2007 AMC 10B Solutions

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1. Isabella's house has 3 bedrooms. Each bedroom is 12 feet long, 10 feet wide, and 8 feet high. Isabella must paint the walls of all the bedrooms. Doorways and windows, which will not be painted, occupy 60 square feet in each bedroom. How many square feet of walls must be painted?

A 678

B 768

C 786

D 867

E 876

Solution:

The walls of one bedroom have area $2(12+10)\cdot 8 = 44\cdot 8 = 352$ square feet. Subtracting the 60 square feet of doorways and windows leaves $352-60=292$ square feet per bedroom.

With 3 bedrooms, the total is $3\cdot 292 = 876$ square feet.

Thus, the correct answer is **E**.

2. Define the operation $\star$ by $a \star b = (a+b)b$. What is $(3 \star 5) - (5 \star 3)$?

A -16

B -8

C 0

D 8

E 16

Solution:

Since $3 \star 5 = (3+5) \cdot 5 = 40$ and $5 \star 3 = (5+3) \cdot 3 = 24$, the difference is $40 - 24 = 16$.

Thus, the correct answer is **E**.

3. A college student drove his compact car 120 miles home for the weekend and averaged 30 miles per gallon. On the return trip the student drove his parents' SUV and averaged only 20 miles per gallon. What was the average gas mileage, in miles per gallon, for the round trip?

A 22

B 24

C 25

D 26

E 28

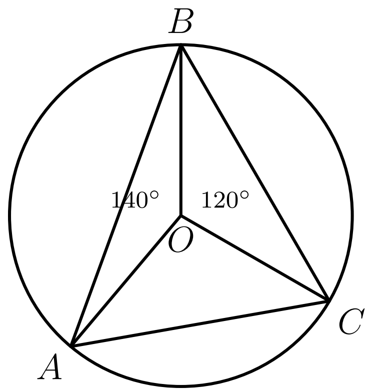
Solution:

The student used $120/30 = 4$ gallons driving home and $120/20 = 6$ gallons returning, for 10 gallons over 240 miles.

The average is $240/10 = 24$ miles per gallon.

Thus, the correct answer is **B**.

4. The point O is the center of the circle circumscribed about $\triangle ABC$, with $\angle BOC = 120^\circ$ and $\angle AOB = 140^\circ$, as shown. What is the degree measure of $\angle ABC$?



A 35

B 40

C 45

D 50

E 60

Solution:

Since $OA = OB = OC$, triangles AOB, BOC, and COA are isosceles. The base angles give $\angle ABO = \dfrac{180^\circ - 140^\circ}{2} = 20^\circ$ and $\angle OBC = \dfrac{180^\circ - 120^\circ}{2} = 30^\circ$.

Therefore $\angle ABC = 20^\circ + 30^\circ = 50^\circ$.

Thus, the correct answer is **D**.

5. In a certain land, all Arogs are Brafs, all Crups are Brafs, all Dramps are Arogs, and all Crups are Dramps. Which of the following statements is implied by these facts?

- A All Dramps are Brafs and are Crups.
- B All Brafs are Crups and are Dramps.
- C All Arogs are Crups and are Dramps.
- D All Crups are Arogs and are Brafs.
- E All Arogs are Dramps and some Arogs may not be Crups.

Solution:

Writing the statements as implications, being a Crup implies being a Dramp, a Dramp implies being an Arog, and an Arog implies being a Braf: $C \Rightarrow D \Rightarrow A \Rightarrow B.$

So every Crup is a Dramp, an Arog, and a Braf. The only listed statement guaranteed true is that all Crups are Arogs and Brafs.

Thus, the correct answer is **D**.

6. The 2007 AMC 10 will be scored by awarding 6 points for each correct response, 0 points for each incorrect response, and 1.5 points for each problem left unanswered. After looking over the 25 problems, Sarah has decided to attempt the first 22 and leave only the last 3 unanswered. How many of the first 22 problems must she solve correctly in order to score at least 100 points?

A 13

B 14

C 15

D 16

E 17

Solution:

The three blank problems give $3 \cdot 1.5 = 4.5$ points, so Sarah needs $100 - 4.5 = 95.5$ points from the first 22.

