2003 AMC 10B Solutions

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1. Which of the following is the same as

$$\frac{2-4+6-8+10-12+14}{3-6+9-12+15-18+21}?$$

A -1

B $-\frac{2}{3}$

C $\frac{2}{3}$

D 1

E $\frac{14}{3}$

Solution:

Factoring gives

$$\frac{2(1-2+3-4+5-6+7)}{3(1-2+3-4+5-6+7)}.$$

The identical alternating sums cancel, leaving $\frac{2}{3}$.

Thus, the correct answer is **C**.

2. Al gets the disease algebritis and must take one green pill and one pink pill each day for two weeks. A green pill costs $\$1$ more than a pink pill, and Al's pills cost a total of $\$546$ for the two weeks. How much does one green pill cost?

- A $\$7$
- B $\$14$
- C $\$19$
- D $\$20$
- E $\$39$

Solution:

Each day's pills cost $546 \div 14 = 39$ dollars. If x is the cost of a green pill, then the pink pill costs $x - 1$, so $x + (x - 1) = 39$. Solving gives $x = 20$.

Thus, the correct answer is **D**.

3. The sum of 5 consecutive even integers is 4 less than the sum of the first 8 consecutive odd counting numbers. What is the smallest of the even integers?

- A 6
- B 8
- C 10
- D 12
- E 14

Solution:

The first 8 odd counting numbers sum to $1 + 3 + \cdots + 15 = 64.$

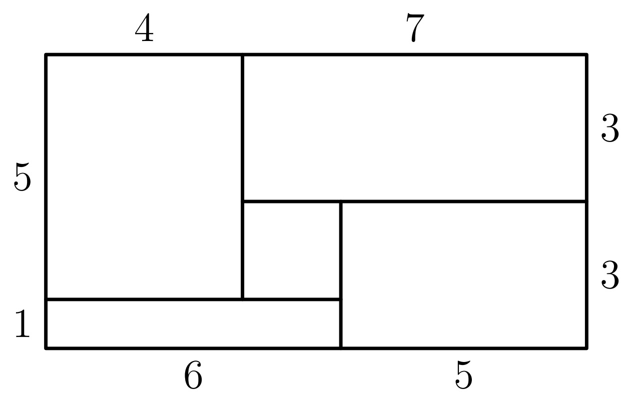
Letting n be the smallest even integer,

$$n + (n + 2) + (n + 4) + (n + 6) + (n + 8) = 5n + 20 = 60,$$

so $n = 8.$

Thus, the correct answer is **B**.

4. Rose fills each of the rectangular regions of her rectangular flower bed with a different type of flower. The lengths, in feet, of the rectangular regions in her flower bed are as shown in the figure. She plants one flower per square foot in each region. Asters cost $\$1$ each, begonias $\$1.50$ each, cannas $\$2$ each, dahlias $\$2.50$ each, and Easter lilies $\$3$ each. What is the least possible cost, in dollars, for her garden?



A 108

B 115

C 132

D 144

E 156

Solution:

The five regions have areas $4, 6, 15, 20,$ and 21 square feet.

Cost is minimized by placing the most expensive flower in the smallest region, so the total is

$$(3)(4) + (2.5)(6) + (2)(15) + (1.5)(20) + (1)(21) = 108.$$

Thus, the correct answer is **A**.

5. Moe uses a mower to cut his rectangular 90-foot by 150-foot lawn. The swath he cuts is 28 inches wide, but he overlaps each cut by 4 inches to make sure that no grass is missed. He walks at the rate of 5000 feet per hour while pushing the mower. Which of the following is closest to the number of hours it will take Moe to mow his lawn?

- A 0.75
- B 0.8
- C 1.35
- D 1.5
- E 3

Solution:

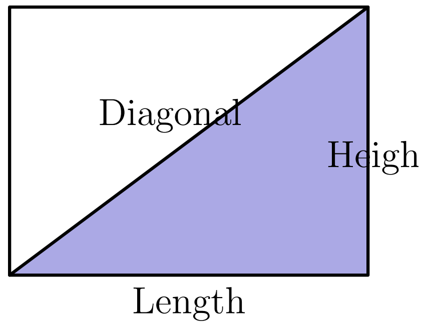
The lawn has area $90 \cdot 150 = 13{,}500$ square feet.

