2004 AMC 10A Solutions

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1. You and five friends need to raise $\$1500$ in donations for a charity, dividing the fundraising equally. How many dollars will each of you need to raise?

A 250

B 300

C 1500

D 7500

E 9000

Solution:

Including you, there are 6 people sharing the fundraising equally. Each must raise

$$\frac{1500}{6} = 250$$

dollars.

Thus, the correct answer is **A**.

2. For any three real numbers $a, b,$ and $c,$ with $b \neq c,$ the operation $\diamond$ is defined by

$$\diamond(a, b, c) = \frac{a}{b - c}.$$

What is $\diamond(\diamond(1, 2, 3), \diamond(2, 3, 1), \diamond(3, 1, 2))$?

A $-\frac{1}{2}$

B $-\frac{1}{4}$

C 0

D $\frac{1}{4}$

E $\frac{1}{2}$

Solution:

The inner values are

$$\diamond(1, 2, 3) = \frac{1}{2 - 3} = -1, \quad \diamond(2, 3, 1) = \frac{2}{3 - 1} = 1, \quad \diamond(3, 1, 2) = \frac{3}{1 - 2} = -3.$$

Therefore

$$\diamond(-1, 1, -3) = \frac{-1}{1 - (-3)} = -\frac{1}{4}.$$

Thus, the correct answer is **B**.

3. Alicia earns $\$20$ per hour, of which 1.45% is deducted to pay local taxes. How many cents per hour of Alicia's wages are used to pay local taxes?

- A 0.0029
- B 0.029
- C 0.29
- D 2.9
- E 29

Solution:

Since $\$20$ equals 2000 cents, the local tax is

$$0.0145 \times 2000 = 29$$

cents per hour.

Thus, the correct answer is **E**.

4. What is the value of x if $|x-1| = |x-2|$?

A $-\frac{1}{2}$

B $\frac{1}{2}$

C 1

D $\frac{3}{2}$

E 2

Solution:

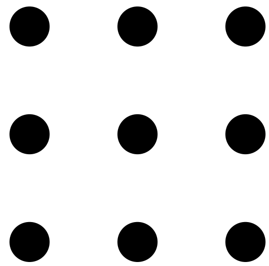
Since $|x-1|$ and $|x-2|$ are the distances from x to 1 and 2, the point x is equidistant from 1 and 2.

That midpoint is

$$x = \frac{1+2}{2} = \frac{3}{2}.$$

Thus, the correct answer is **D**.

5. A set of three points is chosen randomly from the grid shown. Each three-point set has the same probability of being chosen. What is the probability that the points lie on the same straight line?



A $\frac{1}{21}$

B $\frac{1}{14}$

C $\frac{2}{21}$

D $\frac{1}{7}$

E $\frac{2}{7}$

Solution:

The number of three-point sets is

$$\binom{9}{3} = 84.$$

The collinear triples are the 3 rows, the 3 columns, and the 2 main diagonals, for a total of 8.

The probability is therefore

$$\frac{8}{84} = \frac{2}{21}.$$

Thus, the correct answer is **C**.

6. Bertha has 6 daughters and no sons. Some of her daughters have 6 daughters, and the rest have none. Bertha has a total of 30 daughters and granddaughters, and no great-granddaughters. How many of Bertha's daughters and granddaughters have no daughters?

A 22

B 23

C 24

D 25

E 26

Solution:

Bertha has $30 - 6 = 24$ granddaughters, none of whom have daughters.

These granddaughters belong to $24/6 = 4$ of Bertha's daughters. So exactly 4 women have daughters, and the number with no daughters is

$$30 - 4 = 26.$$

Thus, the correct answer is **E**.

7. A grocer stacks oranges in a pyramid-like stack whose rectangular base is 5 oranges by 8 oranges. Each orange above the first level rests in a pocket formed by four oranges in the level below. The stack is completed by a single row of oranges. How many oranges are in the stack?

