2006 AMC 10B Solutions

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https://live.poshenloh.com/past-contests/amc10/2006B/solutions

1. What is $(-1)^1 + (-1)^2 + \cdots + (-1)^{2006}$?

- A -2006
- B -1
- C 0
- D 1
- E 2006

Solution:

There are 2006 terms. Pairing consecutive terms gives $(-1+1) + (-1+1) + \cdots$. Since 2006 is even, every term pairs off and the sum is 0.

Thus, the correct answer is **C**.

2. For real numbers x and y, define $x \spadesuit y = (x+y)(x-y)$. What is $3 \spadesuit (4 \spadesuit 5)$?

A -72

B -27

C -24

D 24

E 72

Solution:

Since $x \spadesuit y = (x+y)(x-y) = x^2 - y^2$, we have $4 \spadesuit 5 = 16 - 25 = -9.$

Then $3 \spadesuit (-9) = 9 - 81 = -72.$

Thus, the correct answer is **A**.

3. A football game was played between two teams, the Cougars and the Panthers. The two teams scored a total of 34 points, and the Cougars won by a margin of 14 points. How many points did the Panthers score?

A 10

B 14

C 17

D 20

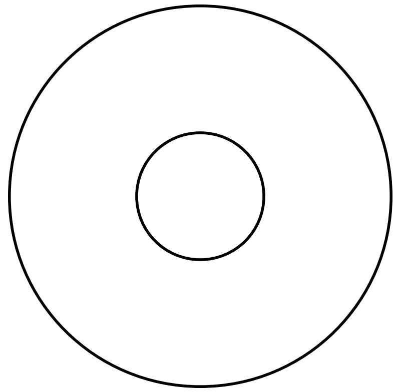
E 24

Solution:

Let c and p be the Cougars' and Panthers' scores. Then $c + p = 34$ and $c - p = 14.$ Subtracting gives $2p = 20,$ so $p = 10.$

Thus, the correct answer is **A**.

4. Circles of diameter 1 inch and 3 inches have the same center. The smaller circle is painted red, and the portion outside the smaller circle and inside the larger circle is painted blue. What is the ratio of the blue-painted area to the red-painted area?



A 2

B 3

C 6

D 8

E 9

Solution:

The red circle has area $\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$, and the large circle has area $\pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$. The blue ring is $\frac{9\pi}{4} - \frac{\pi}{4} = 2\pi$.

The ratio is $2\pi \div \frac{\pi}{4} = 8$.

Thus, the correct answer is **D**.

5. A 2×3 rectangle and a 3×4 rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square?

A 16

B 25

C 36

D 49

E 64

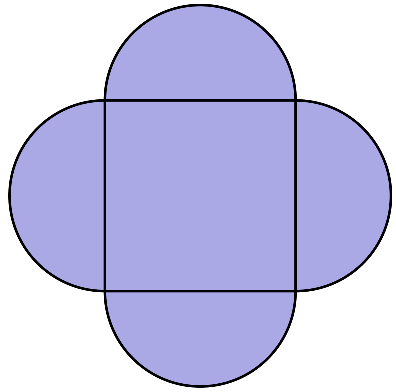
Solution:

Place the rectangles side by side with their 3-length sides vertical. Their widths add to $2 + 3 = 5$, and the heights 3 and 4 both fit within 5.

The side cannot be smaller than 5, since the two smaller dimensions 2 and 3 must be accommodated. The smallest area is $5^2 = 25$.

Thus, the correct answer is **B**.

6. A region is bounded by semicircular arcs constructed on the sides of a square whose sides measure $\frac{2}{\pi}$, as shown. What is the perimeter of this region?



A $\frac{4}{\pi}$

B 2

C $\frac{8}{\pi}$

D 4

E $\frac{16}{\pi}$

Solution:

Each side has length $\frac{2}{\pi}$, the diameter of a semicircular arc, so each arc has length $\frac{1}{2}\pi \cdot \frac{2}{\pi} = 1$.

The boundary consists of four such arcs, so the perimeter is $4 \cdot 1 = 4$.

Thus, the correct answer is **D**.

7. Which of the following is equivalent to

$$\sqrt{\frac{x}{1-\frac{x-1}{x}}}$$

when $x < 0$?

