2003 AMC 10A Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc10/2003A/solutions

1. What is the difference between the sum of the first 2003 even counting numbers and the sum of the first 2003 odd counting numbers?

   A 0

   B 1

   C 2

   D 2003

   E 4006

Solution:

The kth even number $2k$ is exactly 1 more than the kth odd number $2k-1$.

Summing this difference over all 2003 pairs gives $2003 \cdot 1 = 2003$.

Thus, the correct answer is **D**.

2. Members of the Rockham Soccer League buy socks and T-shirts. Socks cost $\$4$ per pair and each T-shirt costs $\$5$ more than a pair of socks. Each member needs one pair of socks and a shirt for home games and another pair of socks and a shirt for away games. If the total cost is $\$2366$, how many members are in the League?

- A 77
- B 91
- C 143
- D 182
- E 286

Solution:

Each T-shirt costs $\$4 + \$5 = \$9$.

Each member needs two pairs of socks and two shirts, costing $2 \cdot 4 + 2 \cdot 9 = \$26$.

The number of members is $2366 \div 26 = 91$.

Thus, the correct answer is **B**.

3. A solid box is 15 cm by 10 cm by 8 cm. A new solid is formed by removing a cube 3 cm on a side from each corner of this box. What percent of the original volume is removed?

A 4.5

B 9

C 12

D 18

E 24

Solution:

The eight removed cubes have total volume $8 \cdot 3^3 = 216$ cubic centimeters.

The original box has volume $15 \cdot 10 \cdot 8 = 1200$ cubic centimeters.

The percent removed is $\frac{216}{1200} \cdot 100\% = 18\%.$

Thus, the correct answer is **D**.

4. It takes Mary 30 minutes to walk uphill 1 km from her home to school, but it takes her only 10 minutes to walk from school to home along the same route. What is her average speed, in km/hr, for the round trip?

- A 3
- B 3.125
- C 3.5
- D 4
- E 4.5

Solution:

Mary walks a total of 2 km in $30 + 10 = 40$ minutes.

Since 40 minutes is $\frac{2}{3}$ hour, her average speed is $2 \div \frac{2}{3} = 3$ km/hr.

Thus, the correct answer is **A**.

5. Let d and e denote the solutions of $2x^2 + 3x - 5 = 0$. What is the value of $(d - 1)(e - 1)$?

- A $-\frac{5}{2}$
- B 0
- C 3
- D 5
- E 6

Solution:

Factoring gives $2x^2 + 3x - 5 = (2x + 5)(x - 1)$, so the roots are $-\frac{5}{2}$ and 1.

Since one root equals 1, the factor $(e - 1) = 0$, making the product 0.

Thus, the correct answer is **B**.

6. Define $x \heartsuit y$ to be $|x - y|$ for all real numbers x and y. Which of the following statements is not true?

- A $x \heartsuit y = y \heartsuit x$ for all x and y
- B $2(x \heartsuit y) = (2x) \heartsuit (2y)$ for all x and y
- C $x \heartsuit 0 = x$ for all x
- D $x \heartsuit x = 0$ for all x
- E $x \heartsuit y > 0$ if $x \neq y$

Solution:

Statement (C) claims $x \heartsuit 0 = x$, but $x \heartsuit 0 = |x - 0| = |x|$, which fails for negative x. For example, $-1 \heartsuit 0 = 1 \neq -1$.

The remaining statements all follow directly from the properties of absolute value.

Thus, the correct answer is **C**.

7. How many non-congruent triangles with perimeter 7 have integer side lengths?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

The longest side cannot exceed $3,$ since otherwise the other two sides could not reach it.

The only possibilities are side lengths 1-3-3 and 2-2-$3,$ giving 2 triangles.

Thus, the correct answer is **B**.

8. What is the probability that a randomly drawn positive factor of 60 is less than 7?

A $\frac{1}{10}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

The factors of 60 are $1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30,$ and $60.$

Six of these twelve factors are less than 7, so the probability is $\frac{6}{12} = \frac{1}{2}.$

Thus, the correct answer is **E**.

9. Simplify

$$\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt{x}}}}.$$

A $\sqrt{x}$

B $\sqrt[3]{x^2}$

C $\sqrt[27]{x^2}$

D $\sqrt[54]{x}$

E $\sqrt[81]{x^{80}}$

Solution:

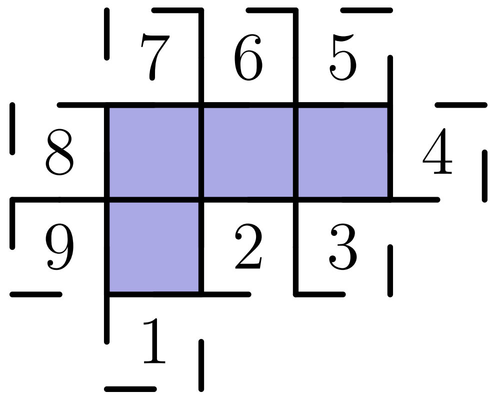
Working outward, $x\sqrt{x} = x^{3/2}$, and its cube root is $x^{1/2}$.

Then $x \cdot x^{1/2} = x^{3/2}$, whose cube root is again $x^{1/2}$.

Repeating once more, $x \cdot x^{1/2} = x^{3/2}$, whose cube root is $x^{1/2} = \sqrt{x}$.

Thus, the correct answer is **A**.

10. The polygon enclosed by the solid lines in the figure consists of 4 congruent squares joined edge-to-edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing?



