2005 AMC 10A Solutions

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1. While eating out, Mike and Joe each tipped their server $\$2$. Mike tipped 10% of his bill and Joe tipped 20% of his bill. What was the difference, in dollars, between their bills?

   A 2

   B 4

   C 5

   D 10

   E 20

   ### Solution:

   Mike's $\$2$ tip is 10% of his bill, so his bill is $2 \times 10 = 20$ dollars. Joe's $\$2$ tip is 20% of his bill, so his bill is $2 \times 5 = 10$ dollars. The difference is $20 - 10 = 10$ dollars.

   Thus, the correct answer is **D**.

2. For each pair of real numbers $a \neq b$, define the operation $\star$ as

$$(a \star b) = \frac{a+b}{a-b}.$$

What is the value of $((1 \star 2) \star 3)$?

A $-\frac{2}{3}$

B $-\frac{1}{5}$

C 0

D $\frac{1}{2}$

E This value is not defined.

Solution:

First $(1 \star 2) = \frac{1+2}{1-2} = \frac{3}{-1} = -3$. Then $(-3 \star 3) = \frac{-3+3}{-3-3} = \frac{0}{-6} = 0.$

Thus, the correct answer is **C**.

3. The equations $2x + 7 = 3$ and $bx - 10 = -2$ have the same solution x. What is the value of b?

- A -8
- B -4
- C -2
- D 4
- E 8

Solution:

From $2x + 7 = 3$ we get $x = -2$. Substituting, $-2b - 10 = -2$, so $-2b = 8$ and $b = -4$.

Thus, the correct answer is **B**.

4. A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle?

A $\frac{1}{4}x^2$

B $\frac{2}{5}x^2$

C $\frac{1}{2}x^2$

D x^2

E $\frac{3}{2}x^2$

Solution:

Let the width be w, so the length is $2w$. Then $x^2 = w^2 + (2w)^2 = 5w^2$, giving $w^2 = \frac{x^2}{5}$. The area is $w \cdot 2w = 2w^2 = \frac{2}{5}x^2$.

Thus, the correct answer is **B**.

5. A store normally sells windows at $\$100$ each. This week the store is offering one free window for each purchase of four. Dave needs seven windows and Doug needs eight windows. How many dollars will they save if they purchase the windows together rather than separately?

- A 100
- B 200
- C 300
- D 400
- E 500

Solution:

Alone, Dave pays for 6 windows and receives one free to reach $7,$ costing $\$600;$ Doug pays for 7 and receives one free to reach $8,$ costing $\$700.$ Separately they pay $\$1300.$ Together they need 15 windows: buying 12 yields 3 free, for $\$1200.$ The savings are $1300 - 1200 = 100$ dollars.

Thus, the correct answer is **A**.

6. The average (mean) of 20 numbers is $30,$ and the average of 30 other numbers is $20.$ What is the average of all 50 numbers?

- A 23
- B 24
- C 25
- D 26
- E 27

Solution:

The combined sum is $20 \cdot 30 + 30 \cdot 20 = 600 + 600 = 1200.$ The average of all 50 numbers is $\frac{1200}{50} = 24.$

Thus, the correct answer is **B**.

7. Josh and Mike live 13 miles apart. Yesterday Josh started to ride his bicycle toward Mike's house. A little later Mike started to ride his bicycle toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike and at four-fifths of Mike's rate. How many miles had Mike ridden when they met?

A 4

B 5

C 6

D 7

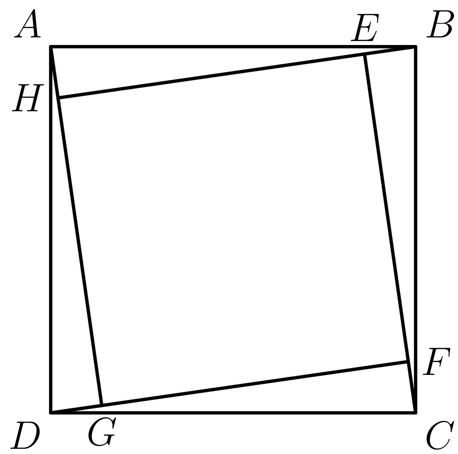
E 8

Solution:

Let Mike ride m miles. Josh rides $\frac{4}{5}$ the rate for 2 times the time, so Josh's distance is $\frac{8}{5}m.$ Together they cover $13,$ so $m + \frac{8}{5}m = \frac{13}{5}m = 13,$ giving $m = 5.$

Thus, the correct answer is **B**.

8. In the figure, the length of side AB of square $ABCD$ is $\sqrt{50}$, E is between B and H, and $BE = 1$. What is the area of the inner square $EFGH$?



