2007 AMC 10A Solutions

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1. One ticket to a show costs $\$20$ at full price. Susan buys 4 tickets using a coupon that gives her a 25% discount. Pam buys 5 tickets using a coupon that gives her a 30% discount. How many more dollars does Pam pay than Susan?

A 2

B 5

C 10

D 15

E 20

Solution:

Susan pays $4 \cdot 0.75 \cdot 20 = 60$ dollars.

Pam pays $5 \cdot 0.70 \cdot 20 = 70$ dollars.

The difference is $70 - 60 = 10$ dollars.

Thus, the correct answer is **C**.

2. Define $a@b = ab - b^2$ and $a\#b = a + b - ab^2$. What is

$$\frac{6@2}{6\#2}?$$

A $-\frac{1}{2}$

B $-\frac{1}{4}$

C $\frac{1}{8}$

D $\frac{1}{4}$

E $\frac{1}{2}$

Solution:

The numerator is $6@2 = 6 \cdot 2 - 2^2 = 12 - 4 = 8$.

The denominator is $6\#2 = 6 + 2 - 6 \cdot 2^2 = 8 - 24 = -16$.

The quotient is

$$\frac{8}{-16} = -\frac{1}{2}.$$

Thus, the correct answer is **A**.

3. An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. It is filled with water to a height of 40 cm. A brick with a rectangular base that measures 40 cm by 20 cm and a height of 10 cm is placed in the aquarium. By how many centimeters does the water rise?

A 0.5

B 1

C 1.5

D 2

E 2.5

Solution:

The brick has volume $40 \cdot 20 \cdot 10 = 8000$ cubic centimeters.

If the water rises by h centimeters, the added volume is $100 \cdot 40 \cdot h = 4000h$ cubic centimeters.

Setting these equal gives $4000h = 8000$, so $h = 2$.

Thus, the correct answer is **D**.

4. The larger of two consecutive odd integers is three times the smaller. What is their sum?

- A 4
- B 8
- C 12
- D 16
- E 20

Solution:

Let the smaller integer be x. Then the larger is $x + 2$, and

$$x + 2 = 3x,$$

so $x = 1$.

The two integers are 1 and 3, and their sum is 4.

Thus, the correct answer is **A**.

5. A school store sells 7 pencils and 8 notebooks for $\$4.15$. It also sells 5 pencils and 3 notebooks for $\$1.77$. How much do 16 pencils and 10 notebooks cost?

- A $\$4.76$
- B $\$5.84$
- C $\$6.00$
- D $\$6.16$
- E $\$6.32$

Solution:

Let p and n be the prices in cents of a pencil and a notebook. Then

$$7p + 8n = 415$$

$$5p + 3n = 177.$$

Solving this system gives $p = 9$ and $n = 44$.

So 16 pencils and 10 notebooks cost $16(9) + 10(44) = 584$ cents, or $\$5.84$.

Thus, the correct answer is **B**.

6. At Euclid High School, the number of students taking the AMC 10 was 60 in $2002, 66$ in $2003, 70$ in $2004, 76$ in $2005, 78$ in $2006,$ and is 85 in $2007.$ Between what two consecutive years was there the largest percentage increase?

A 2002 and 2003

B 2003 and 2004

C 2004 and 2005

D 2005 and 2006

E 2006 and 2007

Solution:

From 2002 to $2003,$ the increase is

$$\frac{6}{60}=\frac{1}{10}=10\%.$$

The other increases are $\frac{4}{66}, \frac{6}{70}, \frac{2}{76},$ and $\frac{7}{78},$ each less than $\frac{1}{10}.$

So the largest percentage increase was between 2002 and $2003.$

Thus, the correct answer is **A**.

7. Last year Mr. John Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of $\$10{,}500$ for both taxes. How many dollars was the inheritance?

- A $30{,}000$
- B $32{,}500$
- C $35{,}000$
- D $37{,}500$
- E $40{,}000$

Solution:

After federal taxes, Mr. Public keeps 80% of his inheritance.

State taxes take 10% of that, which is 8% of the inheritance.

The total tax is $20\% + 8\% = 28\%$ of the inheritance, so the inheritance is

$$\frac{10{,}500}{0.28} = 37{,}500.$$

Thus, the correct answer is **D**.

8. Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside $\triangle ABC$, $\angle ABC = 40^\circ$, and $\angle ADC = 140^\circ$. What is the degree measure of $\angle BAD$?

A 20

B 30

C 40

D 50

E 60

Solution:

Since $\triangle ABC$ is isosceles, $\angle BAC = \frac{1}{2}(180^\circ - 40^\circ) = 70^\circ$.