A $-x$

B x

C 1

D $\sqrt{\frac{x}{2}}$

E $x\sqrt{-1}$

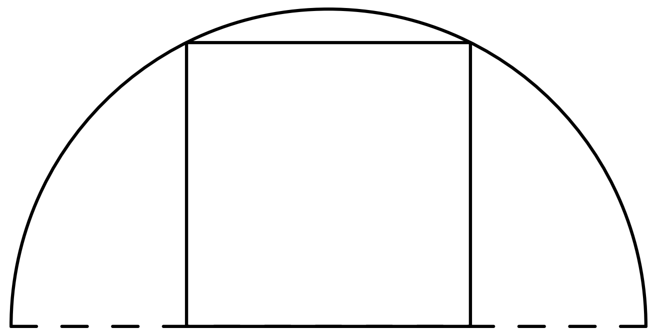
Solution:

The denominator simplifies: $1 - \frac{x-1}{x} = \frac{x-(x-1)}{x} = \frac{1}{x}$.

So the expression is $\sqrt{\frac{x}{1/x}} = \sqrt{x^2} = |x|$. Since $x < 0$, this equals $-x$.

Thus, the correct answer is **A**.

8. A square of area 40 is inscribed in a semicircle as shown. What is the area of the semicircle?



A 20π

B 25π

C 30π

D 40π

E 50π

Solution:

Let the square have side s, so $s^2 = 40$. Its base lies centered on the diameter, and a top corner at $\left(\frac{s}{2}, s\right)$ lies on the circle.

Then $r^2 = \left(\frac{s}{2}\right)^2 + s^2 = \frac{40}{4} + 40 = 50$. The semicircle area is $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(50) = 25\pi$.

Thus, the correct answer is **B**.

9. Francesca uses 100 grams of lemon juice, 100 grams of sugar, and 400 grams of water to make lemonade. There are 25 calories in 100 grams of lemon juice and 386 calories in 100 grams of sugar. Water contains no calories. How many calories are in 200 grams of her lemonade?

A 129

B 137

C 174

D 223

E 411

Solution:

The lemonade totals $100 + 100 + 400 = 600$ grams containing $25 + 386 = 411$ calories.

In 200 grams there are $411 \cdot \frac{200}{600} = \frac{411}{3} = 137$ calories.

Thus, the correct answer is **B**.

10. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

A 43

B 44

C 45

D 46

E 47

Solution:

Let the sides be $x, 3x,$ and 15. The triangle inequality $x + 15 > 3x$ gives $x < 7.5$.

The largest integer is $x = 7$, giving sides $7, 21, 15$ and perimeter $7 + 21 + 15 = 43$.

Thus, the correct answer is **A**.

11. What is the tens digit in the sum $7! + 8! + 9! + \cdots + 2006!$?

A 1

B 3

C 4

D 6

E 9

Solution:

For $n \geq 10, n!$ is divisible by $100,$ so it does not affect the last two digits.

The tens digit comes from $7! + 8! + 9! = 5040 + 40320 + 362880 = 408240,$ whose tens digit is $4.$

Thus, the correct answer is **C**.

12. The lines $x = \frac{1}{4}y + a$ and $y = \frac{1}{4}x + b$ intersect at the point $(1, 2)$. What is $a + b$?

A 0

B $\frac{3}{4}$

C 1

D 2

E $\frac{9}{4}$

Solution:

Substituting $(1, 2)$: from $1 = \frac{1}{4}(2) + a$ we get $a = \frac{1}{2}$, and from $2 = \frac{1}{4}(1) + b$ we get $b = \frac{7}{4}$.

Then $a + b = \frac{1}{2} + \frac{7}{4} = \frac{9}{4}$.

Thus, the correct answer is **E**.

13. Joe and JoAnn each bought 12 ounces of coffee in a 16-ounce cup. Joe drank 2 ounces of his coffee and then added 2 ounces of cream. JoAnn added 2 ounces of cream, stirred the coffee well, and then drank 2 ounces. What is the resulting ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee?

A $\frac{6}{7}$

B $\frac{13}{14}$

C 1

D $\frac{14}{13}$

E $\frac{7}{6}$

Solution:

Joe adds 2 ounces of cream and drinks nothing afterward, so he has 2 ounces of cream.

