2005 AMC 10B Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc10/2005B/solutions

1. A scout troop buys 1000 candy bars at a price of five for $\$2$. They sell all the candy bars at a price of two for $\$1$. What was their profit, in dollars?

A 100

B 200

C 300

D 400

E 500

Solution:

The troop buys $1000 \div 5 = 200$ groups of five bars, costing $200 \cdot 2 = 400$ dollars.

They sell $1000 \div 2 = 500$ pairs of bars, earning $500 \cdot 1 = 500$ dollars.

The profit is $\$500 - \$400 = \$100$.

Thus, **A** is the correct answer.

2. A positive number x has the property that $x\%$ of x is 4. What is x?

A 2

B 4

C 10

D 20

E 40

Solution:

The statement translates to

$$\frac{x}{100} \cdot x = 4,$$

so $x^2 = 400.$

Since x is positive, $x = 20.$

Thus, **D** is the correct answer.

3. A gallon of paint is used to paint a room. One third of the paint is used on the first day. On the second day, one third of the remaining paint is used. What fraction of the original amount of paint is available to use on the third day?

A $\frac{1}{10}$

B $\frac{1}{9}$

C $\frac{1}{3}$

D $\frac{4}{9}$

E $\frac{5}{9}$

Solution:

After the first day, $1 - \frac{1}{3} = \frac{2}{3}$ of the paint remains.

On the second day, $\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$ of the original amount is used.

The fraction available on the third day is

$$\frac{2}{3} - \frac{2}{9} = \frac{4}{9}.$$

Thus, **D** is the correct answer.

4. For real numbers a and b, define $a \diamond b = \sqrt{a^2 + b^2}$. What is the value of

$$(5 \diamond 12) \diamond ((-12) \diamond (-5))?$$

A 0

B $\frac{17}{2}$

C 13

D $13\sqrt{2}$

E 26

Solution:

Each inner expression evaluates to

$$5 \diamond 12 = \sqrt{5^2 + 12^2} = 13$$

and similarly $(-12) \diamond (-5) = \sqrt{144 + 25} = 13.$

Then

$$13 \diamond 13 = \sqrt{13^2 + 13^2} = 13\sqrt{2}.$$

Thus, **D** is the correct answer.

5. Brianna is using part of the money she earned on her weekend job to buy several equally-priced CDs. She used one fifth of her money to buy one third of the CDs. What fraction of her money will she have left after she buys all the CDs?

A $\frac{1}{5}$

B $\frac{1}{3}$

C $\frac{2}{5}$

D $\frac{2}{3}$

E $\frac{4}{5}$

Solution:

Buying all the CDs costs three times as much as buying one third of them, namely $3 \cdot \frac{1}{5} = \frac{3}{5}$ of her money.

The fraction left over is $1 - \frac{3}{5} = \frac{2}{5}$.

Thus, **C** is the correct answer.

6. At the beginning of the school year, Lisa's goal was to earn an A on at least 80% of her 50 quizzes for the year. She earned an A on 22 of the first 30 quizzes. If she is to achieve her goal, on at most how many of the remaining quizzes can she earn a grade lower than an A?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

Lisa needs an A on at least $0.8 \cdot 50 = 40$ quizzes.

She already has $22,$ so she needs $40 - 22 = 18$ A's among the remaining 20 quizzes.

That leaves at most $20 - 18 = 2$ quizzes with a grade lower than an A.

Thus, **B** is the correct answer.

7. A circle is inscribed in a square, then a square is inscribed in this circle, and finally, a circle is inscribed in this square. What is the ratio of the area of the smaller circle to the area of the larger square?

A $\frac{\pi}{16}$

B $\frac{\pi}{8}$

C $\frac{3\pi}{16}$

D $\frac{\pi}{4}$

E $\frac{\pi}{2}$

Solution:

Let the smaller circle have radius r, so its area is πr^2.

The smaller square, which circumscribes this circle, has side $2r$, and its diagonal $2\sqrt{2}\,r$ is the diameter of the larger circle. So the larger circle has radius $\sqrt{2}\,r$.

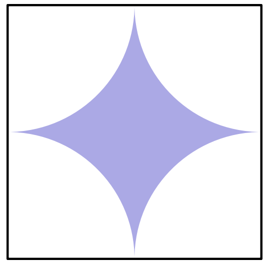
The larger square circumscribes the larger circle, so it has side $2\sqrt{2}\,r$ and area $8r^2$.

The desired ratio is

$$\frac{\pi r^2}{8r^2} = \frac{\pi}{8}.$$

Thus, **B** is the correct answer.

