}{11}$$

miles per hour.

Thus, the correct answer is **A**.

5. What is the sum of the digits of the square of $111{,}111{,}111$?

A 18

B 27

C 45

D 63

E 81

Solution:

The square of the nine-digit repunit is the palindrome

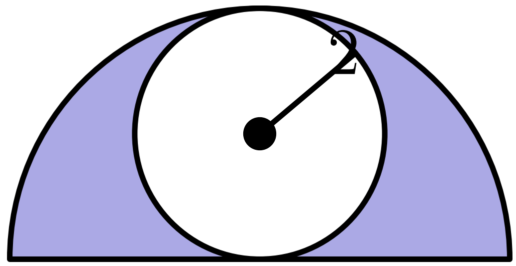
$$111{,}111{,}111^2 = 12{,}345{,}678{,}987{,}654{,}321.$$

Its digits are $1, 2, \ldots, 9, 8, \ldots, 1$, so the sum is

$$2(1 + 2 + \cdots + 8) + 9 = 2 \cdot 36 + 9 = 81.$$

Thus, the correct answer is **E**.

6. A circle of radius 2 is inscribed in a semicircle, as shown. The area inside the semicircle but outside the circle is shaded. What fraction of the semicircle's area is shaded?



A $\frac{1}{2}$

B $\frac{\pi}{6}$

C $\frac{2}{\pi}$

D $\frac{2}{3}$

E $\frac{3}{\pi}$

Solution:

The inscribed circle rests on the diameter and is tangent to the arc, so the semicircle has radius 4. Its area is

$$\frac{1}{2}\pi(4)^2 = 8\pi.$$

The circle's area is $\pi(2)^2 = 4\pi$, so the shaded area is $8\pi - 4\pi = 4\pi$.

The shaded fraction is $\frac{4\pi}{8\pi} = \frac{1}{2}$.

Thus, the correct answer is **A**.

7. A carton contains milk that is 2% fat, an amount that is 40% less fat than the amount contained in a carton of whole milk. What is the percentage of fat in whole milk?

A $\frac{12}{5}$

B 3

C $\frac{10}{3}$

D 38

E 42

Solution:

Let whole milk be $x\%$ fat. Since 2 is 40% less than x, we have

$$0.6x = 2,$$

so

$$x = \frac{2}{0.6} = \frac{10}{3}.$$

Thus, the correct answer is **C**.

8. Three generations of the Wen family are going to the movies, two from each generation. The two members of the youngest generation receive a 50% discount as children. The two members of the oldest generation receive a 25% discount as senior citizens. The two members of the middle generation receive no discount. Grandfather Wen, whose senior ticket costs $\$6.00,$ is paying for everyone. How many dollars must he pay?

A 34

B 36

C 42

D 46

E 48

Solution:

The senior ticket costs $\$6,$ which is $\frac{3}{4}$ of the full price, so a full ticket costs $\frac{4}{3} \cdot 6 = \$8,$ and a child ticket costs $\frac{1}{2} \cdot 8 = \$4.$

The total is

$$2(6 + 8 + 4) = \$36.$$

Thus, the correct answer is **B**.

9. Positive integers $a, b,$ and $2009,$ with $a < b < 2009,$ form a geometric sequence with an integer ratio. What is $a?$

- A 7
- B 41
- C 49
- D 287
- E 2009

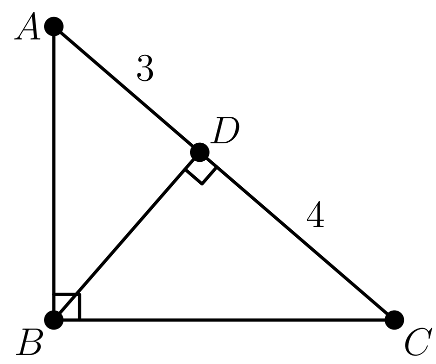
Solution:

Let the common ratio be r. Then $ar^2 = 2009 = 7^2 \cdot 41.$

Since r must be an integer greater than $1,$ the only possibility is $r = 7,$ giving $a = 41$ and the sequence $41, 287, 2009.$

Thus, the correct answer is **B**.

10. Triangle ABC has a right angle at B. Point D is the foot of the altitude from B, $AD = 3$, and $DC = 4$. What is the area of $\triangle ABC$?