JoAnn has 12 ounces of coffee plus 2 ounces of cream, making 14 ounces of uniform mixture. After drinking 2 ounces she keeps $\frac{12}{14} = \frac{6}{7}$ of her cream, which is $\frac{6}{7} \cdot 2 = \frac{12}{7}$ ounces.

The ratio is $2 \div \frac{12}{7} = \frac{7}{6}$.

Thus, the correct answer is **E**.

14. Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?

A $\frac{5}{2}$

B $\frac{7}{2}$

C 4

D $\frac{9}{2}$

E 8

Solution:

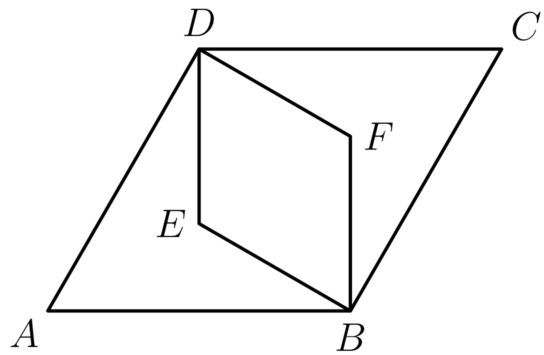
Since a and b are roots of $x^2 - mx + 2$, we have $ab = 2$.

The value q is the product of the new roots:

$$q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 1 + 1 + \frac{1}{ab} = 2 + 2 + \frac{1}{2} = \frac{9}{2}.$$

Thus, the correct answer is **D**.

15. Rhombus $ABCD$ is similar to rhombus $BFDE$. The area of rhombus $ABCD$ is $24,$ and $\angle BAD = 60^\circ$. What is the area of rhombus $BFDE$?



A 6

B $4\sqrt{3}$

C 8

D 9

E $6\sqrt{3}$

Solution:

Because $AB = AD$ and $\angle BAD = 60^\circ$, triangle ABD is equilateral, and so is triangle CBD.

Points E and F split the rhombus into six congruent triangles, each of area $\frac{24}{6} = 4.$

Rhombus $BFDE$ is the union of triangles BED and BFD, so its area is $2 \cdot 4 = 8.$

Thus, the correct answer is **C**.

16. Leap Day, February 29, 2004, occurred on a Sunday. On what day of the week will Leap Day, February 29, 2020, occur?

A Tuesday

B Wednesday

C Thursday

D Friday

E Saturday

Solution:

From one Leap Day to the next is $3 \cdot 365 + 366 = 1461$ days, and $1461 \equiv 5 \pmod{7}$.

Over the four cycles from 2004 to 2020, the weekday advances $4 \cdot 5 = 20 \equiv 6 \pmod{7}$, that is, 6 days forward, which is one day back from Sunday.

So Leap Day 2020 falls on a Saturday.

Thus, the correct answer is **E**.

17. Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?

A $\frac{1}{10}$

B $\frac{1}{6}$

C $\frac{1}{5}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

Alice moves one ball to Bob, so Bob's bag holds 6 balls with exactly one color appearing twice.

The two bags end up identical exactly when Bob returns one of that duplicated pair. Two of the six balls qualify, so the probability is $\frac{2}{6} = \frac{1}{3}$.

Thus, the correct answer is **D**.

18. Let $a_1, a_2, \ldots$ be a sequence for which $a_1 = 2, a_2 = 3,$ and $a_n = \dfrac{a_{n-1}}{a_{n-2}}$ for each positive integer $n \geq 3.$ What is a_{2006}?

A $\dfrac{1}{2}$

B $\dfrac{2}{3}$

C $\dfrac{3}{2}$

D 2

E 3

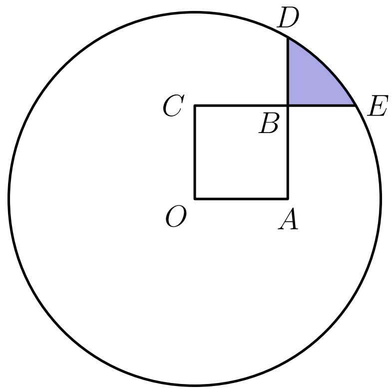
Solution:

The terms are $2,\ 3,\ \frac{3}{2},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3}$, then $2,\ 3, \ldots$, a cycle of length $6.$

Since $2006 = 6 \cdot 334 + 2$, we have $a_{2006} = a_2 = 3.$

Thus, the correct answer is **E**.