8. An 8-foot by 10-foot floor is tiled with square tiles of size 1 foot by 1 foot. Each tile has a pattern consisting of four white quarter circles of radius $\frac{1}{2}$ foot centered at each corner of the tile. The remaining portion of the tile is shaded. How many square feet of the floor are shaded?



A $80 - 20\pi$

B $60 - 10\pi$

C $80 - 10\pi$

D $60 + 10\pi$

E $80 + 10\pi$

Solution:

The four quarter circles on a tile together make one full circle of radius $\frac{1}{2}$, with area $\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.

So each tile has shaded area $1 - \frac{\pi}{4}$ square feet.

There are $8 \cdot 10 = 80$ tiles, so the total shaded area is

$$80\left(1 - \frac{\pi}{4}\right) = 80 - 20\pi.$$

Thus, **A** is the correct answer.

9. One fair die has faces $1, 1, 2, 2, 3, 3$ and another has faces $4, 4, 5, 5, 6, 6$. The dice are rolled and the numbers on the top faces are added. What is the probability that the sum will be odd?

A $\frac{1}{3}$

B $\frac{4}{9}$

C $\frac{1}{2}$

D $\frac{5}{9}$

E $\frac{2}{3}$

Solution:

The first die is odd (a 1 or 3) with probability $\frac{2}{3}$ and even with probability $\frac{1}{3}$. The second die is odd (a 5) with probability $\frac{1}{3}$ and even with probability $\frac{2}{3}$.

The sum is odd when the two parities differ:

$$\frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}.$$

Thus, **D** is the correct answer.

10. In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is BD?

A 3

B $2\sqrt{3}$

C 4

D 5

E $4\sqrt{2}$

Solution:

Let H be the foot of the altitude from C to line AB. Since $AC = BC$, H is the midpoint of AB, so $BH = 1$ and $CH^2 = 7^2 - 1^2 = 48$.

Applying the Pythagorean theorem in $\triangle CHD$, where $HD = BH + BD = 1 + BD$, gives

$$8^2 = 48 + (1 + BD)^2,$$

so $(1 + BD)^2 = 16$.

Then $1 + BD = 4$, so $BD = 3$.

Thus, **A** is the correct answer.

11. The first term of a sequence is 2005. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2005th term of the sequence?

A 29

B 55

C 85

D 133

E 250

Solution:

The sequence begins $2005, 133, 55, 250, 133, \ldots$, so after the first term it repeats the cycle $133, 55, 250$ of length 3.

The terms from position 2 onward follow this cycle. Since $2005 = 2 + 3 \cdot 667 + 2$, the 2005th term matches the third entry of the cycle, 250.

Thus, **E** is the correct answer.

12. Twelve fair dice are rolled. What is the probability that the product of the numbers on the top faces is prime?

A $\left(\frac{1}{12}\right)^{12}$

B $\left(\frac{1}{6}\right)^{12}$

C $2\left(\frac{1}{6}\right)^{11}$

D $\frac{5}{2}\left(\frac{1}{6}\right)^{11}$

E $\left(\frac{1}{6}\right)^{10}$

Solution:

The product is prime exactly when one die shows a prime ($2, 3,$ or 5) and the other eleven all show $1.$

The probability that any single die is the prime one is $\frac{3}{6} = \frac{1}{2}$, and each of the other eleven shows 1 with probability $\frac{1}{6}$. Accounting for which of the twelve dice is prime, the probability is

$$12 \cdot \frac{1}{2} \cdot \left(\frac{1}{6}\right)^{11} = 6 \cdot \left(\frac{1}{6}\right)^{11} = \left(\frac{1}{6}\right)^{10}.$$

Thus, **E** is the correct answer.

13. How many numbers between 1 and 2005 are integer multiples of 3 or 4 but not 12?

A 501

B 668

C 835

D 1002

E 1169

Solution:

Between 1 and 2005 there are 668 multiples of 3, 501 multiples of 4, and 167 multiples of 12.

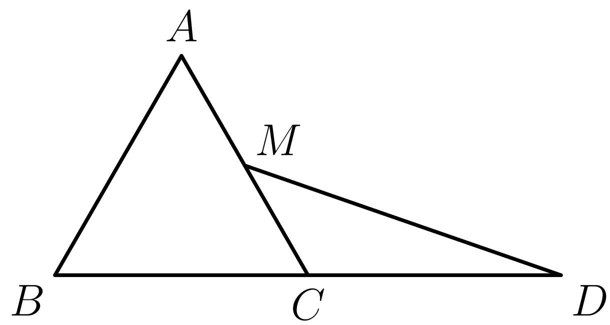
Every multiple of 12 is both a multiple of 3 and of 4, so removing them from each group gives

$$(668 - 167) + (501 - 167) = 835$$

numbers that are multiples of 3 or 4 but not 12.

