2006 AMC 10A Solutions

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1. Sandwiches at Joe's Fast Food cost $\$3$ each and sodas cost $\$2$ each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?

A 31

B 32

C 33

D 34

E 35

Solution:

Five sandwiches cost $5 \cdot 3 = 15$ dollars and eight sodas cost $8 \cdot 2 = 16$ dollars. Together they cost $15 + 16 = 31$ dollars.

Thus, the correct answer is **A**.

2. Define $x \otimes y = x^3 - y$. What is $h \otimes (h \otimes h)$?

- A $-h$
- B 0
- C h
- D $2h$
- E h^3

Solution:

The inner operation gives $h \otimes h = h^3 - h$. Then

$$h \otimes (h^3 - h) = h^3 - (h^3 - h) = h.$$

Thus, the correct answer is **C**.

3. The ratio of Mary's age to Alice's age is $3 : 5$. Alice is 30 years old. How many years old is Mary?

- A 15
- B 18
- C 20
- D 24
- E 50

Solution:

Since the ratio is $3 : 5$ and Alice is 30, Mary is $\frac{3}{5} \cdot 30 = 18$ years old.

Thus, the correct answer is **B**.

4. A digital watch displays hours and minutes with am and pm. What is the largest possible sum of the digits in the display?

A 17

B 19

C 21

D 22

E 23

Solution:

The minutes run from 00 to 59, so the largest digit sum for the minutes is $5+9=14$, at 59 minutes.

For the hour, the single digit 9 beats $1+2=3$ from 12. The largest total is $9+14=23$, occurring at $9:59$.

Thus, the correct answer is **E**.

5. Doug and Dave shared a pizza with 8 equally-sized slices. Doug wanted a plain pizza, but Dave wanted anchovies on half of the pizza. The cost of a plain pizza was $\$8$, and there was an additional cost of $\$2$ for putting anchovies on one half. Dave ate all the slices of anchovy pizza and one plain slice. Doug ate the remainder. Each then paid for what he had eaten. How many more dollars did Dave pay than Doug?

A 1

B 2

C 3

D 4

E 5

Solution:

Each of the 8 slices costs $\$1$. Dave ate 5 slices and also pays the extra $\$2$ for the anchovies, for a total of $5 + 2 = 7$ dollars.

Doug ate 3 slices, paying 3 dollars. So Dave paid $7 - 3 = 4$ dollars more.

Thus, the correct answer is **D**.

6. What non-zero real value for x satisfies $(7x)^{14} = (14x)^7$?

A $\frac{1}{7}$

B $\frac{2}{7}$

C 1

D 7

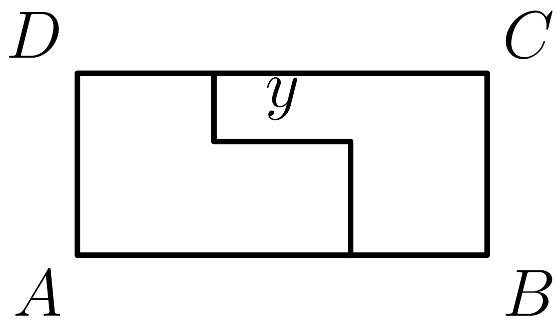
E 14

Solution:

Taking the seventh root of both sides gives $(7x)^2 = 14x$, so $49x^2 = 14x$. Since $x \neq 0$, divide by x to get $49x = 14$, hence $x = \frac{2}{7}$.

Thus, the correct answer is **B**.

7. The 8×18 rectangle $ABCD$ is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be repositioned without overlap to form a square. What is y?



A 6

B 7

C 8

D 9

E 10

Solution:

The rectangle's area is $8 \cdot 18 = 144$, so the square formed has side $\sqrt{144} = 12.$

Along the top edge the three equal horizontal pieces satisfy $DE = y = FB$ with $DE + y + FB = 18$. Hence $3y = 18$, so $y = 6$.

