2008 AMC 10A Solutions

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1. A bakery owner turns on his doughnut machine at $8{:}30$ am. At $11{:}10$ am the machine has completed one third of the day's job. At what time will the doughnut machine complete the job?

A $1{:}50$ pm

B $3{:}00$ pm

C $3{:}30$ pm

D $4{:}30$ pm

E $5{:}50$ pm

Solution:

From $8{:}30$ am to $11{:}10$ am is 2 hours and 40 minutes, or 160 minutes, to finish one third of the job.

The entire job therefore takes $3 \cdot 160 = 480$ minutes, or 8 hours.

Eight hours after $8{:}30$ am is $4{:}30$ pm.

Thus, the correct answer is **D**.

2. A square is drawn inside a rectangle. The ratio of the width of the rectangle to a side of the square is $2:1$. The ratio of the rectangle's length to its width is $2:1$. What percent of the rectangle's area is inside the square?

A 12.5

B 25

C 50

D 75

E 87.5

Solution:

Let the side of the square be s, so its area is s^2.

The width of the rectangle is $2s$, and its length is $2 \cdot 2s = 4s$, giving an area of $8s^2$.

The fraction inside the square is $\frac{s^2}{8s^2} = \frac{1}{8} = 12.5\%$.

Thus, the correct answer is **A**.

3. For the positive integer n, let $\langle n \rangle$ denote the sum of all the positive divisors of n with the exception of n itself. For example, $\langle 4 \rangle = 1 + 2 = 3$ and $\langle 12 \rangle = 1 + 2 + 3 + 4 + 6 = 16$. What is $\langle\langle\langle 6 \rangle\rangle\rangle$?

A 6

B 12

C 24

D 32

E 36

Solution:

The positive divisors of 6 other than 6 are $1, 2,$ and $3,$ so $\langle 6 \rangle = 1 + 2 + 3 = 6.$

Since applying the operation to 6 again returns $6,$ we get $\langle\langle\langle 6 \rangle\rangle\rangle = 6.$

(A number equal to the sum of its proper divisors is called a perfect number, and 6 is the smallest.)

Thus, the correct answer is **A**.

4. Suppose that $\frac{2}{3}$ of 10 bananas are worth as much as 8 oranges. How many oranges are worth as much as $\frac{1}{2}$ of 5 bananas?

- A 2
- B $\frac{5}{2}$
- C 3
- D $\frac{7}{2}$
- E 4

Solution:

Since $\frac{2}{3}$ of 10 bananas is $\frac{20}{3}$ bananas worth 8 oranges, one banana is worth $8 \div \frac{20}{3} = \frac{6}{5}$ oranges.

Now $\frac{1}{2}$ of 5 bananas is $\frac{5}{2}$ bananas, worth $\frac{5}{2} \cdot \frac{6}{5} = 3$ oranges.

Thus, the correct answer is **C**.

5. Which of the following is equal to the product

$$\frac{8}{4}\cdot\frac{12}{8}\cdot\frac{16}{12}\cdots\frac{4n+4}{4n}\cdots\frac{2008}{2004}?$$

- A 251
- B 502
- C 1004
- D 2008
- E 4016

Solution:

Every denominator except the first cancels with the numerator of the previous fraction, so the whole product telescopes to $\dfrac{2008}{4}=502.$

Thus, the correct answer is **B**.

6. A triathlete competes in a triathlon in which the swimming, biking, and running segments are all of the same length. The triathlete swims at a rate of 3 kilometers per hour, bikes at a rate of 20 kilometers per hour, and runs at a rate of 10 kilometers per hour. Which of the following is closest to the triathlete's average speed, in kilometers per hour, for the entire race?

A 3

B 4

C 5

D 6

E 7

Solution:

Let each segment have length x. The total time is

$$\frac{x}{3} + \frac{x}{20} + \frac{x}{10} = \frac{29}{60}x$$

hours for the distance $3x$.