Each foot Moe walks mows an effective strip 2 feet wide, so he mows $2 \cdot 5000 = 10{,}000$ square feet per hour. The time needed is

$$\frac{13{,}500}{10{,}000} = 1.35 \text{ hours.}$$

Thus, the correct answer is **C**.

6. Many television screens are rectangles that are measured by the length of their diagonals. The ratio of the horizontal length to the height in a standard television screen is $4 : 3$. The horizontal length of a "27-inch" television screen is closest, in inches, to which of the following?



A 20

B 20.5

C 21

D 21.5

E 22

Solution:

Since the length and height are in ratio $4 : 3$, the length, height, and diagonal form a $4 : 3 : 5$ right triangle. The diagonal is 27, so the horizontal length is

$$\frac{4}{5}(27) = 21.6,$$

which is closest to 21.5.

Thus, the correct answer is **D**.

7. The symbolism $\lfloor x \rfloor$ denotes the largest integer not exceeding x. For example, $\lfloor 3 \rfloor = 3$, and $\lfloor 9/2 \rfloor = 4$. Compute

$$\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{16} \rfloor.$$

A 35

B 38

C 40

D 42

E 136

Solution:

The value is 1 for $n = 1, 2, 3$; it is 2 for $n = 4, \ldots, 8$; it is 3 for $n = 9, \ldots, 15$; and it is 4 for $n = 16$. The sum is

$$3 \cdot 1 + 5 \cdot 2 + 7 \cdot 3 + 1 \cdot 4 = 38.$$

Thus, the correct answer is **B**.

8. The second and fourth terms of a geometric sequence are 2 and 6. Which of the following is a possible first term?

A $-\sqrt{3}$

B $-\frac{2\sqrt{3}}{3}$

C $-\frac{\sqrt{3}}{3}$

D $\sqrt{3}$

E 3

Solution:

Let the terms be $a, ar, ar^2, ar^3, \ldots$ with $ar = 2$ and $ar^3 = 6$. Dividing gives $r^2 = 3$, so $r = \pm\sqrt{3}$.

Then $a = \frac{2}{r} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$. The choice $-\frac{2\sqrt{3}}{3}$ matches the negative case.

Thus, the correct answer is **B**.

9. Find the value of x that satisfies the equation

$$25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}.$$

- A 2
- B 3
- C 5
- D 6
- E 9

Solution:

Writing everything base 5, the left side is 5^{-4} and the right side is

$$5^{(48-26-34)/x} = 5^{-12/x}.$$

Setting exponents equal, $-4 = -\dfrac{12}{x}$, so $x = 3.$

Thus, the correct answer is **B**.

10. Nebraska, the home of the AMC, changed its license plate scheme. Each old license plate consisted of a letter followed by four digits. Each new license plate consists of three letters followed by three digits. By how many times is the number of possible license plates increased?

A $\frac{26}{10}$

B $\frac{26^2}{10^2}$

C $\frac{26^2}{10}$

D $\frac{26^3}{10^3}$

E $\frac{26^3}{10^2}$

Solution:

The old scheme allows $26 \cdot 10^4$ plates and the new scheme allows $26^3 \cdot 10^3$ plates. The increase factor is

$$\frac{26^3 \cdot 10^3}{26 \cdot 10^4} = \frac{26^2}{10}.$$

Thus, the correct answer is **C**.

11. A line with slope 3 intersects a line with slope 5 at the point $(10, 15)$. What is the distance between the x-intercepts of these two lines?

A 2

B 5

C 7

D 12

E 20

Solution:

The lines are $y - 15 = 3(x - 10)$ and $y - 15 = 5(x - 10)$.