Thus, the correct answer is **A**.

8. A parabola with equation $y = x^2 + bx + c$ passes through the points $(2, 3)$ and $(4, 3)$. What is c?

A 2

B 5

C 7

D 10

E 11

Solution:

Substituting the points gives $3 = 4 + 2b + c$ and $3 = 16 + 4b + c$. Subtracting yields $0 = 12 + 2b$, so $b = -6$.

Then $c = 3 - 4 - 2(-6) = 11.$

Thus, the correct answer is **E**.

9. How many sets of two or more consecutive positive integers have a sum of 15?

A 1

B 2

C 3

D 4

E 5

Solution:

The sum of n consecutive integers equals n times their median. For a sum of 15: $n = 2$ gives $7 + 8$, $n = 3$ gives $4 + 5 + 6$, and $n = 5$ gives $1 + 2 + 3 + 4 + 5$.

No set of 4 works (their sum is even), and 6 or more consecutive positive integers already exceed 15. There are 3 such sets.

Thus, the correct answer is **C**.

10. For how many real values of x is $\sqrt{120-\sqrt{x}}$ an integer?

A 3

B 6

C 9

D 10

E 11

Solution:

Let $k=\sqrt{120-\sqrt{x}}$. Since $\sqrt{x}\geq 0$, we need $0\leq k\leq\sqrt{120}$, so $k\in\{0,1,\ldots,10\}$, giving 11 values.

Each k yields $\sqrt{x}=120-k^2$, and since $120-k^2$ is positive and strictly decreasing, the resulting values $x=(120-k^2)^2$ are distinct.

Thus, the correct answer is **E**.

11. Which of the following describes the graph of the equation $(x+y)^2 = x^2 + y^2$?

- A the empty set
- B one point
- C two lines
- D a circle
- E the entire plane

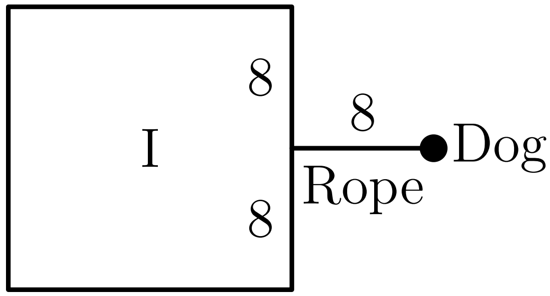
Solution:

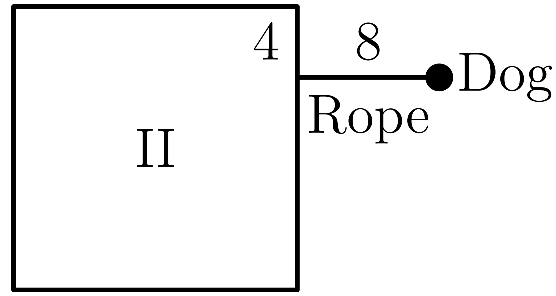
Expanding, $x^2 + 2xy + y^2 = x^2 + y^2$, which reduces to $2xy = 0$, i.e. $xy = 0$.

This holds exactly when $x = 0$ or $y = 0$, the two coordinate axes, so the graph is two lines.

Thus, the correct answer is **C**.

12. Rolly wishes to secure his dog with an 8-foot rope to a square shed that is 16 feet on each side. His preliminary drawings are shown.





Which of these arrangements gives the dog the greater area to roam, and by how many square feet?

A I, by 8π

B I, by 6π

C II, by 4π

D II, by 8π

E II, by 10π

Solution:

In arrangement I the dog is tied at the middle of a side and sweeps a half-disk of radius 8: area $\frac{1}{2}\pi \cdot 8^2 = 32\pi$. The rope reaches exactly to the corners, so nothing wraps.