The average speed is $\frac{3x}{\frac{29}{60}x} = \frac{180}{29} \approx 6.2$, which is closest to 6.

Thus, the correct answer is **D**.

7. The fraction

$$\frac{\left(3^{2008}\right)^2-\left(3^{2006}\right)^2}{\left(3^{2007}\right)^2-\left(3^{2005}\right)^2}$$

simplifies to which of the following?

A 1

B $\frac{9}{4}$

C 3

D $\frac{9}{2}$

E 9

Solution:

Since $\left(3^k\right)^2=9^k$, the fraction is $\frac{9^{2008}-9^{2006}}{9^{2007}-9^{2005}}$.

Factoring 9^{2005} from each part gives

$$\frac{9^{2005}\left(9^3-9\right)}{9^{2005}\left(9^2-1\right)}=\frac{9\left(9^2-1\right)}{9^2-1}=9.$$

Thus, the correct answer is **E**.

8. Heather compares the price of a new computer at two different stores. Store A offers 15% off the sticker price followed by a $\$90$ rebate, and store B offers 25% off the same sticker price with no rebate. Heather saves $\$15$ by buying the computer at store A instead of store B. What is the sticker price of the computer, in dollars?

- A 750
- B 900
- C 1000
- D 1050
- E 1500

Solution:

Let x be the sticker price. Heather pays $0.85x - 90$ at store A and $0.75x$ at store B.

Since store A is $\$15$ cheaper,

$$0.85x - 90 = 0.75x - 15,$$

which gives $0.1x = 75$, so $x = 750$.

Thus, the correct answer is **A**.

9. Suppose that

$$\frac{2x}{3} - \frac{x}{6}$$

is an integer. Which of the following statements must be true about x?

A It is negative.

B It is even, but not necessarily a multiple of 3.

C It is a multiple of 3, but not necessarily even.

D It is a multiple of 6, but not necessarily a multiple of 12.

E It is a multiple of 12.

Solution:

Combining over a common denominator, $\frac{2x}{3} - \frac{x}{6} = \frac{4x - x}{6} = \frac{x}{2}$.

For $\frac{x}{2}$ to be an integer, x must be even.

The example $x = 4$ shows that x need not be a multiple of 3 and rules out the other statements.

Thus, the correct answer is **B**.

10. Each of the sides of a square S_1 with area 16 is bisected, and a smaller square S_2 is constructed using the bisection points as vertices. The same process is carried out on S_2 to construct an even smaller square S_3. What is the area of S_3?

A $\frac{1}{2}$

B 1

C 2

D 3

E 4

Solution:

The side of S_1 is 4. By the Pythagorean theorem, the side of S_2 is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$, so its area is 8.

By the same reasoning, S_3 has half the area of S_2, namely 4.

Thus, the correct answer is **E**.

11. While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing toward the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?

A 2

B 4

C 6

D 8

E 10

Solution:

At 4 miles per hour, Steve rows 1 mile in 15 minutes. During that time $15 \cdot 10 = 150$ gallons enter.

To stay under 30 gallons, LeRoy must bail $150 - 30 = 120$ gallons in 15 minutes, or $\frac{120}{15} = 8$ gallons per minute.

Thus, the correct answer is **D**.

12. In a collection of red, blue, and green marbles, there are 25% more red marbles than blue marbles, and there are 60% more green marbles than red marbles. Suppose that there are r red marbles. What is the total number of marbles in the collection?

- A $2.85r$
- B $3r$
- C $3.4r$
- D $3.85r$
- E $4.25r$

Solution:

Since $r = 1.25b$, the number of blue marbles is $b = \frac{r}{1.25} = 0.8r$.

The number of green marbles is $g = 1.6r$.

The total is $r + 0.8r + 1.6r = 3.4r$.

Thus, the correct answer is **C**.