A $4\sqrt{3}$

B $7\sqrt{3}$

C 21

D $14\sqrt{3}$

E 42

Solution:

For the altitude from the right angle to the hypotenuse,

$$BD^2 = AD \cdot DC = 3 \cdot 4 = 12,$$

so $BD = 2\sqrt{3}$.

The hypotenuse is $AC = 3 + 4 = 7$, so the area is

$$\frac{1}{2} \cdot 7 \cdot 2\sqrt{3} = 7\sqrt{3}.$$

Thus, the correct answer is **B**.

11. One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?

A 8

B 27

C 64

D 125

E 216

Solution:

Let the cube have side length x. The new solid has volume

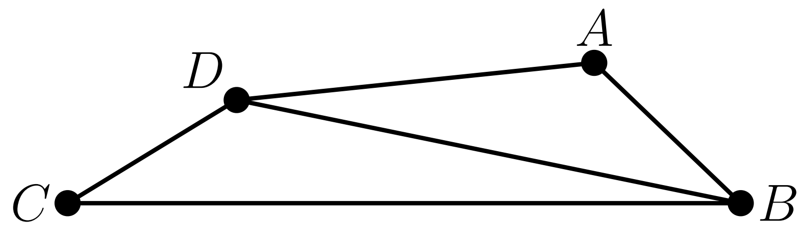
$$x(x+1)(x-1) = x^3 - x.$$

Setting this equal to $x^3 - 5$ gives $x^3 - x = x^3 - 5$, so $x = 5$.

The cube's volume is $5^3 = 125$.

Thus, the correct answer is **D**.

12. In quadrilateral $ABCD$, $AB = 5, BC = 17, CD = 5, DA = 9,$ and BD is an integer. What is BD?



A 11

B 12

C 13

D 14

E 15

Solution:

In $\triangle BCD$, the triangle inequality gives $5 + BD > 17$, so $BD > 12$.

In $\triangle ABD$, it gives $5 + 9 > BD$, so $BD < 14$.

The only integer with $12 < BD < 14$ is $BD = 13$.

Thus, the correct answer is **C**.

13. Suppose that $P = 2^m$ and $Q = 3^n$. Which of the following is equal to 12^{mn} for every pair of integers (m, n)?

A P^2Q

B P^nQ^m

C P^nQ^{2m}

D $P^{2m}Q^n$

E $P^{2n}Q^m$

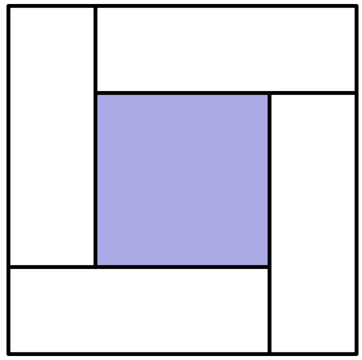
Solution:

Since $12 = 2^2 \cdot 3$,

$$12^{mn} = 2^{2mn} \cdot 3^{mn} = (2^m)^{2n} \cdot (3^n)^m = P^{2n}Q^m.$$

Thus, the correct answer is **E**.

14. Four congruent rectangles are placed as shown. The area of the outer square is 4 times that of the inner square. What is the ratio of the length of the longer side of each rectangle to the length of its shorter side?



A 3

B $\sqrt{10}$

C $2+\sqrt{2}$

D $2\sqrt{3}$

E 4

Solution:

Let each rectangle have shorter side x and longer side y. The outer square has side length $y+x$ and the inner square has side length $y-x$.

Since the area ratio is 4, the side ratio is 2, so

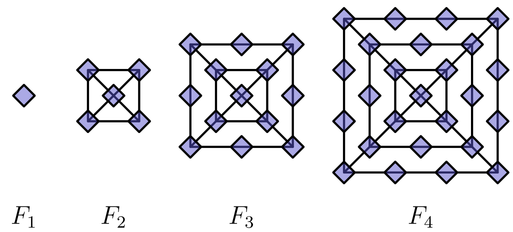
$$y+x=2(y-x),$$

which gives $y=3x$.

The ratio of longer to shorter side is $\frac{y}{x}=3$.

Thus, the correct answer is **A**.

15. The figures $F_1, F_2, F_3,$ and F_4 shown are the first in a sequence of figures. For $n \geq 3,$ F_n is constructed from F_{n-1} by surrounding it with a square and placing one more diamond on each side of the new square than F_{n-1} had on each side of its outside square. For example, figure F_3 has 13 diamonds. How many diamonds are there in figure F_{20}?