- A 96
- B 98
- C 100
- D 101
- E 134

Solution:

There are five layers, each one shorter and narrower than the one below. The total number of oranges is

$$5 \cdot 8 + 4 \cdot 7 + 3 \cdot 6 + 2 \cdot 5 + 1 \cdot 4 = 40 + 28 + 18 + 10 + 4 = 100.$$

Thus, the correct answer is **C**.

8. A game is played with tokens according to the following rule. In each round, the player with the most tokens gives one token to each of the other players and also places one token into a discard pile. The game ends when some player runs out of tokens. Players $A, B,$ and C start with $15, 14,$ and 13 tokens, respectively. How many rounds will there be in the game?

- A 36
- B 37
- C 38
- D 39
- E 40

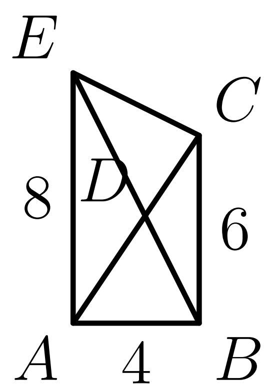
Solution:

After the first three rounds the counts go from $(15, 14, 13)$ to $(14, 13, 12)$. In general, every three rounds each player loses exactly one token.

After 36 rounds the counts are $(3, 2, 1)$. On the 37th round the leader gives away three tokens and drops to 0, ending the game.

Thus, the correct answer is **B**.

9. In the figure, $\angle EAB$ and $\angle ABC$ are right angles, $AB = 4, BC = 6, AE = 8,$ and $\overline{AC}$ and $\overline{BE}$ intersect at $D.$ What is the difference between the areas of $\triangle ADE$ and $\triangle BDC$?



A 2

B 4

C 5

D 8

E 9

Solution:

Let $[ABD]$ be the area shared by both large triangles. Then $[ABE] = [ADE] + [ABD]$ and $[ABC] = [BDC] + [ABD]$.

Subtracting,

$$[ADE] - [BDC] = [ABE] - [ABC].$$

Since $\angle EAB$ and $\angle ABC$ are right angles,

$$[ABE] = \tfrac{1}{2}(4)(8) = 16, \qquad [ABC] = \tfrac{1}{2}(4)(6) = 12.$$

The difference is $16 - 12 = 4.$

Thus, the correct answer is **B**.

10. Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

A $\frac{19}{128}$

B $\frac{23}{128}$

C $\frac{1}{4}$

D $\frac{35}{128}$

E $\frac{1}{2}$

Solution:

The two coins match when both show $0, 1, 2,$ or 3 heads. Coin A has weights $1, 3, 3, 1$ out of 8 and coin B has weights $1, 4, 6, 4, 1$ out of $16.$

The probability is

$$\frac{1 \cdot 1 + 3 \cdot 4 + 3 \cdot 6 + 1 \cdot 4}{8 \cdot 16} = \frac{35}{128}.$$

Thus, the correct answer is **D**.

11. A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. If the diameter of the jars is increased by 25% without altering the volume, by what percent must the height be decreased?

- A 10
- B 25
- C 36
- D 50
- E 60

Solution:

Keeping $\pi r^2 h$ constant while multiplying the radius by 1.25 requires the height to be multiplied by

$$\frac{1}{1.25^2} = \frac{1}{1.5625} = 0.64.$$

So the height becomes 64% of the original, a decrease of 36%.

Thus, the correct answer is **C**.

12. Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?

- A 24
- B 256
- C 768
- D $40{,}320$
- E $120{,}960$

Solution:

Each of the 8 condiments is independently in or out, giving $2^8 = 256$ condiment combinations.

For each of these there are 3 choices of patty count, so the number of hamburgers is

$$3 \times 256 = 768.$$

Thus, the correct answer is **C**.

13. At a party, each man danced with exactly three women and each woman danced with exactly two men. Twelve men attended the party. How many women attended the party?

- A 8
- B 12
- C 16
- D 18
- E 24

Solution:

The number of dancing pairs is $12 \cdot 3 = 36,$ counting from the men's side. Each woman was in exactly 2 pairs, so the number of women is

$$\frac{36}{2} = 18.$$

Thus, the correct answer is **D**.

