2008 AMC 10B Solutions

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1. A basketball player made 5 baskets during a game. Each basket was worth either 2 or 3 points. How many different numbers could represent the total points scored by the player?

A 2

B 3

C 4

D 5

E 6

Solution:

If k of the baskets are worth 3 points and the rest worth $2,$ the total is $2(5-k)+3k=10+k.$

As k ranges over $0, 1, \ldots, 5,$ the total takes every integer value from 10 to $15,$ giving 6 possibilities.

Thus, the correct answer is **E**.

2. A 4×4 block of calendar dates is shown. The order of the numbers in the second row is to be reversed. Then the order of the numbers in the fourth row is to be reversed. Finally, the numbers on each diagonal are to be added. What will be the positive difference between the two diagonal sums?

1	2	3	4
8	9	10	11
15	16	17	18
22	23	24	25

A 2

B 4

C 6

D 8

E 10

Solution:

After reversing the second row to $11, 10, 9, 8$ and the fourth row to $25, 24, 23, 22$, the two diagonals are $1, 10, 17, 22$ and $4, 9, 16, 25$.

Their sums are $1 + 10 + 17 + 22 = 50$ and $4 + 9 + 16 + 25 = 54$, so the positive difference is $54 - 50 = 4$.

Thus, the correct answer is **B**.

3. Assume that x is a positive real number. Which is equivalent to

$$\sqrt[3]{x\sqrt{x}}\,?$$

- A $x^{1/6}$
- B $x^{1/4}$
- C $x^{3/8}$
- D $x^{1/2}$
- E x

Solution:

Since $\sqrt{x} = x^{1/2}$, we have $x\sqrt{x} = x^1 \cdot x^{1/2} = x^{3/2}$.

Taking the cube root multiplies the exponent by $\frac{1}{3}$, giving $\left(x^{3/2}\right)^{1/3} = x^{1/2}$.

Thus, the correct answer is **D**.

4. A semipro baseball league has teams with 21 players each. League rules state that a player must be paid at least $\$15{,}000$, and that the total of all players' salaries for each team cannot exceed $\$700{,}000$. What is the maximum possible salary, in dollars, for a single player?

- A $270{,}000$
- B $385{,}000$
- C $400{,}000$
- D $430{,}000$
- E $700{,}000$

Solution:

One player's salary is largest when the other 20 players each earn the minimum $\$15{,}000$.

That leaves $\$700{,}000 - 20 \cdot \$15{,}000 = \$700{,}000 - \$300{,}000 = \$400{,}000$ for the single player.

Thus, the correct answer is **C**.

5. For real numbers a and b, define $a \,\$\, b = (a-b)^2$. What is $(x-y)^2 \,\$\, (y-x)^2$?

A 0

B $x^2 + y^2$

C $2x^2$

D $2y^2$

E $4xy$

Solution:

Since $(y-x)^2 = (x-y)^2$, the two inputs are identical.

Therefore $(x-y)^2 \,\$\, (y-x)^2 = \left((x-y)^2 - (x-y)^2\right)^2 = 0^2 = 0.$

Thus, the correct answer is **A**.

6. Points B and C lie on $\overline{AD}$. The length of $\overline{AB}$ is 4 times the length of $\overline{BD}$, and the length of $\overline{AC}$ is 9 times the length of $\overline{CD}$. The length of $\overline{BC}$ is what fraction of the length of $\overline{AD}$?

A $\frac{1}{36}$

B $\frac{1}{13}$

C $\frac{1}{10}$

D $\frac{5}{36}$

E $\frac{1}{5}$

Solution:

Since $AB = 4\,BD$ and $AB + BD = AD$, we get $5\,BD = AD$, so $BD = \frac{1}{5}AD$.

Since $AC = 9\,CD$ and $AC + CD = AD$, we get $10\,CD = AD$, so $CD = \frac{1}{10}AD$.

Then $BC = BD - CD = \frac{1}{5}AD - \frac{1}{10}AD = \frac{1}{10}AD$.

Thus, the correct answer is **C**.

7. An equilateral triangle of side length 10 is completely filled in by non-overlapping equilateral triangles of side length 1. How many small triangles are required?

A 10

B 25

C 100

D 250

E 1000

Solution:

The large triangle has side length 10 times that of a small triangle, so its area is $10^2 = 100$ times as large.

Since the small triangles tile it without overlap, exactly 100 of them are required.

Thus, the correct answer is **C**.

8. A class collects $\$50$ to buy flowers for a classmate who is in the hospital. Roses cost $\$3$ each, and carnations cost $\$2$ each. No other flowers are to be used. How many different bouquets could be purchased for exactly $\$50$?