A 401

B 485

C 585

D 626

E 761

Solution:

Going from F_{n-1} to F_n, the new outside square carries $4(n-1)$ diamonds. Starting from the single diamond of F_1,

$$F_n = 1 + 4\big(1 + 2 + \cdots + (n-1)\big) = 1 + 4 \cdot \frac{(n-1)n}{2} = 1 + 2n(n-1).$$

Therefore

$$F_{20} = 1 + 2 \cdot 20 \cdot 19 = 761.$$

Thus, the correct answer is **E**.

16. Let $a, b, c,$ and d be real numbers with $|a-b|=2$, $|b-c|=3$, and $|c-d|=4$. What is the sum of all possible values of $|a-d|$?

A 9

B 12

C 15

D 18

E 24

Solution:

Since $a-d=(a-b)+(b-c)+(c-d)=\pm 2 \pm 3 \pm 4$, the possible absolute values are

$$2+3+4=9, \quad 2+3-4=1, \quad 2-3+4=3, \quad -2+3+4=5.$$

Their sum is $9+1+3+5=18$.

Thus, the correct answer is **D**.

17. Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment EF is constructed through B so that $EF \perp DB$, and A and C lie on DE and DF, respectively. What is EF?

A 9

B 10

C $\frac{125}{12}$

D $\frac{103}{9}$

E 12

Solution:

The diagonal is $DB = \sqrt{4^2 + 3^2} = 5$.

Right triangles EBA, DBC, and BFC are all similar to $\triangle DBA$. From $\triangle EBA$,

$$\frac{EB}{AB} = \frac{DB}{BC} \implies \frac{EB}{4} = \frac{5}{3} \implies EB = \frac{20}{3}.$$

From $\triangle BFC$,

$$\frac{BF}{BC} = \frac{DB}{AB} \implies \frac{BF}{3} = \frac{5}{4} \implies BF = \frac{15}{4}.$$

Therefore

$$EF = EB + BF = \frac{20}{3} + \frac{15}{4} = \frac{125}{12}.$$

Thus, the correct answer is **C**.

18. At Jefferson Summer Camp, 60% of the children play soccer, 30% of the children swim, and 40% of the soccer players swim. To the nearest whole percent, what percent of the non-swimmers play soccer?

A 30%

B 40%

C 49%

D 51%

E 70%

Solution:

Take 100 children: 60 play soccer, and 40% of them, or 24, also swim. So $60 - 24 = 36$ soccer players do not swim.

There are 30 swimmers and 70 non-swimmers, so the fraction of non-swimmers who play soccer is

$$\frac{36}{70} \approx 0.514 \approx 51\%.$$

Thus, the correct answer is **D**.

19. Circle A has radius 100. Circle B has an integer radius $r < 100$ and remains internally tangent to circle A as it rolls once around the circumference of circle A. The two circles have the same points of tangency at the beginning and end of circle B's trip. How many possible values can r have?

A 4

B 8

C 9

D 50

E 90

Solution:

The circumferences are 200π and $2\pi r$, so the initial point of tangency returns after

$$\frac{200\pi}{2\pi r} = \frac{100}{r}$$

rolls.

For this to be an integer greater than 1, r must be a divisor of 100 less than $100:$ namely $1, 2, 4, 5, 10, 20, 25,$ and 50. That is 8 values.

Thus, the correct answer is **B**.

20. Andrea and Lauren are 20 kilometers apart. They bike toward one another with Andrea traveling three times as fast as Lauren, and the distance between them decreasing at a rate of 1 kilometer per minute. After 5 minutes, Andrea stops biking because of a flat tire and waits for Lauren. After how many minutes from the time they started to bike does Lauren reach Andrea?

- A 20
- B 30
- C 55
- D 65
- E 80

Solution:

Let Lauren's rate be r km/min. Then $r + 3r = 1$, so $r = \frac{1}{4}$.

In the first 5 minutes the gap shrinks by 5 km, leaving 15 km. Lauren covers this alone at $\frac{1}{4}$ km/min, taking

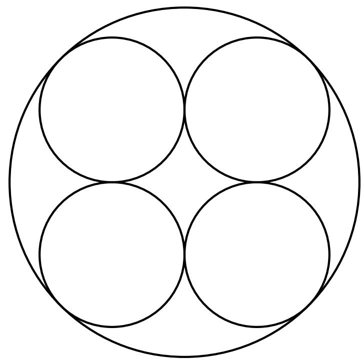
$$\frac{15}{1/4} = 60$$

minutes.