Since $95.5/6$ lies between 15 and 16, she must answer at least 16 correctly, which would give a score of 100.5.

Thus, the correct answer is **D**.

7. All sides of the convex pentagon $ABCDE$ are of equal length, and $\angle A = \angle B = 90^\circ$. What is the degree measure of $\angle E$?

- A 90
- B 108
- C 120
- D 144
- E 150

Solution:

Because $AB = BC = EA$ and $\angle A = \angle B = 90^\circ$, quadrilateral $ABCE$ is a square, so $\angle AEC = 90^\circ$.

The remaining sides satisfy $CD = DE = EC$, so $\triangle CDE$ is equilateral and $\angle CED = 60^\circ$.

Therefore $\angle E = \angle AEC + \angle CED = 90^\circ + 60^\circ = 150^\circ$.

Thus, the correct answer is **E**.

8. On the trip home from the meeting where this AMC 10 was constructed, the Contest Chair noted that his airport parking receipt had digits of the form $bbcac$, where $0 \leq a < b < c \leq 9$, and b was the average of a and c. How many different five-digit numbers satisfy all these properties?

A 12

B 16

C 18

D 20

E 24

Solution:

Once a and c are chosen, $b = \dfrac{a+c}{2}$ is determined, and $a < b < c$ holds automatically. For b to be an integer, a and c must share parity.

Choosing two even digits from $\{0, 2, 4, 6, 8\}$ gives $\binom{5}{2} = 10$ pairs, and choosing two odd digits from $\{1, 3, 5, 7, 9\}$ gives another $10.$

This yields 20 valid numbers.

Thus, the correct answer is **D**.

9. A cryptographic code is designed as follows. The first time a letter appears in a given message it is replaced by the letter that is 1 place to its right in the alphabet (assuming that the letter A is one place to the right of the letter Z). The second time this same letter appears in the given message, it is replaced by the letter that is $1+2$ places to the right, the third time it is replaced by the letter that is $1+2+3$ places to the right, and so on. For example, with this code the word "banana" becomes "cbodqg". What letter will replace the last letter s in the message

"Lee's sis is a Mississippi miss, Chriss!"?

A g

B h

C o

D s

E t

Solution:

The final s is the 12th appearance of the letter s in the message, so it is shifted $1+2+\cdots+12=\frac{12\cdot 13}{2}=78$ places to the right.

Since $78=3\cdot 26$ is a multiple of the alphabet length $26,$ the shift returns to the same letter, s.

Thus, the correct answer is **D**.

10. Two points B and C are in a plane. Let S be the set of all points A in the plane for which $\triangle ABC$ has area 1. Which of the following describes S?

A two parallel lines

B a parabola

C a circle

D a line segment

E two points

Solution:

Taking BC as the base, the area is $\frac{1}{2}(BC)d$, where d is the distance from A to line BC. The area equals 1 exactly when $d = \frac{2}{BC}$.

The points at this fixed distance from line BC form two lines parallel to BC, one on each side.

Thus, the correct answer is **A**.

11. A circle passes through the three vertices of an isosceles triangle that has two sides of length 3 and a base of length 2. What is the area of this circle?

A 2π

B $\frac{5}{2}\pi$

C $\frac{81}{32}\pi$

D 3π

E $\frac{7}{2}\pi$

Solution:

The triangle has sides $3, 3, 2$. Its area is $\frac{1}{2} \cdot 2 \cdot \sqrt{3^2 - 1^2} = 2\sqrt{2}$.

The circumradius is $R = \frac{abc}{4K} = \frac{3 \cdot 3 \cdot 2}{4 \cdot 2\sqrt{2}} = \frac{9}{4\sqrt{2}} = \frac{9\sqrt{2}}{8}$.

The area of the circle is $\pi R^2 = \pi \cdot \frac{81 \cdot 2}{64} = \frac{81}{32}\pi$.

Thus, the correct answer is **C**.

12. Tom's age is T years, which is also the sum of the ages of his three children. His age N years ago was twice the sum of their ages then. What is T/N?

A 2

B 3

C 4

D 5

E 6

Solution:

N years ago Tom's age was $T-N$, and the sum of his three children's ages was $T-3N$.