13. Doug can paint a room in 5 hours. Dave can paint the same room in 7 hours. Doug and Dave paint the room together and take a one-hour break for lunch. Let t be the total time, in hours, required for them to complete the job working together, including lunch. Which of the following equations is satisfied by t?

A $\left(\frac{1}{5}+\frac{1}{7}\right)(t+1)=1$

B $\left(\frac{1}{5}+\frac{1}{7}\right)t+1=1$

C $\left(\frac{1}{5}+\frac{1}{7}\right)t=1$

D $\left(\frac{1}{5}+\frac{1}{7}\right)(t-1)=1$

E $(5+7)t=1$

Solution:

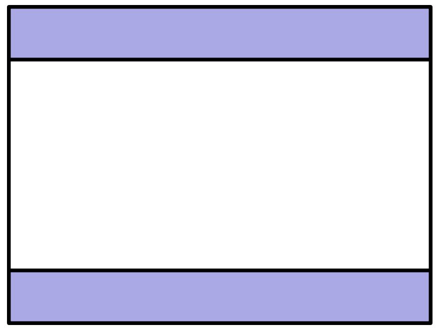
Working together, Doug and Dave paint $\frac{1}{5}+\frac{1}{7}$ of the room per hour.

Because they break for one hour, they work for only $t-1$ hours, and this must complete the whole room:

$$\left(\frac{1}{5}+\frac{1}{7}\right)(t-1)=1.$$

Thus, the correct answer is **D**.

14. Older television screens have an aspect ratio of $4:3$. That is, the ratio of the width to the height is $4:3$. The aspect ratio of many movies is not $4:3$, so they are sometimes shown on a television screen by "letterboxing" — darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of $2:1$ and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?



- A 2
- B 2.25
- C 2.5
- D 2.7
- E 3

Solution:

Since the screen is $4:3$ with a 27-inch diagonal, $h:w:27=3:4:5$, giving height $h=\frac{3}{5}\cdot 27=16.2$ and width $w=\frac{4}{5}\cdot 27=21.6$.

The lit $2:1$ region has the full width 21.6 and height $\frac{21.6}{2}=10.8$.

The two strips share the remaining height, so each has height $\frac{16.2-10.8}{2}=2.7$.

Thus, the correct answer is **D**.

15. Yesterday Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?

- A 120
- B 130
- C 140
- D 150
- E 160

Solution:

Let Ian drive t hours at rate r, covering rt miles.

Han drove $(r+5)(t+1)-rt=5t+r+5=70$, so $5t+r=65$.

Jan drove $(r+10)(t+2)-rt=10t+2r+20=2(5t+r)+20=2\cdot 65+20=150$ miles more than Ian.

Thus, the correct answer is **D**.

16. Points A and B lie on a circle centered at O, and $\angle AOB = 60^\circ$. A second circle is internally tangent to the first and tangent to both OA and OB. What is the ratio of the area of the smaller circle to that of the larger circle?

A $\frac{1}{16}$

B $\frac{1}{9}$

C $\frac{1}{8}$

D $\frac{1}{6}$

E $\frac{1}{4}$

Solution:

Let the radii be r and R. The small circle's center E lies on the bisector of $\angle AOB$, so the angle to a tangent line is 30°.

The perpendicular from E to OA has length r, and in the resulting 30-60-90 triangle $OE = 2r$.

Since $OE = R - r$, we get $2r = R - r$, so $R = 3r$ and the area ratio is $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$.

Thus, the correct answer is **B**.

17. An equilateral triangle has side length 6. What is the area of the region containing all points that are outside the triangle and not more than 3 units from a point of the triangle?

A $36 + 24\sqrt{3}$

B $54 + 9\pi$

C $54 + 18\sqrt{3} + 6\pi$

D $\left(2\sqrt{3} + 3\right)^2 \pi$

E $9\left(\sqrt{3} + 1\right)^2 \pi$

Solution:

Along each of the three sides is a 6×3 rectangle, contributing $3 \cdot 6 \cdot 3 = 54.$

At each vertex is a 120° sector of radius 3; the three together form a full circle of area $\pi \cdot 3^2 = 9\pi.$

The total area is $54 + 9\pi.$

Thus, the correct answer is **B**.