A 25

B 32

C 36

D 40

E 42

Solution:

The triangles ABH, BCE, CDF, and DAG are congruent right triangles. In $\triangle BCE$ the hypotenuse is $BC = \sqrt{50}$ and $BE = 1$, so $CE = \sqrt{50-1} = 7$. Since $BH = CE = 7$ and E lies on BH with $BE = 1$, the inner square's side is $EH = 7 - 1 = 6$, giving area $6^2 = 36$.

Thus, the correct answer is **C**.

9. Three tiles are marked X and two other tiles are marked O. The five tiles are randomly arranged in a row. What is the probability that the arrangement reads XOXOX?

A $\frac{1}{12}$

B $\frac{1}{10}$

C $\frac{1}{6}$

D $\frac{1}{4}$

E $\frac{1}{3}$

Solution:

The three X positions can be any of $\binom{5}{3} = 10$ equally likely choices, and exactly one of them produces XOXOX. So the probability is $\frac{1}{10}$.

Thus, the correct answer is **B**.

10. There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of those values of a?

- A -16
- B -8
- C 0
- D 8
- E 20

Solution:

Writing the equation as $4x^2 + (a+8)x + 9 = 0$, there is one solution exactly when the discriminant $(a+8)^2 - 144 = 0$. Then $a + 8 = \pm 12$, so $a = 4$ or $a = -20$, and their sum is -16.

Thus, the correct answer is **A**.

11. A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

A 3

B 4

C 5

D 6

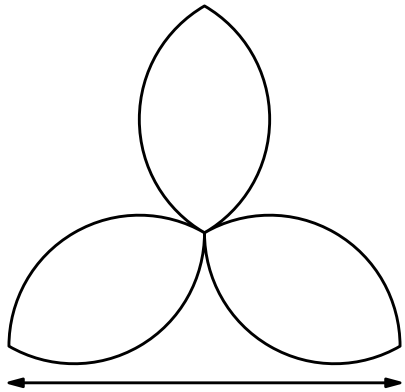
E 7

Solution:

The unit cubes have $6n^3$ faces total, of which the original surface accounts for $6n^2$ red faces. Then $\frac{6n^2}{6n^3} = \frac{1}{n} = \frac{1}{4}$, so $n = 4.$

Thus, the correct answer is **B**.

12. The figure shown is called a *trefoil* and is constructed by drawing circular sectors about sides of the congruent equilateral triangles. What is the area of a trefoil whose horizontal base has length 2?



A $\frac{1}{3}\pi + \frac{\sqrt{3}}{2}$

B $\frac{2}{3}\pi$

C $\frac{2}{3}\pi + \frac{\sqrt{3}}{4}$

D $\frac{2}{3}\pi + \frac{\sqrt{3}}{3}$

E $\frac{2}{3}\pi + \frac{\sqrt{3}}{2}$

Solution:

Since the base 2 equals two radii, the radius is 1. The trefoil is made of four equilateral triangles and four circular segments, which reassemble into four 60° sectors of a circle of radius 1. Their total area is $4 \cdot \frac{60}{360}\pi(1)^2 = \frac{2}{3}\pi$.

Thus, the correct answer is **B**.

13. How many positive integers n satisfy the following condition:

$$(130n)^{50} > n^{100} > 2^{200}?$$

- A 0
- B 7
- C 12
- D 65
- E 125

Solution:

Taking 50th roots, the condition becomes $130n > n^2 > 2^4 = 16$. From $n^2 > 16$ we get $n > 4$, and from $130n > n^2$ we get $n < 130$. So n ranges over the integers $5, 6, \ldots, 129$, which is 125 values.

Thus, the correct answer is **E**.

14. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

A 41

B 42

C 43

D 44

E 45

Solution:

The first and last digits must have the same parity so their average is a digit. Both odd gives $5 \cdot 5 = 25$ pairs. Both even, with a nonzero leading digit, gives $4 \cdot 5 = 20$ pairs. Each pair fixes the middle digit, for a total of $25 + 20 = 45$ numbers.

Thus, the correct answer is **E**.

15. How many positive cubes divide $3! \cdot 5! \cdot 7!$?

A 2

B 3

C 4

D 5

E 6

Solution:

As a product of primes, $3! \cdot 5! \cdot 7! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$ A cube divisor uses exponents that are multiples of $3:$ the exponent of 2 can be $0, 3,$ or 6 (3 choices), the exponent of 3 can be 0 or 3 (2 choices), and the exponents of 5 and 7 must be $0.$ That gives $3 \cdot 2 \cdot 1 \cdot 1 = 6$ cubes.