Setting $y = 0$ gives x-intercepts $x = 5$ and $x = 7$. The distance between $(5, 0)$ and $(7, 0)$ is 2.

Thus, the correct answer is **A**.

12. Al, Betty, and Clare split $\$1000$ among them to be invested in different ways. Each begins with a different amount. At the end of one year they have a total of $\$1500.$ Betty and Clare have both doubled their money, whereas Al has managed to lose $\$100.$ What was Al's original portion?

A $\$250$

B $\$350$

C $\$400$

D $\$450$

E $\$500$

Solution:

Let a, b, c be the original portions. Then $a + b + c = 1000$ and $(a - 100) + 2(b + c) = 1500.$

Substituting $b + c = 1000 - a$ into the second equation,

$$a - 100 + 2(1000 - a) = 1500,$$

so $a = 400.$

Thus, the correct answer is **C**.

13. Let $\clubsuit(x)$ denote the sum of the digits of the positive integer x. For example, $\clubsuit(8) = 8$ and $\clubsuit(123) = 1 + 2 + 3 = 6$. For how many two-digit values of x is $\clubsuit(\clubsuit(x)) = 3$?

A 3

B 4

C 6

D 9

E 10

Solution:

Let $y = \clubsuit(x)$. Since $x \leq 99$, we have $y \leq 18$, so $\clubsuit(y) = 3$ forces $y = 3$ or $y = 12$.

The two-digit numbers with digit sum 3 are $12, 21, 30$ (3 of them). Those with digit sum 12 are $39, 48, 57, 66, 75, 84, 93$ (7 of them). In all there are 10.

Thus, the correct answer is **E**.

14. Given that $3^8 \cdot 5^2 = a^b$, where both a and b are positive integers, find the smallest possible value for $a + b$.

- A 25
- B 34
- C 351
- D 407
- E 900

Solution:

Because a must be divisible by 5, and $3^8 \cdot 5^2$ is divisible by 5^2 but not 5^3, we need $b \leq 2$.

Taking $b = 2$ gives $a = \sqrt{3^8 \cdot 5^2} = 3^4 \cdot 5 = 405$, so $a + b = 407$. This beats $b = 1$, which gives $a + b = 164{,}026$.

Thus, the correct answer is **D**.

15. There are 100 players in a singles tennis tournament. The tournament is single elimination, meaning that a player who loses a match is eliminated. In the first round, the strongest 28 players are given a bye, and the remaining 72 players are paired off to play. After each round, the remaining players play in the next round. The match continues until only one player remains unbeaten. The total number of matches played is

- A a prime number
- B divisible by 2
- C divisible by 5
- D divisible by 7
- E divisible by 11

Solution:

Each match eliminates exactly one player. Since 100 players start and all but the champion are eliminated, there are 99 matches.

Because $99 = 9 \cdot 11$, it is divisible by 11 but satisfies none of the other options.

Thus, the correct answer is **E**.

16. A restaurant offers three desserts, and exactly twice as many appetizers as main courses. A dinner consists of an appetizer, a main course, and a dessert. What is the least number of main courses that the restaurant should offer so that a customer could have a different dinner each night in the year 2003?

A 4

B 5

C 6

D 7

E 8

Solution:

With m main courses, the number of dinners is $3 \cdot m \cdot 2m = 6m^2$. This must be at least 365.

So $m^2 \geq \dfrac{365}{6} \approx 60.8$. Since $7^2 = 49$ is too small but $8^2 = 64$ works, $m = 8$.

Thus, the correct answer is **E**.

17. An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. What is the ratio of the cone's height to its radius? (Note: A cone with radius r and height h has volume $\pi r^2 h/3$, and a sphere with radius r has volume $4\pi r^3/3$.)

- A $2:1$
- B $3:1$
- C $4:1$
- D $16:3$
- E $6:1$

Solution:

The melted volume equals the cone's volume, so

$$\frac{3}{4} \cdot \frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^2 h.$$

Simplifying gives $\pi r^3 = \dfrac{1}{3}\pi r^2 h$, so $h = 3r$. The ratio of height to radius is $3:1$.