A 1

B 7

C 9

D 16

E 17

Solution:

If r roses and c carnations are bought, then $3r + 2c = 50$. Because $2c$ and 50 are even, $3r$ must be even, so r is even.

Also $3r \leq 50$, so $r \leq 16$. The even values $r = 0, 2, 4, \ldots, 16$ each give a valid c, which is 9 bouquets.

Thus, the correct answer is **C**.

9. A quadratic equation $ax^2 - 2ax + b = 0$ has two real solutions. What is the average of the solutions?

A 1

B 2

C $\frac{b}{a}$

D $\frac{2b}{a}$

E $\sqrt{2a - b}$

Solution:

By Vieta's formulas, the sum of the roots of $ax^2 - 2ax + b = 0$ is $\frac{-(-2a)}{a} = 2.$

The average is half of this, namely $1.$

Thus, the correct answer is **A**.

10. Points A and B are on a circle of radius 5 and $AB = 6$. Point C is the midpoint of the minor arc AB. What is the length of the line segment AC?

A $\sqrt{10}$

B $\frac{7}{2}$

C $\sqrt{14}$

D $\sqrt{15}$

E 4

Solution:

Let O be the center and D the midpoint of AB. Then $OD \perp AB$ with $AD = 3$, so $OD = \sqrt{5^2 - 3^2} = 4$.

Since C is the midpoint of the minor arc, O, D, C are collinear and $DC = OC - OD = 5 - 4 = 1$.

Then $AC = \sqrt{AD^2 + DC^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$.

Thus, the correct answer is **A**.

11. Suppose that (u_n) is a sequence of real numbers satisfying $u_{n+2} = 2u_{n+1} + u_n$, and that $u_3 = 9$ and $u_6 = 128$. What is u_5?

- A 40
- B 53
- C 68
- D 88
- E 104

Solution:

Using the recurrence, $u_5 = 2u_4 + u_3 = 2u_4 + 9$ and $u_6 = 2u_5 + u_4 = 2(2u_4 + 9) + u_4 = 5u_4 + 18$.

Setting $5u_4 + 18 = 128$ gives $u_4 = 22$, so $u_5 = 2 \cdot 22 + 9 = 53$.

Thus, the correct answer is **B**.

12. Postman Pete has a pedometer to count his steps. The pedometer records up to 99999 steps, then flips over to 00000 on the next step. Pete plans to determine his mileage for a year. On January 1 Pete sets the pedometer to $00000.$ During the year, the pedometer flips from 99999 to 00000 forty-four times. On December 31 the pedometer reads $50000.$ Pete takes 1800 steps per mile. Which of the following is closest to the number of miles Pete walked during the year?

A 2500

B 3000

C 3500

D 4000

E 4500

Solution:

Each flip counts 100000 steps, so Pete took $44 \cdot 100000 + 50000 = 4{,}450{,}000$ steps.

Dividing by 1800 gives about 2472 miles, which is closest to $2500.$

Thus, the correct answer is **A**.

13. For each positive integer $n,$ the mean of the first n terms of a sequence is $n.$ What is the 2008th term of the sequence?

A 2008

B 4015

C 4016

D $4{,}030{,}056$

E $4{,}032{,}064$

Solution:

Since the mean of the first n terms is $n,$ their sum is $n^2.$

The nth term is $n^2 - (n-1)^2 = 2n - 1,$ so the 2008th term is $2 \cdot 2008 - 1 = 4015.$

Thus, the correct answer is **B**.

14. Triangle OAB has $O = (0,0)$, $B = (5,0)$, and A in the first quadrant. In addition, $\angle ABO = 90^\circ$ and $\angle AOB = 30^\circ$. Suppose that $\overline{OA}$ is rotated 90° counterclockwise about O. What are the coordinates of the image of A?

A $\left(-\frac{10}{3}\sqrt{3},\ 5\right)$

B $\left(-\frac{5}{3}\sqrt{3},\ 5\right)$

C $(\sqrt{3},\ 5)$

D $\left(\frac{5}{3}\sqrt{3},\ 5\right)$

E $\left(\frac{10}{3}\sqrt{3},\ 5\right)$

Solution:

Because $\angle ABO = 90^\circ$, segment AB is vertical, so $A = (5,\ 5\tan 30^\circ) = \left(5,\ \frac{5\sqrt{3}}{3}\right)$.

A 90° counterclockwise rotation about the origin sends (x, y) to $(-y, x)$, so the image of A is $\left(-\frac{5\sqrt{3}}{3},\ 5\right)$.

Thus, the correct answer is **B**.