The condition gives $T-N=2(T-3N)$, so $T-N=2T-6N$, which simplifies to $5N=T$.

Therefore $T/N=5$.

Thus, the correct answer is **D**.

13. Two circles of radius 2 are centered at $(2, 0)$ and at $(0, 2)$. What is the area of the intersection of the interiors of the two circles?

A $\pi - 2$

B $\frac{\pi}{2}$

C $\frac{\pi\sqrt{3}}{3}$

D $2(\pi - 2)$

E π

Solution:

The two circles intersect at $(0, 0)$ and $(2, 2)$.

By symmetry, half the intersection is formed by removing an isosceles right triangle of leg length 2 from a quarter of one circle. The quarter-circle has area $\frac{1}{4}\pi(2)^2 = \pi$ and the triangle has area $\frac{1}{2}(2)^2 = 2$.

Therefore the whole region has area $2(\pi - 2)$.

Thus, the correct answer is **D**.

14. Some boys and girls are having a car wash to raise money for a class trip to China. Initially 40% of the group are girls. Shortly thereafter two girls leave and two boys arrive, and then 30% of the group are girls. How many girls were initially in the group?

A 4

B 6

C 8

D 10

E 12

Solution:

Two girls leave and two boys arrive, so the group size is unchanged. The two girls who left therefore represent $40\% - 30\% = 10\%$ of the group.

Thus the group has $2 \div 0.10 = 20$ people, and the original number of girls was 40% of 20, or 8.

Thus, the correct answer is **C**.

15. The angles of quadrilateral $ABCD$ satisfy $\angle A = 2\angle B = 3\angle C = 4\angle D$. What is the degree measure of $\angle A$, rounded to the nearest whole number?

A 125

B 144

C 153

D 173

E 180

Solution:

Let $x = \angle A$. Then $\angle B = \frac{x}{2}$, $\angle C = \frac{x}{3}$, and $\angle D = \frac{x}{4}$.

The angles sum to 360°, so $x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{25x}{12} = 360.$

Thus $x = \frac{12 \cdot 360}{25} = 172.8 \approx 173.$

Thus, the correct answer is **D**.

16. A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was $84.$ The juniors all received the same score, and the average score of the seniors was $83.$ What score did each of the juniors receive on the test?

A 85

B 88

C 93

D 94

E 98

Solution:

Suppose the class has 10 students: one junior and nine seniors. The total of all scores is $10 \cdot 84 = 840.$

The nine seniors total $9 \cdot 83 = 747,$ so the junior's score is $s = 840 - 747 = 93.$

Thus, the correct answer is **C**.

17. Point P is inside equilateral $\triangle ABC$. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB?

A 4

B $3\sqrt{3}$

C 6

D $4\sqrt{3}$

E 9

Solution:

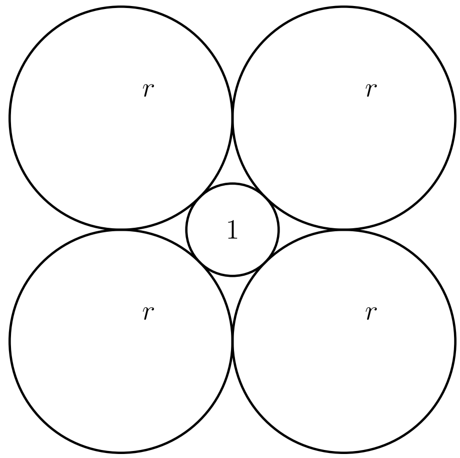
Let the side length be s. The perpendiculars from P are the heights of triangles APB, BPC, and CPA, so their areas are $\frac{s}{2}$, s, and $\frac{3s}{2}$.

Their sum equals the area of $\triangle ABC$, which is also $\frac{\sqrt{3}}{4}s^2$. Hence $3s = \frac{\sqrt{3}}{4}s^2$.

The positive solution is $s = 4\sqrt{3}$.

Thus, the correct answer is **D**.