Thus, the correct answer is **B**.

18. What is the largest integer that is a divisor of

$$(n+1)(n+3)(n+5)(n+7)(n+9)$$

for all positive even integers n?

A 3

B 5

C 11

D 15

E 165

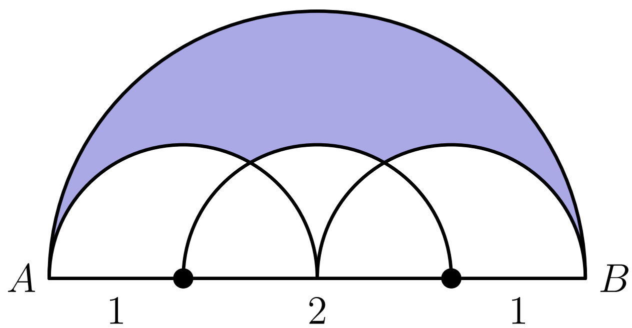
Solution:

For even n, the factors are five consecutive odd numbers. Among any five consecutive odd numbers, at least one is divisible by 3 and exactly one by 5, so the product is always divisible by 15.

No larger fixed divisor works: $n = 2$ gives $3 \cdot 5 \cdot 7 \cdot 9 \cdot 11$, whose greatest common divisor with other cases such as $n = 10$ is exactly 15.

Thus, the correct answer is **D**.

19. Three semicircles of radius 1 are constructed on diameter $\overline{AB}$ of a semicircle of radius 2. The centers of the small semicircles divide $\overline{AB}$ into four line segments of equal length, as shown. What is the area of the shaded region that lies within the large semicircle but outside the smaller semicircles?



A $\pi - \sqrt{3}$

B $\pi - \sqrt{2}$

C $\dfrac{\pi + \sqrt{2}}{2}$

D $\dfrac{\pi + \sqrt{3}}{2}$

E $\dfrac{7}{6}\pi - \dfrac{\sqrt{3}}{2}$

Solution:

The large semicircle has area $\frac{1}{2}\pi(2)^2 = 2\pi$.

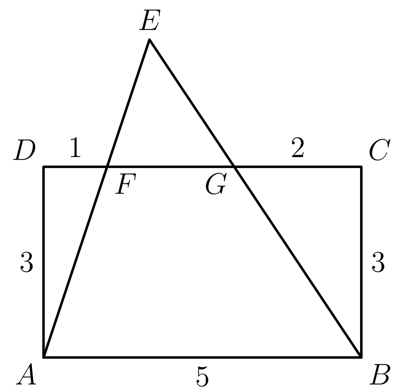
Removing the small semicircles deletes a region equal to five congruent 60° sectors of radius 1 plus two equilateral triangles of side 1. Each sector has area $\frac{\pi}{6}$ and each triangle has area $\frac{\sqrt{3}}{4}$.

The shaded area is

$$2\pi - 5 \cdot \frac{\pi}{6} - 2 \cdot \frac{\sqrt{3}}{4} = \frac{7}{6}\pi - \frac{\sqrt{3}}{2}.$$

Thus, the correct answer is **E**.

20. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of $\triangle AEB$.



A 10

B $\frac{21}{2}$

C 12

D $\frac{25}{2}$

E 15

Solution:

Here $FG = CD - DF - GC = 5 - 1 - 2 = 2$. Let h be the distance from E down to line CD. Since $\triangle FEG \sim \triangle AEB$ with ratio $\frac{FG}{AB} = \frac{2}{5}$, we have

$$\frac{h}{h+3} = \frac{2}{5},$$

so $h = 2$.

The height of $\triangle AEB$ from E to AB is $h + 3 = 5$, giving area $\frac{1}{2} \cdot 5 \cdot 5 = \frac{25}{2}$.

Thus, the correct answer is **D**.

21. A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out. What is the probability that all beads in the bag are red after three such replacements?