14. The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

With n coins the total value is $20n$ cents. Adding a quarter gives

$$20n + 25 = 21(n + 1),$$

so $n = 4.$

Four coins worth a total of 80 cents must be three quarters and one nickel. Hence the number of dimes is $0.$

Thus, the correct answer is **A**.

15. Given that $-4 \le x \le -2$ and $2 \le y \le 4$, what is the largest possible value of

$$\frac{x+y}{x}?$$

A -1

B $-\frac{1}{2}$

C 0

D $\frac{1}{2}$

E 1

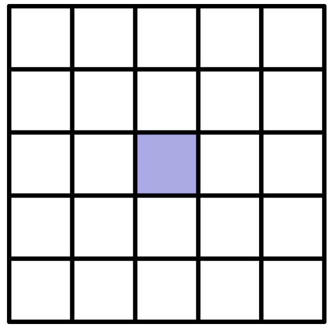
Solution:

Write $\frac{x+y}{x} = 1 + \frac{y}{x}$. Here $\frac{y}{x} < 0$, so the expression is largest when $\left|\frac{y}{x}\right|$ is smallest. That happens with $y = 2$ and $x = -4$, giving

$$1 + \frac{2}{-4} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Thus, the correct answer is **D**.

16. The 5×5 grid shown contains a collection of squares with sizes from 1×1 to 5×5. How many of these squares contain the shaded center square?



A 12

B 15

C 17

D 19

E 20

Solution:

Every $5 \times 5, 4 \times 4,$ and 3×3 square contains the center cell, and there are

$$1^2 + 2^2 + 3^2 = 14$$

of them.

Among the smaller squares, 4 of the 2×2 squares and 1 of the 1×1 squares cover the center, giving

$$14 + 4 + 1 = 19.$$

Thus, the correct answer is **D**.

17. Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

- A 250
- B 300
- C 350
- D 400
- E 500

Solution:

Before the first meeting the two together cover half the track. Between the first and second meetings they together cover a full track, which is twice as far, so Brenda runs $2 \cdot 100 = 200$ meters in that stretch.

Sally runs 150 meters in the same stretch, so the full track length is

$$200 + 150 = 350.$$

Thus, the correct answer is **C**.

18. A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

A 1

B 4

C 36

D 49

E 81

Solution:

The arithmetic progression is $9, 9+d, 9+2d$, so the geometric progression is 9, $11+d, 29+2d$.

The geometric condition gives

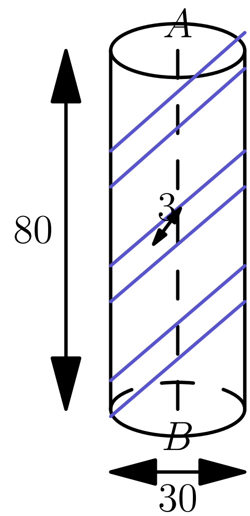
$$(11+d)^2 = 9(29+2d),$$

which simplifies to $d^2+4d-140=0$, so $d=10$ or $d=-14$.

The third terms $29+2d$ are 49 and 1. The smallest is 1.

Thus, the correct answer is **A**.

19. A cylindrical silo has a diameter of 30 feet and a height of 80 feet. A stripe with a horizontal width of 3 feet is painted on the silo, as shown, making two complete revolutions around it. What is the area of the stripe in square feet?



A 120

B 180

C 240

D 360

E 480

Solution:

Unrolling the stripe flattens it into a parallelogram. Its base (the horizontal width) is 3 feet and its height spans the full 80 feet of the silo.

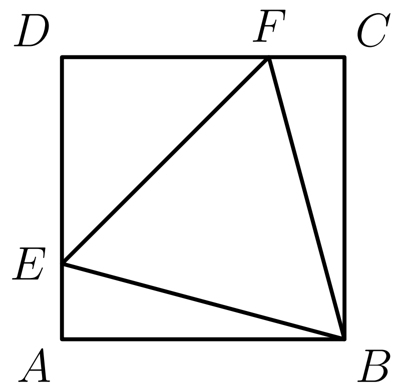
The area is therefore

$$3 \times 80 = 240$$

square feet.