Thus, **C** is the correct answer.

14. Equilateral $\triangle ABC$ has side length $2,$ M is the midpoint of $\overline{AC},$ and C is the midpoint of $\overline{BD}.$ What is the area of $\triangle CDM?$



A $\frac{\sqrt{2}}{2}$

B $\frac{3}{4}$

C $\frac{\sqrt{3}}{2}$

D 1

E $\sqrt{2}$

Solution:

Take $\overline{CD}$ as the base. Since C is the midpoint of $\overline{BD}$ and $BC = 2,$ we have $CD = 2.$

The height of $\triangle CDM$ is the distance from M to line $BD.$ Because M is the midpoint of $\overline{AC},$ this distance is half the height of $\triangle ABC,$ which is $\frac{1}{2} \cdot \sqrt{3} = \frac{\sqrt{3}}{2}.$

The area is

$$\frac{1}{2} \cdot 2 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$

Thus, **C** is the correct answer.

15. An envelope contains eight bills: 2 ones, 2 fives, 2 tens, and 2 twenties. Two bills are drawn at random without replacement. What is the probability that their sum is $\$20$ or more?

A $\frac{1}{4}$

B $\frac{2}{5}$

C $\frac{3}{7}$

D $\frac{1}{2}$

E $\frac{2}{3}$

Solution:

There are $\binom{8}{2} = 28$ equally likely pairs.

A sum of at least $\$20$ comes from both twenties (1 way), a twenty paired with any of the six smaller bills ($2 \cdot 6 = 12$ ways), or both tens (1 way).

The probability is

$$\frac{1+12+1}{28} = \frac{14}{28} = \frac{1}{2}.$$

Thus, **D** is the correct answer.

16. The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of $m, n,$ and p is zero. What is the value of $\frac{n}{p}$?

- A 1
- B 2
- C 4
- D 8
- E 16

Solution:

Let r_1 and r_2 be the roots of $x^2 + px + m = 0$, so $m = r_1 r_2$ and $p = -(r_1 + r_2)$.

The roots of $x^2 + mx + n = 0$ are $2r_1$ and $2r_2$, so $n = 4r_1 r_2$ and $-m = 2(r_1 + r_2)$.

Then $n = 4m$ and $m = -2(r_1 + r_2) = 2p$, so $p = \frac{m}{2}$. Therefore

$$\frac{n}{p} = \frac{4m}{m/2} = 8.$$

Thus, **D** is the correct answer.

17. Suppose that $4^a = 5, 5^b = 6, 6^c = 7$, and $7^d = 8$. What is $a \cdot b \cdot c \cdot d$?

A 1

B $\frac{3}{2}$

C 2

D $\frac{5}{2}$

E 3

Solution:

Chaining the equations,

$$4^{abcd} = \left(\left(\left(4^a\right)^b\right)^c\right)^d = \left(\left(5^b\right)^c\right)^d = \left(6^c\right)^d = 7^d = 8.$$

Since $8 = 4^{3/2}$, we conclude $a \cdot b \cdot c \cdot d = \frac{3}{2}$.

Thus, **B** is the correct answer.

18. All of David's telephone numbers have the form $555–abc–defg$, where $a, b, c, d, e, f,$ and g are distinct digits and in increasing order, and none is either 0 or 1. How many different telephone numbers can David have?

- A 1
- B 2
- C 7
- D 8
- E 9

Solution:

The seven digits are chosen from $\{2, 3, 4, 5, 6, 7, 8, 9\}$, and once chosen they must be written in increasing order, so only the choice of digits matters.

Choosing seven of these eight digits is the same as choosing the one digit to leave out, which can be done in 8 ways.

Thus, **D** is the correct answer.

19. On a certain math exam, 10% of the students got 70 points, 25% got 80 points, 20% got 85 points, 15% got 90 points, and the rest got 95 points. What is the difference between the mean and the median score on this exam?

A 0

B 1

C 2

D 4

E 5

Solution:

The percentage scoring 95 is $100 - 10 - 25 - 20 - 15 = 30.$

The mean is

$$0.10(70) + 0.25(80) + 0.20(85) + 0.15(90) + 0.30(95) = 86.$$

Since 35% scored below 85 and 35% scored above $85,$ the middle student scored $85,$ so the median is $85.$

The difference is $86 - 85 = 1.$

Thus, **B** is the correct answer.

20. What is the average (mean) of all 5-digit numbers that can be formed by using each of the digits $1, 3, 5, 7,$ and 8 exactly once?