Thus, the correct answer is **E**.

16. The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property?

A 5

B 7

C 9

D 10

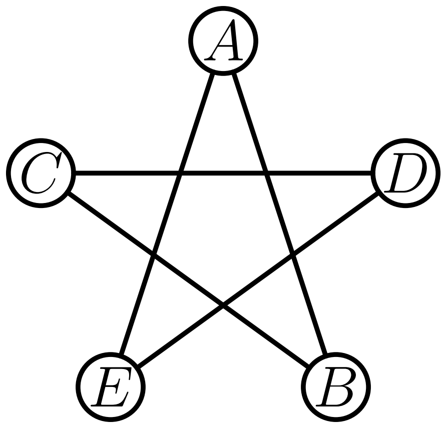
E 19

Solution:

If the number is $10a + b$, then $(10a + b) - (a + b) = 9a$. The units digit of $9a$ is 6 only when $a = 4$, since $9 \cdot 4 = 36$. The digit b can then be anything from 0 to 9, giving the ten numbers 40 through 49.

Thus, the correct answer is **D**.

17. In the five-sided star shown, the letters $A, B, C, D,$ and E are replaced by the numbers $3, 5, 6, 7,$ and $9,$ although not necessarily in this order. The sums of the numbers at the ends of the line segments $AB, BC, CD, DE,$ and EA form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?



A 9

B 10

C 11

D 12

E 13

Solution:

Every number is an endpoint of two segments, so the five segment sums total $2(3 + 5 + 6 + 7 + 9) = 60.$ The middle term of a five-term arithmetic sequence equals its mean, which is $\frac{60}{5} = 12.$

Thus, the correct answer is **D**.

18. Team A and team B play a series. The first team to win three games wins the series. Each team is equally likely to win each game, there are no ties, and the outcomes of the individual games are independent. If team B wins the second game and team A wins the series, what is the probability that team B wins the first game?

A $\frac{1}{5}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E $\frac{2}{3}$

Solution:

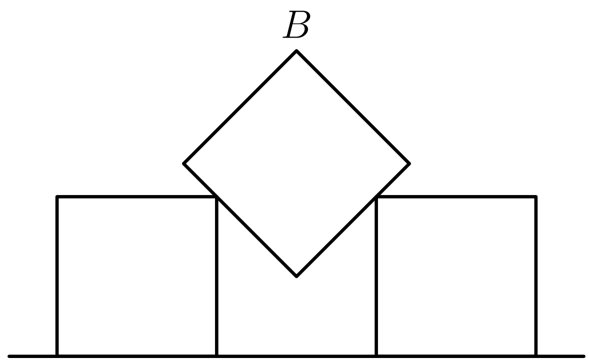
Suppose all five games are played, so every sequence of five results is equally likely. Requiring that B wins game 2 and A ends up with the series (three wins) leaves the equally likely sequences

$$\text{BBAAA}, \quad \text{ABBAA}, \quad \text{ABABA}, \quad \text{ABAAB}, \quad \text{ABAAA}.$$

Only in BBAAA does team B win the first game, so the probability is $\frac{1}{5}$.

Thus, the correct answer is **A**.

19. Three one-inch squares are placed with their bases on a line. The center square is lifted out and rotated 45°, as shown. Then it is centered and lowered into its original location until it touches both of the adjoining squares. How many inches is the point B from the line on which the bases of the original squares were placed?



A 1

B $\sqrt{2}$

C $\frac{3}{2}$

D $\sqrt{2}+\frac{1}{2}$

E 2

Solution:

When lowered, the rotated square's two lower edges rest on the inner top corners of the adjoining squares, which are at height 1. Working out the geometry, the square's bottom vertex settles at height $\frac{1}{2}$. The point B is the opposite vertex, a full vertical diagonal of length $\sqrt{2}$ higher, so its height is $\frac{1}{2} + \sqrt{2}$.

Thus, the correct answer is **D**.

20. An equiangular octagon has four sides of length 1 and four sides of length $\frac{\sqrt{2}}{2}$, arranged so that no two consecutive sides have the same length. What is the area of the octagon?

A $\frac{7}{2}$

B $\frac{7\sqrt{2}}{2}$

C $\frac{5+4\sqrt{2}}{2}$

D $\frac{4+5\sqrt{2}}{2}$

E 7

Solution:

Extend the four sides of length 1 to form a square. Each short side $\frac{\sqrt{2}}{2}$ is the hypotenuse of an isosceles right triangle with legs $\frac{1}{2}$, and cutting these four corners from a square of side $1+2\cdot\frac{1}{2}=2$ gives the octagon. Its area is $2^2-4\cdot\frac{1}{2}\left(\frac{1}{2}\right)^2=4-\frac{1}{2}=\frac{7}{2}$.