Thus, the correct answer is **C**.

20. Points E and F are located on square $ABCD$ so that $\triangle BEF$ is equilateral. What is the ratio of the area of $\triangle DEF$ to that of $\triangle ABE$?



A $\frac{4}{3}$

B $\frac{3}{2}$

C $\sqrt{3}$

D 2

E $1+\sqrt{3}$

Solution:

Let the square have side 1, and by symmetry let $ED = DF = x$, so $AE = 1 - x$.

Since $\triangle BEF$ is equilateral, $EF^2 = EB^2$, giving

$$2x^2 = 1 + (1-x)^2,$$

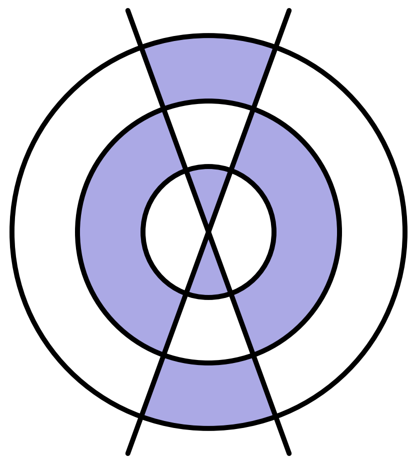
which simplifies to $x^2 = 2(1-x)$.

The right triangles have areas $[DEF] = \frac{1}{2}x^2$ and $[ABE] = \frac{1}{2}(1-x)$, so

$$\frac{[DEF]}{[ABE]} = \frac{x^2}{1-x} = \frac{2(1-x)}{1-x} = 2.$$

Thus, the correct answer is **D**.

21. Two distinct lines pass through the center of three concentric circles of radii $3, 2,$ and 1. The area of the shaded region in the diagram is $\frac{8}{13}$ of the area of the unshaded region. What is the radian measure of the acute angle formed by the two lines? (Note: π radians is 180 degrees.)



A $\frac{\pi}{8}$

B $\frac{\pi}{7}$

C $\frac{\pi}{6}$

D $\frac{\pi}{5}$

E $\frac{\pi}{4}$

Solution:

Let θ be the acute angle. The shaded region has three parts: two acute sectors of the unit disk with total area θ, two obtuse sectors of the ring between radii 1 and 2 with total area $3(\pi - \theta)$, and two acute sectors of the ring between radii 2 and 3 with total area 5θ.

Adding these gives a shaded area of

$$\theta + 3(\pi - \theta) + 5\theta = 3\pi + 3\theta.$$

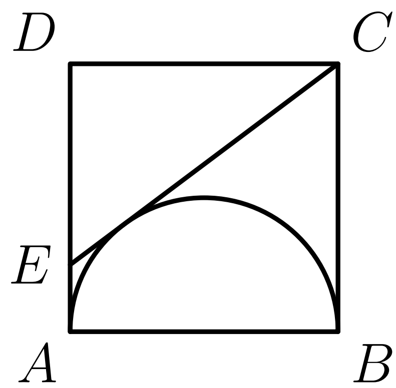
The shaded region is $\frac{8}{13}$ of the unshaded region, so it is $\frac{8}{21}$ of the total area 9π. Then

$$3\pi + 3\theta = \frac{8}{21}(9\pi) = \frac{24\pi}{7},$$

which gives $\theta = \frac{\pi}{7}$.

Thus, the correct answer is **B**.

22. Square $ABCD$ has side length $2.$ A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at $E.$ What is the length of $\overline{CE}$?