The total time is $5 + 60 = 65$ minutes.

Thus, the correct answer is **D**.

21. Many Gothic cathedrals have windows with portions containing a ring of congruent circles that are circumscribed by a larger circle. In the figure shown, the number of smaller circles is four. What is the ratio of the sum of the areas of the four smaller circles to the area of the larger circle?



A $3-2\sqrt{2}$

B $2-\sqrt{2}$

C $4(3-2\sqrt{2})$

D $\frac{1}{2}(3-\sqrt{2})$

E $2\sqrt{2}-2$

Solution:

Let each small circle have radius 1. Their centers form a square of side 2, whose diagonal is $2\sqrt{2}$.

The large circle's diameter is $2+2\sqrt{2}$, so its radius is $1+\sqrt{2}$.

The desired ratio is

$$\frac{4\cdot\pi(1)^2}{\pi(1+\sqrt{2})^2}=\frac{4}{3+2\sqrt{2}}=4(3-2\sqrt{2}).$$

Thus, the correct answer is **C**.

22. Two cubical dice each have removable numbers 1 through $6.$ The twelve numbers on the two dice are removed, put into a bag, then drawn one at a time and randomly reattached to the faces of the cubes, one number to each face. The dice are then rolled and the numbers on the two top faces are added. What is the probability that the sum is $7?$

A $\frac{1}{9}$

B $\frac{1}{8}$

C $\frac{1}{6}$

D $\frac{2}{11}$

E $\frac{1}{5}$

Solution:

Randomly attaching the tiles and then rolling is equivalent to choosing two of the twelve numbers at random and adding them.

Suppose the first top face shows $N.$ For a sum of $7,$ the second must be $7 - N,$ and there are exactly 2 tiles equal to $7 - N$ among the remaining $11.$

So the probability is $\frac{2}{11}.$

Thus, the correct answer is **D**.

23. Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE?

A $\frac{9}{2}$

B $\frac{50}{11}$

C $\frac{21}{4}$

D $\frac{17}{3}$

E 6

Solution:

Since $[AED] = [BEC]$, adding $[CED]$ to both gives $[ACD] = [BCD]$. These share base CD, so A and B are equidistant from line CD, meaning $AB \parallel CD$.

Then $\triangle ABE \sim \triangle CDE$ with ratio $\frac{AB}{CD} = \frac{9}{12} = \frac{3}{4}$, so $\frac{AE}{EC} = \frac{3}{4}$.

With $AE + EC = AC = 14$, we get $AE = \frac{3}{7} \cdot 14 = 6$.

Thus, the correct answer is **E**.

24. Three distinct vertices of a cube are chosen at random. What is the probability that the plane determined by these three vertices contains points inside the cube?

A $\frac{1}{4}$

B $\frac{3}{8}$

C $\frac{4}{7}$

D $\frac{5}{7}$

E $\frac{3}{4}$

Solution:

Three vertices determine a plane that cuts through the interior unless all three lie on a single face.

Each of the 6 faces gives $\binom{4}{3} = 4$ triples, so $6 \cdot 4 = 24$ triples lie on a face out of $\binom{8}{3} = 56$ total.

The probability of hitting the interior is

$$1 - \frac{24}{56} = \frac{4}{7}.$$

Thus, the correct answer is **C**.

25. For $k > 0$, let $I_k = 10\ldots064$, where there are k zeros between the 1 and the 6. Let $N(k)$ be the number of factors of 2 in the prime factorization of I_k. What is the maximum value of $N(k)$?

A 6

B 7

C 8

D 9

E 10

Solution:

Write

$$I_k = 10^{k+2} + 64 = 2^{k+2} \cdot 5^{k+2} + 2^6.$$

If $k < 4$, the first term has fewer than 6 factors of 2, so $N(k) = k + 2 < 6$.

If $k > 4$, the first term has at least 7 factors of 2 while the second has exactly 6, so their sum has exactly $6 : N(k) = 6$.

If $k = 4$, then $I_4 = 2^6(5^6 + 1)$. Since $5^6 + 1 = (5^2 + 1)\big((5^2)^2 - 5^2 + 1\big) = 26 \cdot 601$, it contributes exactly one more factor of 2. Thus $N(4) = 7$.

The maximum value is $N(4) = 7$.

Thus, the correct answer is **B**.