18. A circle of radius 1 is surrounded by 4 circles of radius r as shown. What is r?



A $\sqrt{2}$

B $1+\sqrt{2}$

C $\sqrt{6}$

D 3

E $2+\sqrt{2}$

Solution:

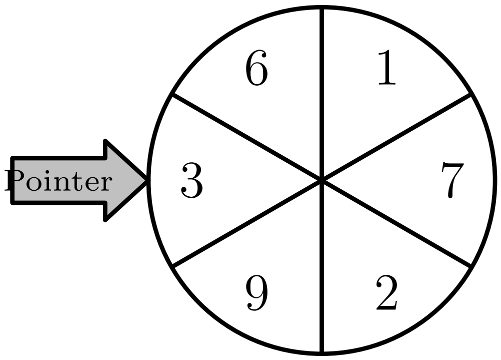
Connect the centers of the four outer circles to form a square. Adjacent outer circles are tangent, so each side has length $2r$.

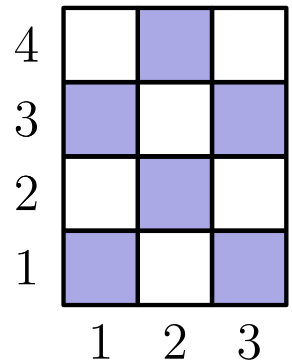
The diagonal of the square passes through the center circle, giving length $1+r+r+1=2+2r$. Since a square with side $2r$ has diagonal $2r\sqrt{2}$, we get $2(2r)^2=(2+2r)^2$.

Expanding gives $1+2r+r^2=2r^2$, so $r^2-2r-1=0$. The positive root is $r=1+\sqrt{2}$.

Thus, the correct answer is **B**.

19. The wheel shown is spun twice, and the randomly determined numbers opposite the pointer are recorded. The first number is divided by 4, and the second number is divided by 5. The first remainder designates a column, and the second remainder designates a row on the checkerboard shown. What is the probability that the pair of numbers designates a shaded square?





A $\frac{1}{3}$

B $\frac{4}{9}$

C $\frac{1}{2}$

D $\frac{5}{9}$

E $\frac{2}{3}$

Solution:

The shaded squares are those where the two remainders are both odd or both even.

The first remainder is even (from the numbers 2 and 6) with probability $\frac{1}{3}$ and odd with probability $\frac{2}{3}$.

The second remainder is even with probability $\frac{1}{2}$ and odd with probability $\frac{1}{2}$.

The probability that they share parity is $\frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}$.

Thus, the correct answer is **C**.

20. A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

- A 100
- B 125
- C 600
- D 2300
- E 3600

Solution:

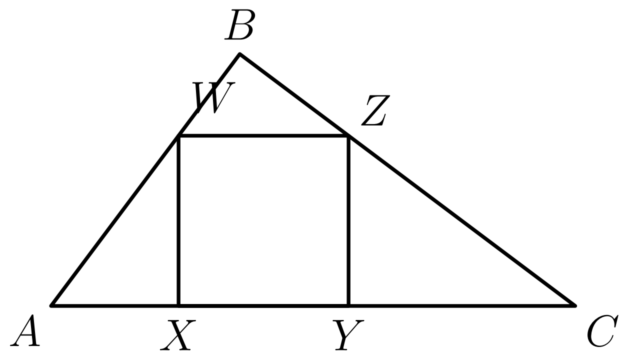
Choose 3 of the 5 rows in $\binom{5}{3} = 10$ ways and 3 of the 5 columns in $\binom{5}{3} = 10$ ways.

The three chosen blocks must occupy distinct rows and columns, so they form a matching between the three rows and three columns, which can be done in $3! = 6$ ways.

The total is $10 \cdot 10 \cdot 6 = 600$.

Thus, the correct answer is **C**.

21. Right $\triangle ABC$ has $AB = 3, BC = 4,$ and $AC = 5$. Square $XYZW$ is inscribed in $\triangle ABC$ with X and Y on $\overline{AC}, W$ on $\overline{AB},$ and Z on $\overline{BC}$. What is the side length of the square?



A $\frac{3}{2}$

B $\frac{60}{37}$

C $\frac{12}{7}$

D $\frac{23}{13}$

E 2

Solution:

Let s be the side of the square and h the altitude from B to AC. Then $h = \frac{AB \cdot BC}{AC} = \frac{3 \cdot 4}{5} = \frac{12}{5}$.