A $\dfrac{2+\sqrt{5}}{2}$

B $\sqrt{5}$

C $\sqrt{6}$

D $\dfrac{5}{2}$

E $5-\sqrt{5}$

Solution:

Let F be the point where CE touches the semicircle and let $x = AE.$ Since tangents from a point are equal, $CF = CB = 2$ and $EF = EA = x,$ so $CE = 2 + x.$

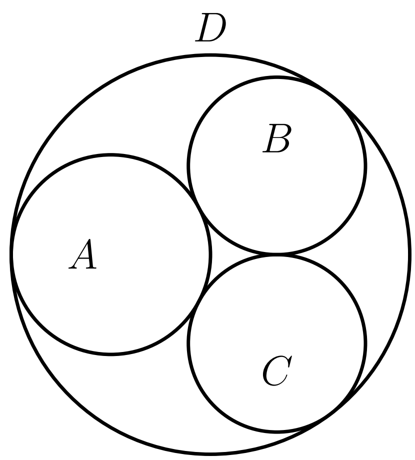
In right triangle $CDE,$ we have $DE = 2 - x$ and $DC = 2,$ so

$$(2-x)^2 + 2^2 = (2+x)^2.$$

This gives $x = \dfrac{1}{2},$ hence $CE = 2 + \dfrac{1}{2} = \dfrac{5}{2}.$

Thus, the correct answer is **D**.

23. Circles $A, B,$ and C are externally tangent to each other and internally tangent to circle D. Circles B and C are congruent. Circle A has radius 1 and passes through the center of D. What is the radius of circle B?



A $\frac{2}{3}$

B $\frac{\sqrt{3}}{2}$

C $\frac{7}{8}$

D $\frac{8}{9}$

E $\frac{1+\sqrt{3}}{3}$

Solution:

Because circle A passes through D's center and is internally tangent to D, circle D has radius 2. Place D's center at the origin and A's center at $(-1, 0)$.

Let circle B have radius r and center (x, r), using the symmetry of B and C about the horizontal axis. Tangency gives

$$(x+1)^2 + r^2 = (1+r)^2, \qquad x^2 + r^2 = (2-r)^2.$$

Subtracting yields $x = 3r - 2$. Substituting into the second equation gives $9r^2 - 8r = 0$, so $r = \frac{8}{9}$.

Thus, the correct answer is **D**.

24. Let $a_1, a_2, \ldots$ be a sequence with the following properties: $a_1 = 1$, and $a_{2n} = n \cdot a_n$ for any positive integer n. What is the value of $a_{2^{100}}$?

- A 1
- B 2^{99}
- C 2^{100}
- D 2^{4950}
- E 2^{9999}

Solution:

Applying the rule repeatedly,

$$a_{2^1} = 2^0, \quad a_{2^2} = 2^1, \quad a_{2^3} = 2^{1+2}, \quad a_{2^4} = 2^{1+2+3}, \ldots$$

so in general $a_{2^n} = 2^{1+2+\cdots+(n-1)} = 2^{n(n-1)/2}$.

For $n = 100$, the exponent is $\frac{100 \cdot 99}{2} = 4950$, so $a_{2^{100}} = 2^{4950}$.

Thus, the correct answer is **D**.

25. Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

A $3+\dfrac{\sqrt{30}}{2}$

B $3+\dfrac{\sqrt{69}}{3}$

C $3+\dfrac{\sqrt{123}}{4}$

D $\dfrac{52}{9}$

E $3+2\sqrt{2}$

Solution:

The three small centers form an equilateral triangle of side 2, each 1 unit above the plane. Its centroid D is at distance $\dfrac{2\sqrt{3}}{3}$ from each vertex.

The large sphere's center E sits directly above D, and the distance between E and a small center is $1+2=3$. Thus

$$DE=\sqrt{3^2-\left(\frac{2\sqrt{3}}{3}\right)^2}=\sqrt{9-\frac{4}{3}}=\frac{\sqrt{69}}{3}.$$

Adding the 1 unit from the plane to D and the 2 units from E to the top of the large sphere gives

$$1+\frac{\sqrt{69}}{3}+2=3+\frac{\sqrt{69}}{3}.$$

Thus, the correct answer is **B**.

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