In arrangement II the dog is tied 4 feet from a corner. It sweeps the same 32π half-disk, and after the rope reaches the corner, 4 feet remain to sweep a quarter-disk of radius 4: $\frac{1}{4}\pi \cdot 4^2 = 4\pi$.

So II gives 36π, exceeding I by 4π.

Thus, the correct answer is **C**.

13. A player pays $\$5$ to play a game. A die is rolled. If the number on the die is odd, the game is lost. If the number on the die is even, the die is rolled again. In this case the player wins if the second number matches the first and loses otherwise. How much should the player win if the game is fair? (In a fair game the probability of winning times the amount won is what the player should pay.)

A $\$12$

B $\$30$

C $\$50$

D $\$60$

E $\$100$

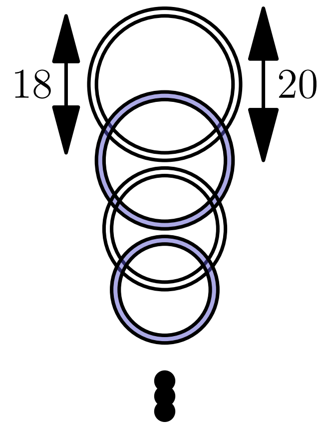
Solution:

The player wins only if the first roll is even (probability $\frac{1}{2}$) and the second roll matches it (probability $\frac{1}{6}$), so $P(\text{win}) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$.

For a fair game, $\frac{1}{12}x = 5$, so $x = 60$.

Thus, the correct answer is **D**.

14. A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the other rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?



A 171

B 173

C 182

D 188

E 210

Solution:

The top ring contributes its full outside diameter, 20 cm. Because the rings are 1 cm thick, each ring hangs 2 cm below the top of the ring above it, so each lower ring adds its outside diameter minus 2.

The outside diameters run $20, 19, \ldots, 3$, so the added distances are $17, 16, \ldots, 1$. The total is

$$20 + (17 + 16 + \cdots + 1) = 20 + \frac{17 \cdot 18}{2} = 20 + 153 = 173.$$

Thus, the correct answer is **B**.

15. Odell and Kershaw run for 30 minutes on a circular track. Odell runs clockwise at 250 m/min and uses the inner lane with a radius of 50 meters. Kershaw runs counterclockwise at 300 m/min and uses the outer lane with a radius of 60 meters, starting on the same radial line as Odell. How many times after the start do they pass each other?

A 29

B 42

C 45

D 47

E 50

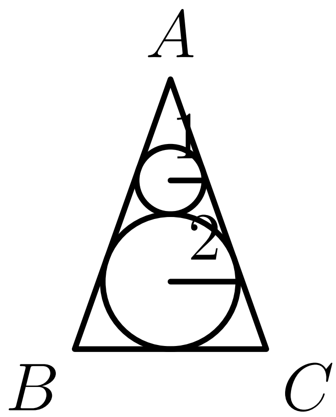
Solution:

Odell's lap is $2\pi(50) = 100\pi$ m at 250 m/min, taking $\frac{100\pi}{250} = 0.4\pi$ min. Kershaw's lap is $2\pi(60) = 120\pi$ m at 300 m/min, also $\frac{120\pi}{300} = 0.4\pi$ min.

Their periods are equal. Running in opposite directions, they meet at times $t = \frac{k}{2}(0.4\pi)$ for $k = 1, 2, \ldots$ Requiring $t \leq 30$ gives $k \leq \frac{60}{0.4\pi} = \frac{150}{\pi} \approx 47.7$, so they pass 47 times.

Thus, the correct answer is **D**.

16. A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides AB and AC are congruent. What is the area of $\triangle ABC$?