15. How many right triangles have integer leg lengths a and b and a hypotenuse of length $b+1$, where $b<100$?

- A 6
- B 7
- C 8
- D 9
- E 10

Solution:

From $a^2+b^2=(b+1)^2$ we get $a^2=2b+1$, so a is odd and a^2 is an odd perfect square.

Since $b<100$, we need $a^2=2b+1<201$, and $a^2\geq 9$ for $b\geq 4$. The odd squares $9, 25, 49, 81, 121, 169$ give $a=3, 5, 7, 9, 11, 13$, which is 6 triangles.

Thus, the correct answer is **A**.

16. Two fair coins are to be tossed once. For each head that results, one fair die is to be rolled. What is the probability that the sum of the die rolls is odd? (Note that if no die is rolled, the sum is 0.)

A $\frac{3}{8}$

B $\frac{1}{2}$

C $\frac{43}{72}$

D $\frac{5}{8}$

E $\frac{2}{3}$

Solution:

Whenever at least one die is rolled, by symmetry the sum is odd with probability $\frac{1}{2}$.

No die is rolled only when both coins are tails, with probability $\frac{1}{4}$; that sum 0 is even. So the answer is $\left(1-\frac{1}{4}\right)\cdot\frac{1}{2}=\frac{3}{8}$.

Thus, the correct answer is **A**.

17. A poll shows that 70% of all voters approve of the mayor's work. On three separate occasions a pollster selects a voter at random. What is the probability that on exactly one of these three occasions the voter approves of the mayor's work?

A 0.063

B 0.189

C 0.233

D 0.333

E 0.441

Solution:

Exactly one approval among three occasions arises in $\binom{3}{1} = 3$ ways, each with probability $(0.7)(0.3)(0.3) = 0.063.$

The total is $3 \cdot 0.063 = 0.189.$

Thus, the correct answer is **B**.

18. Bricklayer Brenda would take 9 hours to build a chimney alone, and bricklayer Brandon would take 10 hours to build it alone. When they work together, they talk a lot, and their combined output is decreased by 10 bricks per hour. Working together, they build the chimney in 5 hours. How many bricks are in the chimney?

A 500

B 900

C 950

D 1000

E 1900

Solution:

Let n be the number of bricks. Brenda lays $\frac{n}{9}$ per hour and Brandon $\frac{n}{10}$, so together they lay $\frac{n}{9} + \frac{n}{10} - 10$ per hour.

Over 5 hours this equals n :

$$5\left(\frac{n}{9} + \frac{n}{10} - 10\right) = n.$$

Solving, $\frac{5n}{9} + \frac{n}{2} - 50 = n,$ which gives $n = 900.$

Thus, the correct answer is **B**.

19. A cylindrical tank with radius 4 feet and height 9 feet is lying on its side. The tank is filled with water to a depth of 2 feet. What is the volume of the water, in cubic feet?

- A $24\pi - 36\sqrt{2}$
- B $24\pi - 24\sqrt{3}$
- C $36\pi - 36\sqrt{3}$
- D $36\pi - 24\sqrt{2}$
- E $48\pi - 36\sqrt{3}$

Solution:

The submerged cross-section is a circular segment. The chord is $4 - 2 = 2$ feet below the center, and $\cos\theta = \frac{2}{4} = \frac{1}{2}$, so the half-angle is 60° and the central angle is 120°.

The sector area is $\frac{120}{360}\pi(4)^2 = \frac{16\pi}{3}$, and the triangle formed by the two radii has area $\frac{1}{2}(4)^2 \sin 120^\circ = 4\sqrt{3}$. The segment area is $\frac{16\pi}{3} - 4\sqrt{3}$.
Multiplying by the length 9 gives $9\left(\frac{16\pi}{3} - 4\sqrt{3}\right) = 48\pi - 36\sqrt{3}$.

Thus, the correct answer is **E**.

20. The faces of a cubical die are marked with the numbers $1, 2, 2, 3, 3,$ and $4.$ The faces of a second cubical die are marked with the numbers $1, 3, 4, 5, 6,$ and $8.$ Both dice are thrown. What is the probability that the sum of the two top numbers will be $5, 7,$ or $9?$

A $\frac{5}{18}$

B $\frac{7}{18}$

C $\frac{11}{18}$

D $\frac{3}{4}$

E $\frac{8}{9}$

Solution:

Of the 36 equally likely outcomes, the pairs giving sum 5 are $(1,4),(2,3),(2,3),(4,1),$ which is 4 outcomes.