Since $\triangle ADC$ is isosceles, $\angle DAC = \frac{1}{2}(180^\circ - 140^\circ) = 20^\circ$.

Therefore $\angle BAD = \angle BAC - \angle DAC = 70^\circ - 20^\circ = 50^\circ$.

Thus, the correct answer is **D**.

9. Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. What is ab?

A -60

B -17

C 9

D 12

E 60

Solution:

The equations become $3^a = 3^{4(b+2)}$ and $5^{3b} = 5^{a-3}$.

So $a = 4(b+2)$ and $3b = a - 3$.

Solving gives $a = -12$ and $b = -5$, so $ab = 60$.

Thus, the correct answer is **E**.

10. The Dunbar family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

A 2

B 3

C 4

D 5

E 6

Solution:

Let N be the number of children and T the total age of the family.

Then $20 = \dfrac{T}{N+2}$ and $16 = \dfrac{T-48}{N+1}$.

These give $20N + 40 = T$ and $16N + 64 = T$, so $20N + 40 = 16N + 64$.

Hence $4N = 24$ and $N = 6$.

Thus, the correct answer is **E**.

11. The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

A 14

B 16

C 18

D 20

E 24

Solution:

Each vertex belongs to exactly three faces, so summing the numbers over all six faces gives

$$3(1+2+\cdots+8)=3\cdot 36=108.$$

There are six faces, so the common sum is $108 \div 6 = 18$.

Thus, the correct answer is **C**.

12. Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

A 56

B 58

C 60

D 62

E 64

Solution:

Each tourist independently chooses one of the two guides, giving $2^6 = 64$ arrangements.

Exactly two of these leave a guide with no tourists, so the answer is $64 - 2 = 62.$

Thus, the correct answer is **D**.

13. Yan is somewhere between his home and the stadium. To get to the stadium he can walk directly to the stadium, or else he can walk home and then ride his bicycle to the stadium. He rides 7 times as fast as he walks, and both choices require the same amount of time. What is the ratio of Yan's distance from his home to his distance from the stadium?

A $\frac{2}{3}$

B $\frac{3}{4}$

C $\frac{4}{5}$

D $\frac{5}{6}$

E $\frac{6}{7}$

Solution:

Let x and y be Yan's distances from home and from the stadium, and let w be his walking speed.

Walking to the stadium takes $\frac{y}{w}$. Walking home and biking takes $\frac{x}{w} + \frac{x+y}{7w}$.

Setting these equal gives $7y = 8x + y$, so $8x = 6y$ and $\frac{x}{y} = \frac{3}{4}$.

Thus, the correct answer is **B**.

14. A triangle with side lengths in the ratio $3 : 4 : 5$ is inscribed in a circle of radius 3. What is the area of the triangle?

A 8.64

B 12

C 5π

D 17.28

E 18

Solution:

Let the sides be $3x, 4x, 5x$. The triangle is right-angled, so its hypotenuse is a diameter.

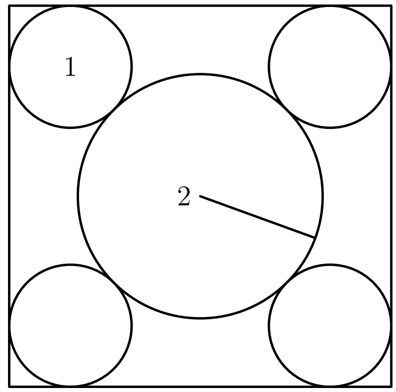
Thus $5x = 2 \cdot 3 = 6$, giving $x = \frac{6}{5}$.

The area is

$$\frac{1}{2}(3x)(4x) = 6x^2 = 6 \cdot \frac{36}{25} = 8.64.$$

Thus, the correct answer is **A**.

15. Four circles of radius 1 are each tangent to two sides of a square and externally tangent to a circle of radius $2,$ as shown. What is the area of the square?



A 32

B $22 + 12\sqrt{2}$

C $16 + 16\sqrt{3}$

D 48

E $36 + 16\sqrt{2}$

Solution:

Consider the isosceles right triangle joining the center of the radius-2 circle to the centers of two adjacent small circles. Its legs have length $2 + 1 = 3,$ so its hypotenuse is $3\sqrt{2}.$

The side of the square exceeds this hypotenuse by 2 (one radius on each end), so $s = 2 + 3\sqrt{2}.$

The area is

$$(2 + 3\sqrt{2})^2 = 4 + 12\sqrt{2} + 18 = 22 + 12\sqrt{2}.$$

Thus, the correct answer is **B**.