A $\frac{35}{2}$

B $15\sqrt{2}$

C $\frac{64}{3}$

D $16\sqrt{2}$

E 24

Solution:

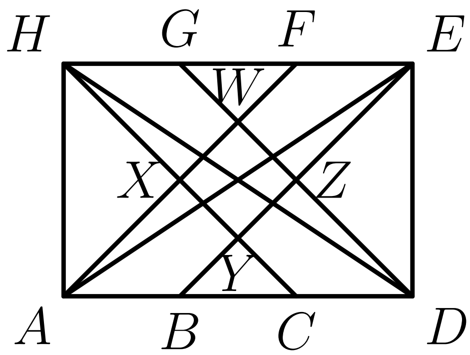
Let O, O' be the centers of the small and large circles, and let D be the point where the small circle touches AC. The right triangles cut off along AC are similar, so $\frac{AO}{1} = \frac{AO+3}{2}$, giving $AO = 3$ and $AO' = 6$.

The tangent length is $AD = \sqrt{AO^2 - 1^2} = \sqrt{3^2 - 1^2} = 2\sqrt{2}$. Let F be the midpoint of BC; then $AF = AO' + 2 = 8$.

Since $\triangle ADO \sim \triangle AFC$, we get $\frac{FC}{1} = \frac{AF}{2\sqrt{2}} = \frac{8}{2\sqrt{2}} = 2\sqrt{2}$. Thus $BC = 4\sqrt{2}$, and the area is $\frac{1}{2} \cdot BC \cdot AF = \frac{1}{2} \cdot 4\sqrt{2} \cdot 8 = 16\sqrt{2}$.

Thus, the correct answer is **D**.

17. In rectangle $ADEH$, points B and C trisect $\overline{AD}$, and points G and F trisect $\overline{HE}$. In addition, $AH = AC = 2$. What is the area of quadrilateral $WXYZ$ shown in the figure?



A $\frac{1}{2}$

B $\frac{\sqrt{2}}{2}$

C $\frac{\sqrt{3}}{2}$

D $\frac{2\sqrt{2}}{3}$

E $\frac{2\sqrt{3}}{3}$

Solution:

Set $A = (0,0), D = (3,0), H = (0,2)$, so $B = (1,0), C = (2,0), G = (1,2)$, $F = (2,2)$, and $E = (3,2)$.

The drawn segments meet at $W = (1.5, 1.5), X = (1,1), Y = (1.5, 0.5)$, and $Z = (2,1)$. These form a square whose perpendicular diagonals WY and XZ each have length 1.

Its area is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$.

Thus, the correct answer is **A**.

18. A license plate in a certain state consists of 4 digits, not necessarily distinct, and 2 letters, also not necessarily distinct. These six characters may appear in any order, except that the two letters must appear next to each other. How many distinct license plates are possible?

A $10^4 \cdot 26^2$

B $10^3 \cdot 26^3$

C $5 \cdot 10^4 \cdot 26^2$

D $10^2 \cdot 26^4$

E $5 \cdot 10^3 \cdot 26^3$

Solution:

Since the two letters must be adjacent, treat them as one block. A plate is then 4 digits plus this block—5 objects—and the block can occupy 5 positions.

There are 10^4 choices for the digits and 26^2 for the two letters, so the total is $5 \cdot 10^4 \cdot 26^2$.

Thus, the correct answer is **C**.

19. How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?

A 0

B 1

C 59

D 89

E 178

Solution:

Let the angles be $n-d, n, n+d$. Their sum is $3n = 180$, so $n = 60$.

The measures are distinct positive integers, so $d \geq 1$, and $n - d > 0$ forces $d < 60$. Thus $d \in \{1, 2, \ldots, 59\}$, giving 59 non-similar triangles.

Thus, the correct answer is **C**.

20. Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

A $\frac{1}{2}$

B $\frac{3}{5}$

C $\frac{2}{3}$

D $\frac{4}{5}$

E 1

Solution:

Group the integers by their remainder modulo 5. There are only 5 possible remainders but 6 integers, so by the Pigeonhole Principle two share a remainder.

Their difference is then a multiple of 5. This always happens, so the probability is 1.

Thus, the correct answer is **E**.