19. A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides $\overline{AB}$ and $\overline{CB}$ are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by $\overline{BD}$, $\overline{BE}$, and the minor arc connecting D and E?



A $\dfrac{\pi}{3}+1-\sqrt{3}$

B $\dfrac{\pi}{2}(2-\sqrt{3})$

C $\pi(2-\sqrt{3})$

D $\dfrac{\pi}{6}+\dfrac{\sqrt{3}-1}{2}$

E $\dfrac{\pi}{3}-1+\sqrt{3}$

Solution:

Since $OA=1$ and $OD=2$ with D on the line $x=1$, we get $\angle AOD=60^\circ$, and likewise $\angle COE=60^\circ$, so $\angle DOE=30^\circ$.

The sector DOE has area $\frac{30}{360}\pi(2^2)=\frac{\pi}{3}$.

The region is this sector minus triangles OBD and OBE. With $BD=BE=\sqrt{3}-1$, each triangle has area $\frac{1}{2}(\sqrt{3}-1)(1)$, totaling $\sqrt{3}-1$.

So the shaded area is $\frac{\pi}{3}-(\sqrt{3}-1)=\frac{\pi}{3}+1-\sqrt{3}$.

Thus, the correct answer is **A**.

20. In rectangle $ABCD$, we have $A=(6,-22)$, $B=(2006,178)$, and $D=(8,y)$ for some integer y. What is the area of rectangle $ABCD$?

- A 4000
- B 4040
- C 4400
- D $40{,}000$
- E $40{,}400$

Solution:

The slope of $\overline{AB}$ is $\frac{178-(-22)}{2006-6}=\frac{200}{2000}=\frac{1}{10}$. Since $\overline{AD}\perp\overline{AB}$, its slope is -10, so $\frac{y+22}{8-6}=-10$ gives $y=-42$.

Then $AB=\sqrt{2000^2+200^2}=200\sqrt{101}$ and $AD=\sqrt{2^2+20^2}=2\sqrt{101}$.

The area is $200\sqrt{101}\cdot 2\sqrt{101}=400\cdot 101=40{,}400$.

Thus, the correct answer is **E**.

21. For a particular peculiar pair of dice, the probabilities of rolling $1, 2, 3, 4, 5,$ and 6 on each die are in the ratio $1 : 2 : 3 : 4 : 5 : 6$. What is the probability of rolling a total of 7 on the two dice?

A $\frac{4}{63}$

B $\frac{1}{8}$

C $\frac{8}{63}$

D $\frac{1}{6}$

E $\frac{2}{7}$

Solution:

Each die shows k with probability $\frac{k}{1+2+\cdots+6} = \frac{k}{21}$.

For a total of 7, the ordered pairs $(1,6), (2,5), \ldots, (6,1)$ contribute

$$\frac{1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1}{21^2} = \frac{56}{441} = \frac{8}{63}.$$

Thus, the correct answer is **C**.

22. Elmo makes N sandwiches for a fundraiser. For each sandwich he uses B globs of peanut butter at $4¢$ per glob and J blobs of jam at $5¢$ per blob. The cost of the peanut butter and jam to make all the sandwiches is $\$2.53$. Assume that B, J, and N are positive integers with $N > 1$. What is the cost of the jam Elmo uses to make the sandwiches?

A $\$1.05$

B $\$1.25$

C $\$1.45$

D $\$1.65$

E $\$1.85$

Solution:

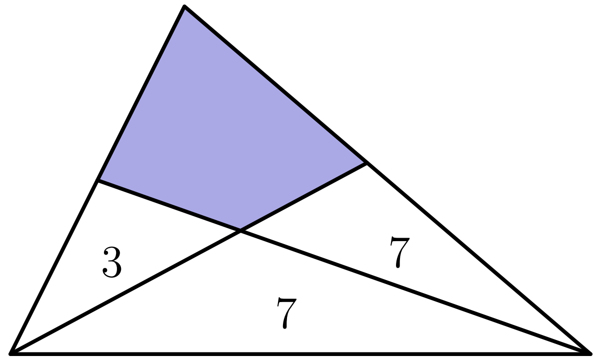
The total cost is $N(4B + 5J) = 253$ cents $= 11 \cdot 23$. Since $N > 1, N \in \{11, 23, 253\}$.