A 48000

B 49999.5

C 53332.8

D 55555

E 56432.8

Solution:

By symmetry, each of the five digits appears equally often in each place, so the average digit in every place is

$$\frac{1+3+5+7+8}{5} = 4.8.$$

The average number is therefore

$$4.8(1+10+100+1000+10000) = 4.8 \cdot 11111 = 53332.8.$$

Thus, **C** is the correct answer.

21. Forty slips are placed into a hat, each bearing a number $1, 2, 3, 4, 5, 6, 7, 8, 9,$ or $10,$ with each number entered on four slips. Four slips are drawn from the hat at random and without replacement. Let p be the probability that all four slips bear the same number. Let q be the probability that two of the slips bear a number a and the other two bear a number $b \neq a.$ What is the value of $\frac{q}{p}$?

- A 162
- B 180
- C 324
- D 360
- E 720

Solution:

Both events draw from $\binom{40}{4}$ equally likely selections, so $\frac{q}{p}$ is the ratio of their favorable counts.

Exactly 10 draws give four slips of the same number, one for each value.

For two a's and two b's, choose the two values in $\binom{10}{2}$ ways, then two of the four a-slips and two of the four b-slips:

$$\binom{10}{2}\binom{4}{2}\binom{4}{2} = 45 \cdot 6 \cdot 6 = 1620.$$

Therefore $\frac{q}{p} = \frac{1620}{10} = 162.$

Thus, **A** is the correct answer.

22. For how many positive integers n less than or equal to 24 is $n!$ evenly divisible by $1 + 2 + \cdots + n$?

- A 8
- B 12
- C 16
- D 17
- E 21

Solution:

Since $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$, divisibility is equivalent to

$$\frac{n!}{n(n+1)/2} = \frac{2(n-1)!}{n+1}$$

being an integer.

If $n+1$ is not prime (and $n \geq 1$), its factors appear among $1, 2, \ldots, n-1$ or in the factor 2, so the fraction is an integer. If $n+1$ is an odd prime, it divides neither $(n-1)!$ nor 2, so the fraction is not an integer.

The odd primes at most 25 are $3, 5, 7, 11, 13, 17, 19, 23$, giving 8 failing values of n. Hence $24 - 8 = 16$ values work.

Thus, **C** is the correct answer.

23. In trapezoid $ABCD$ we have $\overline{AB}$ parallel to $\overline{DC}$, E as the midpoint of $\overline{BC}$, and F as the midpoint of $\overline{DA}$. The area of $ABEF$ is twice the area of $FECD$. What is $\dfrac{AB}{DC}$?

A 2

B 3

C 5

D 6

E 8

Solution:

Let $AB = a$ and $DC = c$. The midsegment $\overline{FE}$ has length $\dfrac{a+c}{2}$, and $ABEF$ and $FECD$ have the same height.

Their areas are proportional to the averages of their parallel sides, so

$$\frac{a + \frac{a+c}{2}}{\frac{a+c}{2} + c} = \frac{3a+c}{a+3c} = 2.$$

Then $3a + c = 2a + 6c$, so $a = 5c$ and $\dfrac{AB}{DC} = 5$.

Thus, **C** is the correct answer.

24. Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m. What is $x + y + m$?

A 88

B 112

C 116

D 144

E 154

Solution:

Write $x = 10a + b$ and $y = 10b + a$ with $a > b$. Then

$$m^2 = x^2 - y^2 = 99(a^2 - b^2) = 99(a + b)(a - b).$$

Since $99 = 9 \cdot 11$, for m^2 to be a perfect square we need $11 \mid (a + b)(a - b)$. As $a + b \leq 17$, this forces $a + b = 11$, and then $a - b$ must itself be a perfect square.

With $a - b \leq 8$, the only workable case is $a - b = 1$, giving $(a, b) = (6, 5)$. Then $x = 65, y = 56$, and $m^2 = 99 \cdot 11 = 33^2$, so $m = 33$.

Therefore $x + y + m = 65 + 56 + 33 = 154.$

Thus, **E** is the correct answer.

25. A subset B of the set of integers from 1 to 100, inclusive, has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B?

- A 50
- B 51
- C 62
- D 65
- E 68

Solution:

The pairs summing to 125 are $(25, 100), (26, 99), \ldots, (62, 63)$, which is $62 - 25 + 1 = 38$ pairs. From each pair, B may contain at most one element.

The numbers 1 through 24 cannot pair with anything in range to sum to 125, so all 24 of them may be included.

Thus B has at most $38 + 24 = 62$ elements, and the set $\{1, 2, \ldots, 62\}$ achieves this.

Thus, **C** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc10/2005B