A $\frac{1}{8}$

B $\frac{5}{32}$

C $\frac{9}{32}$

D $\frac{3}{8}$

E $\frac{7}{16}$

Solution:

The bag always holds 4 beads. All are red at the end precisely when both greens are drawn.

Drawing green then green has probability $\frac{2}{4} \cdot \frac{1}{4} = \frac{1}{8}$. Green, red, green has probability $\frac{2}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{32}$. Red, green, green has probability $\frac{2}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

The total is

$$\frac{1}{8} + \frac{3}{32} + \frac{1}{16} = \frac{9}{32}.$$

Thus, the correct answer is **C**.

22. A clock chimes once at 30 minutes past each hour and chimes on the hour according to the hour. For example, at 1 PM there is one chime and at noon and midnight there are twelve chimes. Starting at 11:15 AM on February 26, 2003, on what date will the 2003rd chime occur?

- A March 8
- B March 9
- C March 10
- D March 20
- E March 21

Solution:

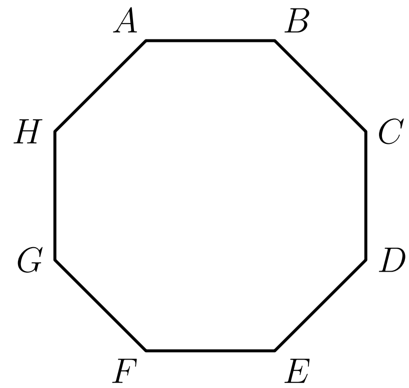
Each 12-hour period has $12 + 78 = 90$ chimes, so each full day has $180.$

After 11:15 AM on February 26, the rest of that day has 91 chimes. Adding 180 for each following full day, the count reaches 1891 by the end of March 8.

The remaining chimes fall on March 9: the 2003rd chime is the 112th chime of that day, occurring at 3:30 PM on March 9.

Thus, the correct answer is **B**.

23. A regular octagon $ABCDEFGH$ has an area of one square unit. What is the area of the rectangle $ABEF$?



A $1-\frac{\sqrt{2}}{2}$

B $\frac{\sqrt{2}}{4}$

C $\sqrt{2}-1$

D $\frac{1}{2}$

E $\frac{1+\sqrt{2}}{4}$

Solution:

Let O be the center. The octagon splits into 8 congruent triangles from O, so $\triangle AOB$ has area $\frac{1}{8}$.

Since O is the midpoint of AE, triangles AOB and BOE have equal area, so $\triangle ABE$ has area $\frac{1}{4}$. The rectangle $ABEF$ is twice this, namely $\frac{1}{2}$.

Thus, the correct answer is **D**.

24. The first four terms in an arithmetic sequence are $x+y, x-y, xy,$ and $x/y,$ in that order. What is the fifth term?

A $-\frac{15}{8}$

B $-\frac{6}{5}$

C 0

D $\frac{27}{20}$

E $\frac{123}{40}$

Solution:

The common difference is $-2y,$ so the third and fourth terms must be $x-3y$ and $x-5y.$ Thus $xy=x-3y$ and $\frac{x}{y}=x-5y.$

From $\frac{x}{y}=x-5y$ we get $x=xy-5y^2,$ and substituting $xy=x-3y$ gives $-3y-5y^2=0.$ Since $y\neq 0, y=-\frac{3}{5}$ and then $x=-\frac{9}{8}.$

The fifth term is $x-7y=-\frac{9}{8}+\frac{21}{5}=\frac{123}{40}.$

Thus, the correct answer is **E**.

25. How many distinct four-digit numbers are divisible by 3 and have 23 as their last two digits?

A 27

B 30

C 33

D 81

E 90

Solution:

Write the number as $\overline{ab23}$. It is divisible by 3 when $a+b+2+3=a+b+5$ is divisible by 3, that is, when $a+b \equiv 1 \pmod{3}$.

The two-digit prefix $\overline{ab}$ ranges over the 90 values from 10 to 99, and exactly one third of them satisfy this, giving $\frac{90}{3}=30$.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc10/2003B