If $N = 253$ or $N = 23$, then $4B + 5J$ equals 1 or 11, impossible for positive integers.

So $N = 11$ and $4B + 5J = 23$, whose only positive solution is $B = 2, J = 3$. The jam costs $11 \cdot 3 \cdot 5 = 165$ cents $= \$1.65$.

Thus, the correct answer is **D**.

23. A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are $3, 7,$ and $7,$ as shown. What is the area of the shaded quadrilateral?



A 15

B 17

C $\frac{35}{2}$

D 18

E $\frac{55}{3}$

Solution:

Split the quadrilateral into two triangles of areas R and S, so the shaded area is $T = R + S$.

Comparing triangles that share an altitude, base ratios give $\frac{R}{3} = \frac{T+7}{10}$ and $\frac{S}{7} = \frac{T+3}{14}$.

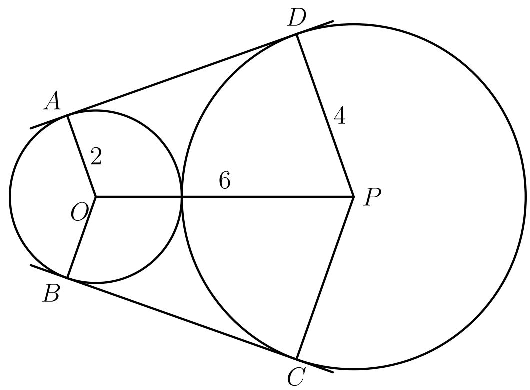
Then

$$T = R + S = 3 \cdot \tfrac{T+7}{10} + 7 \cdot \tfrac{T+3}{14},$$

so $10T = 3(T+7) + 5(T+3) = 8T + 36$, giving $T = 18$.

Thus, the correct answer is **D**.

24. Circles with centers at O and P have radii 2 and 4, respectively, and are externally tangent. Points A and B on the circle with center O and points C and D on the circle with center P are such that $\overline{AD}$ and $\overline{BC}$ are common external tangents to the circles. What is the area of the concave hexagon $AOBCPD$?



A $18\sqrt{3}$

B $24\sqrt{2}$

C 36

D $24\sqrt{3}$

E $32\sqrt{2}$

Solution:

The hexagon is symmetric about $\overline{OP}$, so its area is twice that of trapezoid $AOPD$.

Draw $OF \parallel AD$ with F on $\overline{PD}$. Then $AOFD$ is a rectangle, so $DF = OA = 2$ and $FP = PD - DF = 4 - 2 = 2.$

Since the circles are externally tangent, $OP = 2 + 4 = 6,$ so in right triangle OFP, $OF = \sqrt{36 - 4} = 4\sqrt{2}.$

Trapezoid $AOPD$ has parallel sides $OA = 2$ and $PD = 4$ with height $OF = 4\sqrt{2},$ giving area $\frac{1}{2}(2 + 4)(4\sqrt{2}) = 12\sqrt{2}.$ The hexagon area is $2 \cdot 12\sqrt{2} = 24\sqrt{2}.$

Thus, the correct answer is **B**.

25. Mr. Jones has eight children of different ages. On a family trip his oldest child, who is 9, spots a license plate with a 4-digit number in which each of two digits appears two times. "Look, daddy!" she exclaims. "That number is evenly divisible by the age of each of us kids!" "That's right," replies Mr. Jones, "and the last two digits just happen to be my age." Which of the following is *not* the age of one of Mr. Jones's children?

A 4

B 5

C 6

D 7

E 8

Solution:

Since a child is 9, the number is divisible by 9, so its digit sum $2(a+b)$ is a multiple of 9, which forces $a+b=9$.

There is also a 4- or 8-year-old, so the number is divisible by 4. Among numbers with two repeated digits summing to 9 and divisible by 4, the number 5544 is divisible by $1, 2, 3, 4, 6, 7, 8,$ and 9, but not by 5.

So the eight ages can be $\{1, 2, 3, 4, 6, 7, 8, 9\}$, and 5 need not be among them. The age that is not necessarily a child's age is 5.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc10/2006B