Thus, the correct answer is **A**.

21. For how many positive integers n does $1+2+\cdots+n$ evenly divide $6n$?

A 3

B 5

C 7

D 9

E 11

Solution:

Since $1+2+\cdots+n=\dfrac{n(n+1)}{2}$, the quotient is $\dfrac{6n}{n(n+1)/2}=\dfrac{12}{n+1}$, which is an integer exactly when $n+1$ divides $12.$ The divisors of 12 that are at least 2 are $2, 3, 4, 6, 12,$ giving $n=1, 2, 3, 5, 11$ — five values.

Thus, the correct answer is **B**.

22. Let S be the set of the 2005 smallest positive multiples of $4,$ and let T be the set of the 2005 smallest positive multiples of $6.$ How many elements are common to S and T?

A 166

B 333

C 500

D 668

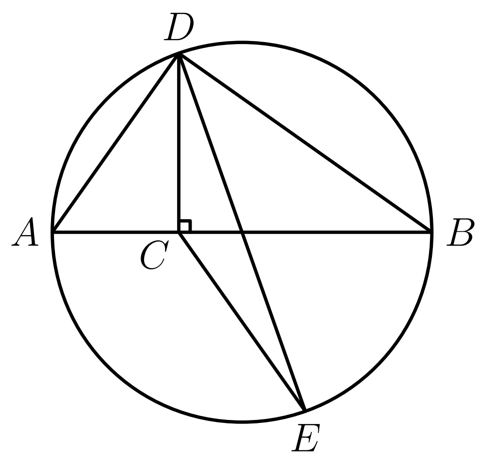
E 1001

Solution:

The elements common to S and T are the multiples of $\operatorname{lcm}(4, 6) = 12.$ Now S contains multiples of 4 up to $4 \cdot 2005 = 8020,$ while T reaches up to $6 \cdot 2005 = 12{,}030,$ so the common elements are the multiples of 12 not exceeding $8020.$ There are $\left\lfloor \frac{8020}{12} \right\rfloor = 668$ of them.

Thus, the correct answer is **D**.

23. Let AB be a diameter of a circle and C be a point on AB with $2 \cdot AC = BC$. Let D and E be points on the circle such that $DC \perp AB$ and DE is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



A $\frac{1}{6}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E $\frac{2}{3}$

Solution:

Let O be the center. From $2 \cdot AC = BC$ and $AC + BC = AB$, we get $AC = \dfrac{AB}{3}$, so $CO = \dfrac{AB}{2} - \dfrac{AB}{3} = \dfrac{AB}{6}$. Triangles DCO and DAB share the apex D with bases CO and AB on the same line, so $[\triangle DCO] = \dfrac{CO}{AB}[\triangle DAB] = \dfrac{1}{6}[\triangle DAB]$. Because O is the midpoint of DE, $[\triangle DCE] = 2\,[\triangle DCO] = \dfrac{1}{3}[\triangle DAB]$.

Thus, the correct answer is **C**.

24. For each positive integer $m > 1$, let $P(m)$ denote the greatest prime factor of m. For how many positive integers n is it true that both $P(n) = \sqrt{n}$ and $P(n+48) = \sqrt{n+48}$?

A 0

B 1

C 3

D 4

E 5

Solution:

The condition $P(n) = \sqrt{n}$ means n is the square of a prime q, and likewise $n + 48 = p^2$ for a prime p. Then $48 = p^2 - q^2 = (p-q)(p+q)$. Checking the same-parity factorizations of 48, only $p - q = 2,\ p + q = 24$ yields primes, giving $(p, q) = (13, 11)$ and $n = 121$. So there is exactly one such n.

Thus, the correct answer is **B**.

25. In $\triangle ABC$ we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on AB and AC respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of the quadrilateral $BCED$?

A $\frac{266}{1521}$

B $\frac{19}{75}$

C $\frac{1}{3}$

D $\frac{19}{56}$

E 1

Solution:

Triangles ADE and ABC share angle A, so $\frac{[ADE]}{[ABC]} = \frac{AD \cdot AE}{AB \cdot AC} = \frac{19 \cdot 14}{25 \cdot 42} = \frac{266}{1050} = \frac{19}{75}$. Since $[BCED] = [ABC] - [ADE]$, we get $\frac{[ADE]}{[BCED]} = \frac{19}{75 - 19} = \frac{19}{56}$.

Thus, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc10/2005A