Sum 7 comes from $(1,6),(2,5),(2,5),(3,4),(3,4),(4,3),$ which is $6,$ and sum 9 from $(1,8),(3,6),(3,6),(4,5),$ which is $4.$

The probability is $\frac{4+6+4}{36}=\frac{14}{36}=\frac{7}{18}.$

Thus, the correct answer is **B**.

21. Ten chairs are evenly spaced around a round table and numbered clockwise from 1 through 10. Five married couples are to sit in the chairs with men and women alternating, and no one is to sit either next to or directly across from his or her spouse. How many seating arrangements are possible?

- A 240
- B 360
- C 480
- D 540
- E 720

Solution:

Seat the women first. The first woman may take any of the 10 chairs, and since seats alternate, the remaining women fill their four seats in $4!$ ways, giving $10 \cdot 4! = 240$ arrangements.

Fix a woman in chair 1. Her spouse must sit in chair 4 or chair 8; each choice then forces the placement of every other man consistently. So each seating of the women yields exactly 2 valid seatings of the men.

The total is $2 \cdot 240 = 480.$

Thus, the correct answer is **C**.

22. Three red beads, two white beads, and one blue bead are placed in a line in random order. What is the probability that no two neighboring beads are the same color?

A $\frac{1}{12}$

B $\frac{1}{10}$

C $\frac{1}{6}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

There are $\frac{6!}{3!\,2!} = 60$ distinguishable orderings. The three reds must occupy non-adjacent positions, and the possible red placements are $\{1,3,5\}, \{2,4,6\}, \{1,3,6\}$, and $\{1,4,6\}$.

For $\{1,3,5\}$ and $\{2,4,6\}$, the remaining seats are mutually non-adjacent, so the blue bead can go in any of the 3, giving $3+3=6$. For $\{1,3,6\}$ and $\{1,4,6\}$, two remaining seats are adjacent, so the blue must separate the whites, giving $2+2=4$.

That is 10 valid orderings, so the probability is $\frac{10}{60} = \frac{1}{6}$.

Thus, the correct answer is **C**.

23. A rectangular floor measures a feet by b feet, where a and b are positive integers with $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half the area of the entire floor. How many possibilities are there for the ordered pair (a, b)?

A 1

B 2

C 3

D 4

E 5

Solution:

The painted rectangle is $(a-2) \times (b-2)$, and it is half the floor, so $ab = 2(a-2)(b-2)$.

Expanding gives $0 = ab - 4a - 4b + 8$, and adding 8 yields $(a-4)(b-4) = 8$.

With $b > a > 0$, the factorizations $8 = 1 \cdot 8 = 2 \cdot 4$ give $(a, b) = (5, 12)$ and $(6, 8)$. So there are 2 possibilities.

Thus, the correct answer is **B**.

24. Quadrilateral $ABCD$ has $AB = BC = CD, \angle ABC = 70^\circ$, and $\angle BCD = 170^\circ$. What is the degree measure of $\angle BAD$?

- A 75
- B 80
- C 85
- D 90
- E 95

Solution:

Let M be the point with $\triangle BMC$ equilateral, on the same side of BC as A. Then $\angle ABM = 70^\circ - 60^\circ = 10^\circ$ and $\angle MCD = 170^\circ - 60^\circ = 110^\circ$.

Since $AB = BM$ and $MC = CD$, triangles ABM and MCD are isosceles, giving $\angle AMB = 85^\circ$ and $\angle CMD = 35^\circ$.

Then $\angle AMD = 360^\circ - 85^\circ - 60^\circ - 35^\circ = 180^\circ$, so M lies on $\overline{AD}$ and $\angle BAD = \angle BAM = 85^\circ$.

Thus, the correct answer is **C**.

25. Michael walks at the rate of 5 feet per second on a long straight path. Trash pails are located every 200 feet along the path. A garbage truck travels at 10 feet per second in the same direction as Michael and stops for 30 seconds at each pail. As Michael passes a pail, he notices the truck ahead of him just leaving the next pail. How many times will Michael and the truck meet?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

Number the pails so Michael is at pail 0 and the truck at pail $1.$ Michael reaches pail n at $40n$ seconds. The truck leaves pail n at $50(n-1)$ seconds and arrives there at $50(n-1)-30$ seconds.

Michael and the truck are together at pail n when $50(n-1)-30 \leq 40n \leq 50(n-1),$ which simplifies to $5 \leq n \leq 8.$

At pail 5 they meet as the truck departs, at pails 6 and 7 Michael passes it, and at pail 8 they meet as the truck arrives. Between pails 6 and 7 the truck must overtake Michael once more, so in total they meet 5 times.

Thus, the correct answer is **B**.

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