16. Integers $a, b, c,$ and $d,$ not necessarily distinct, are chosen independently and at random from 0 to $2007,$ inclusive. What is the probability that $ad - bc$ is even?

A $\frac{3}{8}$

B $\frac{7}{16}$

C $\frac{1}{2}$

D $\frac{9}{16}$

E $\frac{5}{8}$

Solution:

Half the integers from 0 to 2007 are odd, so each of ad and bc is odd with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and even with probability $\frac{3}{4}$.

The difference $ad - bc$ is even when both products have the same parity:

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{1}{16} + \frac{9}{16} = \frac{5}{8}.$$

Thus, the correct answer is **E**.

17. Suppose that m and n are positive integers such that $75m = n^3$. What is the minimum possible value of $m + n$?

A 15

B 30

C 50

D 60

E 5700

Solution:

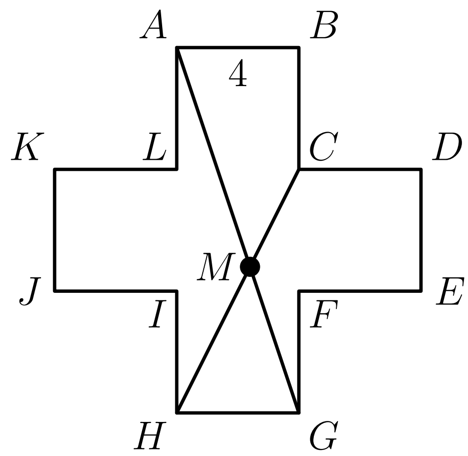
Since $n^3 = 75m = 3 \cdot 5^2 \cdot m$, every prime factor must occur a multiple of three times.

The smallest such m is $3^2 \cdot 5 = 45$, giving $n^3 = 3^3 \cdot 5^3$ and $n = 15$.

Then $m + n = 45 + 15 = 60$.

Thus, the correct answer is **D**.

18. Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 4, and each two consecutive sides form a right angle. Suppose that $\overline{AG}$ and $\overline{CH}$ meet at M. What is the area of quadrilateral $ABCM$?



A $\frac{44}{3}$

B 16

C $\frac{88}{5}$

D 20

E $\frac{62}{3}$

Solution:

Put the figure on coordinates with $A = (-2, 6), B = (2, 6), C = (2, 2), G = (2, -6)$, and $H = (-2, -6)$.

Line AG is $y = -3x$, and line CH is $y = 2x - 2$.

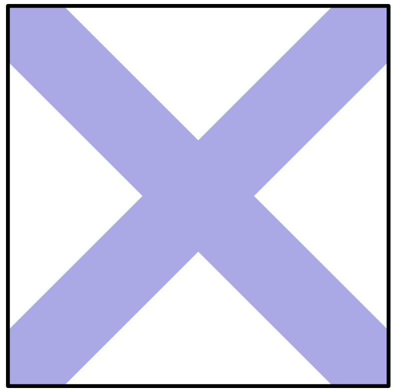
Their intersection is $M = \left(\frac{2}{5}, -\frac{6}{5}\right)$.

Applying the shoelace formula to A, B, C, M gives area

$$\frac{|-35.2|}{2} = \frac{88}{5}.$$

Thus, the correct answer is **C**.

19. A paint brush is swept along both diagonals of a square to produce the symmetric painted area, as shown. Half the area of the square is painted. What is the ratio of the side length of the square to the brush width?



A $2\sqrt{2}+1$

B $3\sqrt{2}$

C $2\sqrt{2}+2$

D $3\sqrt{2}+1$

E $3\sqrt{2}+2$

Solution:

Let s be the side, w the brush width, and x the leg of one unpainted isosceles right triangle. Each triangle has area $\frac{1}{8}s^2$, so $\frac{1}{2}x^2 = \frac{1}{8}s^2$ and $x = \frac{s}{2}$.

The leg plus the brush width is half the diagonal: $x + w = \frac{\sqrt{2}}{2}s$. Thus $w = \frac{\sqrt{2}}{2}s - \frac{s}{2}$.

Therefore

$$\frac{s}{w} = \frac{2}{\sqrt{2}-1} = 2\sqrt{2}+2.$$

Thus, the correct answer is **C**.

20. Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

A 164

B 172

C 192

D 194

E 212

Solution:

Squaring $a + a^{-1} = 4$ gives $a^2 + 2 + a^{-2} = 16$, so $a^2 + a^{-2} = 14$.

Squaring again gives $a^4 + 2 + a^{-4} = 196$, so $a^4 + a^{-4} = 194$.

Thus, the correct answer is **D**.