21. How many four-digit positive integers have at least one digit that is a 2 or a 3?

A 2439

B 4096

C 4903

D 4904

E 5416

Solution:

There are 9000 four-digit integers. For those avoiding 2 and $3,$ the leading digit is one of $\{1,4,5,6,7,8,9\}$ (7 choices) and each remaining digit is one of $\{0,1,4,5,6,7,8,9\}$ (8 choices): $7 \cdot 8^3 = 3584.$

So $9000 - 3584 = 5416$ have at least one 2 or $3.$

Thus, the correct answer is **E**.

22. Two farmers agree that pigs are worth $\$300$ and that goats are worth $\$210$. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a $\$390$ debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

- A $\$5$
- B $\$10$
- C $\$30$
- D $\$90$
- E $\$210$

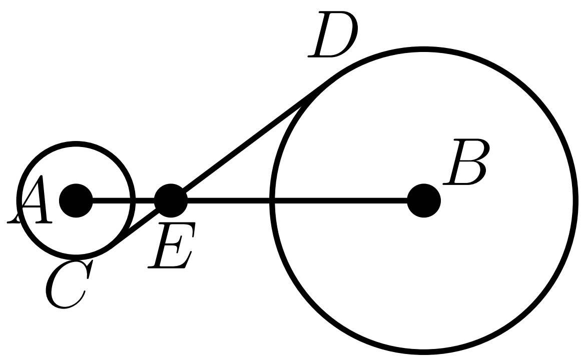
Solution:

A resolvable debt is $D = 300p + 210g$ for integers p, g, where a negative value means change received. Since $D = 30(10p + 7g)$ and $\gcd(10, 7) = 1$, the value $10p + 7g$ can be any integer, so D is any multiple of 30.

The smallest positive one is 30, achieved by $30 = 300(-2) + 210(3)$ (give 3 goats, receive 2 pigs).

Thus, the correct answer is **C**.

23. Circles with centers A and B have radii 3 and 8, respectively. A common internal tangent touches the circles at C and D, as shown. Lines AB and CD intersect at E, and $AE = 5$. What is CD?



A 13

B $\frac{44}{3}$

C $\sqrt{221}$

D $\sqrt{255}$

E $\frac{55}{3}$

Solution:

Since $AC \perp CD$, we have $CE = \sqrt{AE^2 - AC^2} = \sqrt{25 - 9} = 4.$

Because $\triangle ACE \sim \triangle BDE$, $\frac{DE}{CE} = \frac{BD}{AC}$, so $DE = 4 \cdot \frac{8}{3} = \frac{32}{3}.$

Then $CD = CE + DE = 4 + \frac{32}{3} = \frac{44}{3}.$

Thus, the correct answer is **B**.

24. Centers of adjacent faces of a unit cube are joined to form a regular octahedron. What is the volume of this octahedron?

A $\frac{1}{8}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

The six face centers form a regular octahedron, viewed as two congruent square pyramids sharing a base. Adjacent face centers are $\frac{\sqrt{2}}{2}$ apart, so the square base has area $\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$.

Each pyramid has height $\frac{1}{2}$, so its volume is $\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}$. The octahedron has volume $2 \cdot \frac{1}{12} = \frac{1}{6}$.

Thus, the correct answer is **B**.

25. A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once?

A $\frac{1}{2187}$

B $\frac{1}{729}$

C $\frac{2}{243}$

D $\frac{1}{81}$

E $\frac{5}{243}$

Solution:

After 7 moves there are $3^7 = 2187$ equally likely walks. A successful walk visits every vertex exactly once.

From the start there are 3 choices for the first move and 2 for the second (not returning). Labeling the first three vertices A, B, C, the bug must move to one of two vertices, after which the route is forced except for a single binary choice, giving $3 \cdot 2 \cdot 3 = 18$ such paths.

The probability is $\frac{18}{2187} = \frac{2}{243}$.

Thus, the correct answer is **C**.

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