The small triangle above the square is similar to $\triangle ABC$ with the square's top side as its base, giving $\frac{h-s}{h} = \frac{s}{AC}$, so $s = \frac{AC \cdot h}{AC + h}$.

Substituting, $s = \frac{5 \cdot \frac{12}{5}}{5 + \frac{12}{5}} = \frac{12}{37/5} = \frac{60}{37}$.

Thus, the correct answer is **B**.

22. A player chooses one of the numbers 1 through $4.$ After the choice has been made, two regular four-sided (tetrahedral) dice are rolled, with the sides of the dice numbered 1 through $4.$ If the number chosen appears on the bottom of exactly one die after it is rolled, then the player wins $1. If the number chosen appears on the bottom of both of the dice, then the player wins $2. If the number chosen does not appear on the bottom of either of the dice, the player loses $1. What is the expected return to the player, in dollars, for one roll of the dice?

A $-\dfrac{1}{8}$

B $-\dfrac{1}{16}$

C 0

D $\dfrac{1}{16}$

E $\dfrac{1}{8}$

Solution:

Each die shows the chosen number on the bottom with probability $\dfrac{1}{4}$. So the number appears $0, 1,$ or 2 times with probabilities $P(0) = \dfrac{9}{16},\ P(1) = \dfrac{6}{16},\ P(2) = \dfrac{1}{16}.$

The expected return is $(-1) \cdot \dfrac{9}{16} + (1) \cdot \dfrac{6}{16} + (2) \cdot \dfrac{1}{16} = \dfrac{-9+6+2}{16} = -\dfrac{1}{16}.$

Thus, the correct answer is **B**.

23. A pyramid with a square base is cut by a plane that is parallel to its base and is 2 units from the base. The surface area of the smaller pyramid that is cut from the top is half the surface area of the original pyramid. What is the altitude of the original pyramid?

- A 2
- B $2+\sqrt{2}$
- C $1+2\sqrt{2}$
- D 4
- E $4+2\sqrt{2}$

Solution:

Let h be the altitude of the original pyramid; the smaller pyramid has altitude $h-2$. The two pyramids are similar, so the ratio of their surface areas is the square of the ratio of their altitudes.

The smaller surface area is half the original, so $\left(\frac{h-2}{h}\right)^2=\frac{1}{2}$, giving $\frac{h}{h-2}=\sqrt{2}$.

Then $h=\sqrt{2}(h-2)$, so $h(\sqrt{2}-1)=2\sqrt{2}$ and $h=\frac{2\sqrt{2}}{\sqrt{2}-1}=4+2\sqrt{2}$.

Thus, the correct answer is **E**.

24. Let n denote the smallest positive integer that is divisible by both 4 and 9, and whose base-10 representation consists of only 4's and 9's, with at least one of each. What are the last four digits of n?

A 4444

B 4494

C 4944

D 9444

E 9944

Solution:

Since n is divisible by 9, its digit sum is a multiple of 9. With k fours and m nines, the digit sum is $4k + 9m$, so $9 \mid 4k$, forcing $9 \mid k$. Thus $k \geq 9$, and with at least one 9, the number has at least ten digits.

For divisibility by 4, the last two digits must form a multiple of 4, and among $44, 49, 94, 99$ only 44 works, so n ends in 44.

The smallest such ten-digit number places the single 9 in the lowest available position, giving $4{,}444{,}444{,}944$. Its last four digits are 4944.

Thus, the correct answer is **C**.

25. How many pairs of positive integers (a, b) are there such that a and b have no common factors greater than 1 and

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer?

A 4

B 6

C 9

D 12

E infinitely many

Solution:

Combining, the expression is $\frac{9a^2 + 14b^2}{9ab}$. For this to be an integer, a must divide $9a^2 + 14b^2$, hence $a \mid 14b^2$. Since $\gcd(a, b) = 1$, we get $a \mid 14$. Similarly $b \mid 9a^2$ gives $b \mid 9$.

So $a \in \{1, 2, 7, 14\}$ and $b \in \{1, 3, 9\}$. Checking these, only $b = 3$ makes the expression an integer for each allowed a.

The valid pairs are $(1, 3)$, $(2, 3)$, $(7, 3)$, and $(14, 3)$, for a total of 4.

Thus, the correct answer is **A**.

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