18. A right triangle has perimeter 32 and area 20. What is the length of its hypotenuse?

A $\frac{57}{4}$

B $\frac{59}{4}$

C $\frac{61}{4}$

D $\frac{63}{4}$

E $\frac{65}{4}$

Solution:

Let the legs be y, z and the hypotenuse x. Then $y^2 + z^2 = x^2, y + z = 32 - x$, and $yz = 40$.

Squaring the second equation,

$$(32 - x)^2 = y^2 + z^2 + 2yz = x^2 + 80.$$

This gives $1024 - 64x = 80$, so $x = \frac{59}{4}$.

Thus, the correct answer is **B**.

19. Rectangle $PQRS$ lies in a plane with $PQ = RS = 2$ and $QR = SP = 6$. The rectangle is rotated 90° clockwise about R, then rotated 90° clockwise about the point that S moved to after the first rotation. What is the length of the path traveled by point P?

A $\left(2\sqrt{3} + \sqrt{5}\right)\pi$

B 6π

C $\left(3 + \sqrt{10}\right)\pi$

D $\left(\sqrt{3} + 2\sqrt{5}\right)\pi$

E $2\sqrt{10}\pi$

Solution:

In the first rotation, P moves on a quarter circle about R with radius $PR = \sqrt{2^2 + 6^2} = 2\sqrt{10}$. The arc length is $\frac{1}{4}\left(2\pi \cdot 2\sqrt{10}\right) = \sqrt{10}\,\pi$.

In the second rotation, P moves on a quarter circle about the new position of S with radius 6. The arc length is $\frac{1}{4}(2\pi \cdot 6) = 3\pi$.

The total path length is $\left(3 + \sqrt{10}\right)\pi$.

Thus, the correct answer is **C**.

20. Trapezoid $ABCD$ has bases AB and CD and diagonals intersecting at K. Suppose that $AB = 9, DC = 12,$ and the area of $\triangle AKD$ is $24.$ What is the area of trapezoid $ABCD$?

A 92

B 94

C 96

D 98

E 100

Solution:

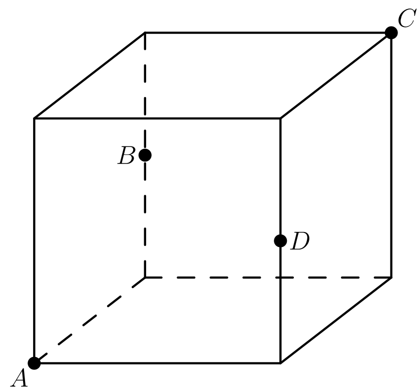
Triangles AKB and CKD are similar with ratio $\frac{9}{12} = \frac{3}{4}$.

Since $\triangle AKD$ and $\triangle KCD$ share the base and have collinear vertices, $\frac{[KCD]}{[AKD]} = \frac{KC}{AK} = \frac{4}{3}$, so $[KCD] = 32.$ Similarly $[AKB] = 18.$

Also $[BKC] = [AKD] = 24.$ The total is $24 + 32 + 18 + 24 = 98.$

Thus, the correct answer is **D**.

21. A cube with side length 1 is sliced by a plane that passes through two diagonally opposite vertices A and C and the midpoints B and D of two opposite edges not containing A or C, as shown. What is the area of quadrilateral $ABCD$?



A $\frac{\sqrt{6}}{2}$

B $\frac{5}{4}$

C $\sqrt{2}$

D $\frac{3}{2}$

E $\sqrt{3}$

Solution:

Each side of $ABCD$ joins a vertex of the cube to the midpoint of an edge, so all four sides are equal and $ABCD$ is a rhombus.