21. A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

A 3

B 6

C 8

D 9

E 12

Solution:

Each face of the outer cube has area $24 \div 6 = 4,$ so its side is $2,$ and the sphere has diameter $2.$

This diameter is the space diagonal of the inner cube, so $l\sqrt{3} = 2,$ giving $l^2 = \frac{4}{3}.$

The inner cube's surface area is $6l^2 = 6 \cdot \frac{4}{3} = 8.$

Thus, the correct answer is **C**.

22. A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms $247, 475,$ and 756 and end with the term $824.$ Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

A 3

B 7

C 13

D 37

E 43

Solution:

Each digit appears as a hundreds digit, a tens digit, and a units digit the same number of times across the sequence.

If k is the sum of the units digits of all terms, then $S = 111k = 3 \cdot 37 \cdot k,$ so S is always divisible by $37.$

The sequence $123, 231, 312$ gives $S = 666 = 2 \cdot 3^2 \cdot 37,$ which has no larger prime factor forced, so 37 is the answer.

Thus, the correct answer is **D**.

23. How many ordered pairs (m, n) of positive integers, with $m > n$, have the property that their squares differ by 96?

A 3

B 4

C 6

D 9

E 12

Solution:

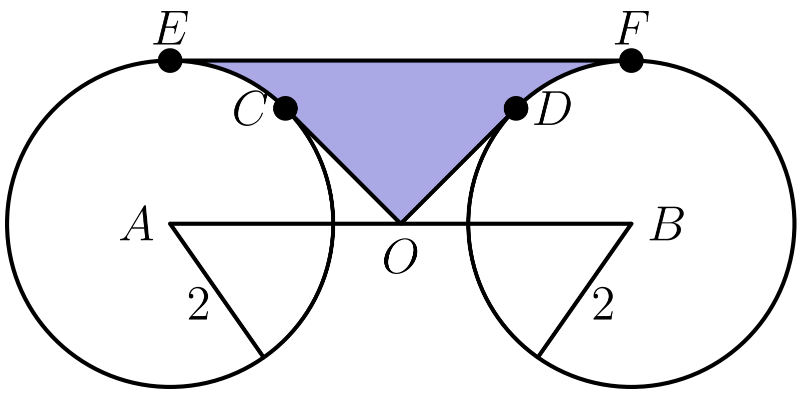
Since $(m+n)(m-n) = 96$ and 96 is even, both factors must be even.

The even factor pairs are $(48, 2)$, $(24, 4)$, $(16, 6)$, and $(12, 8)$, giving $(m, n) = (25, 23)$, $(14, 10)$, $(11, 5)$, and $(10, 2)$.

So there are 4 ordered pairs.

Thus, the correct answer is **B**.

24. Circles centered at A and B each have radius 2, as shown. Point O is the midpoint of $\overline{AB}$, and $OA = 2\sqrt{2}$. Segments OC and OD are tangent to the circles centered at A and B, respectively, and $\overline{EF}$ is a common tangent. What is the area of the shaded region $ECODF$?



A $\dfrac{8\sqrt{2}}{3}$

B $8\sqrt{2} - 4 - \pi$

C $4\sqrt{2}$

D $4\sqrt{2} + \dfrac{\pi}{8}$

E $8\sqrt{2} - 2 - \dfrac{\pi}{2}$

Solution:

Rectangle $ABFE$ has area $AE \cdot AB = 2 \cdot 4\sqrt{2} = 8\sqrt{2}.$

Right triangles ACO and BDO each have hypotenuse $2\sqrt{2}$ and a leg of 2, so each is isosceles right with area $2.$

Angles CAE and DBF are each 45°, so sectors CAE and DBF each have area $\frac{1}{8}\pi \cdot 2^2 = \frac{\pi}{2}.$

The shaded area is

$$8\sqrt{2} - 2 \cdot 2 - 2 \cdot \tfrac{\pi}{2} = 8\sqrt{2} - 4 - \pi.$$

Thus, the correct answer is **B**.

25. For each positive integer n, let $S(n)$ denote the sum of the digits of n. For how many values of n is $n + S(n) + S(S(n)) = 2007$?

A 1

B 2

C 3

D 4

E 5

Solution:

If $n \leq 2007$, then $S(n) \leq 28$ and $S(S(n)) \leq 10$, so $n \geq 2007 - 28 - 10 = 1969$.

Since n, $S(n)$, and $S(S(n))$ all leave the same remainder modulo 9 and 2007 is a multiple of 9, each must be a multiple of 3.

Checking the multiples of 3 between 1969 and 2007, the condition holds for 1977, 1980, 1983, and 2001.

So there are 4 values of n.

Thus, the correct answer is **D**.

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