Its diagonals are the space diagonal $AC = \sqrt{3}$ and the face diagonal $BD = \sqrt{2}$.

The area of a rhombus is half the product of its diagonals: $\frac{1}{2} \cdot \sqrt{3} \cdot \sqrt{2} = \frac{\sqrt{6}}{2}$.

Thus, the correct answer is **A**.

22. Jacob uses the following procedure to write down a sequence of numbers. First he chooses the first term to be 6. To generate each succeeding term, he flips a fair coin. If it comes up heads, he doubles the previous term and subtracts 1. If it comes up tails, he takes half of the previous term and subtracts 1. What is the probability that the fourth term in Jacob's sequence is an integer?

A $\frac{1}{6}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{5}{8}$

E $\frac{3}{4}$

Solution:

Starting from 6, the second terms are 11 (heads) and 2 (tails).

Continuing the tree, the eight equally likely fourth terms are $41, 9.5, 8, 1.25, 5, 0.5, -1, -1.$

Of these, $41, 8, 5, -1, -1$ are integers, so the probability is $\frac{5}{8}$.

Thus, the correct answer is **D**.

23. Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

- A 20
- B 40
- C 60
- D 160
- E 320

Solution:

Choose the two common elements in $\binom{5}{2} = 10$ ways.

Each of the remaining 3 elements must lie in exactly one subset, giving $2^3 = 8$ assignments, for 80 ordered pairs.

Since the order of the two subsets does not matter, divide by 2 to get $\frac{80}{2} = 40.$

Thus, the correct answer is **B**.

24. Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

A 0

B 2

C 4

D 6

E 8

Solution:

The units digit of 2^n cycles $2, 4, 8, 6$, so 2^{2008} ends in 6. Also 2008^2 ends in 4.

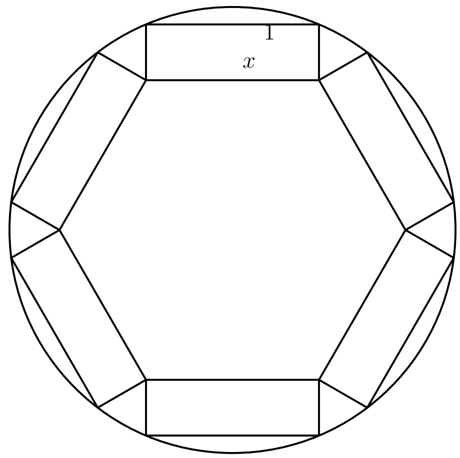
Thus k ends in 0, so k^2 ends in 0.

Both 2008^2 and 2^{2008} are multiples of 4, so $k \equiv 0 \pmod 4$, which makes 2^k end in 6.

The units digit of $k^2 + 2^k$ is $0 + 6 = 6$.

Thus, the correct answer is **D**.

25. A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being end points of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x?



A $2\sqrt{5}-\sqrt{3}$

B 3

C $\dfrac{3\sqrt{7}-\sqrt{3}}{2}$

D $2\sqrt{3}$

E $\dfrac{5+2\sqrt{3}}{2}$

Solution:

Pick a mat with outer corners P and Q, and let R be the point on the circle diametrically opposite P. Then $\triangle PQR$ is right-angled at Q with hypotenuse $PR = 8$.

The inner corners of adjacent mats meet in isosceles triangles with vertex angle 120° and sides x, whose base is $\sqrt{3}\,x$. Together with the two mat widths, $QR = \sqrt{3}\,x + 2$.

By the Pythagorean theorem,

$$\left(\sqrt{3}\,x + 2\right)^2 + x^2 = 64,$$

which simplifies to $x^2 + \sqrt{3}\,x - 15 = 0$.

Taking the positive root,

$$x = \frac{3\sqrt{7} - \sqrt{3}}{2}.$$

Thus, the correct answer is **C**.

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