A 2

B 3

C 4

D 5

E 6

Solution:

When the four given squares are folded, two pairs of their edges meet to form a band of four faces around the cube, leaving two faces open.

The fifth square folds into one of these open faces exactly when it is attached along a free edge. This works for 6 of the 9 positions; the other 3 would fold onto a face already covered.

Thus, the correct answer is **E**.

11. The sum of the two 5-digit numbers $AMC10$ and $AMC12$ is 123422. What is $A + M + C$?

A 10

B 11

C 12

D 13

E 14

Solution:

The two numbers equal $100 \cdot \overline{AMC} + 10$ and $100 \cdot \overline{AMC} + 12$, so their sum is $200 \cdot \overline{AMC} + 22 = 123422$.

Then $200 \cdot \overline{AMC} = 123400$, so $\overline{AMC} = 617$.

Therefore $A + M + C = 6 + 1 + 7 = 14$.

Thus, the correct answer is **E**.

12. A point (x, y) is randomly picked from inside the rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 1)$, and $(0, 1)$. What is the probability that $x < y$?

A $\frac{1}{8}$

B $\frac{1}{4}$

C $\frac{3}{8}$

D $\frac{1}{2}$

E $\frac{3}{4}$

Solution:

The condition $x < y$ holds in the triangle bounded by $y = x$, $y = 1$, and $x = 0$, which has vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$.

This triangle has area $\frac{1}{2}$, while the rectangle has area 4.

The probability is $\frac{1/2}{4} = \frac{1}{8}$.

Thus, the correct answer is **A**.

13. The sum of three numbers is 20. The first is 4 times the sum of the other two. The second is seven times the third. What is the product of all three?

A 28

B 40

C 100

D 400

E 800

Solution:

Let the numbers be a, b, c. Since $a = 4(b + c)$, we get $4(b + c) + (b + c) = 20$, so $b + c = 4$ and $a = 16$.

With $b = 7c$, we have $7c + c = 4$, so $c = \frac{1}{2}$ and $b = \frac{7}{2}$.

The product is $16 \cdot \frac{7}{2} \cdot \frac{1}{2} = 28$.

Thus, the correct answer is **A**.

14. Let n be the largest integer that is the product of exactly 3 distinct prime numbers, d, e, and $10d + e$, where d and e are single digits. What is the sum of the digits of n?

A 12

B 15

C 18

D 21

E 24

Solution:

Both d and e are single-digit primes, and $10d + e$ must be prime. Testing the largest options, 75 and 57 are not prime.

Using $d = 7, e = 3$ gives the prime 73, and $n = 7 \cdot 3 \cdot 73 = 1533.$

The sum of its digits is $1 + 5 + 3 + 3 = 12.$

Thus, the correct answer is **A**.

15. What is the probability that an integer in the set $\{1, 2, 3, ..., 100\}$ is divisible by 2 and not divisible by 3?

A $\frac{1}{6}$

B $\frac{33}{100}$

C $\frac{17}{50}$

D $\frac{1}{2}$

E $\frac{18}{25}$

Solution:

Of the 100 integers, 50 are divisible by 2.

Among those, the ones also divisible by 3 are the multiples of 6, of which there are 16.

So $50 - 16 = 34$ qualify, giving probability $\frac{34}{100} = \frac{17}{50}$.

Thus, the correct answer is **C**.

16. What is the units digit of 13^{2003}?

A 1

B 3

C 7

D 8

E 9

Solution:

The units digit of 13^{2003} matches that of 3^{2003}.

Powers of 3 have units digits cycling $3, 9, 7, 1$ with period 4.

Since $2003 = 4 \cdot 500 + 3$, the units digit is the third in the cycle, which is 7.

Thus, the correct answer is **C**.

17. The number of inches in the perimeter of an equilateral triangle equals the number of square inches in the area of its circumscribed circle. What is the radius, in inches, of the circle?

A $\frac{3\sqrt{2}}{\pi}$

B $\frac{3\sqrt{3}}{\pi}$

C $\sqrt{3}$

D $\frac{6}{\pi}$

E $\sqrt{3}\pi$

Solution:

Let the side length be s and the circumradius be R. From a 30-60-90 triangle formed by the center and a side, $R = \frac{s}{\sqrt{3}}$, so $s = R\sqrt{3}$.

The perimeter is $3s = 3R\sqrt{3}$ and the circle's area is πR^2.

Setting them equal, $3R\sqrt{3} = \pi R^2$, so $R = \frac{3\sqrt{3}}{\pi}$.

Thus, the correct answer is **B**.

18. What is the sum of the reciprocals of the roots of the equation

$$\frac{2003}{2004}x + 1 + \frac{1}{x} = 0?$$

A $-\frac{2004}{2003}$

B -1

C $\frac{2003}{2004}$

D 1

E $\frac{2004}{2003}$

Solution:

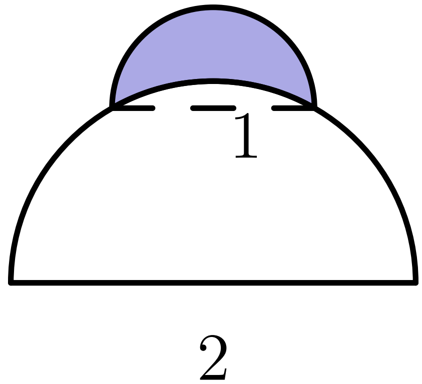
Let $a = \frac{2003}{2004}$. Multiplying the equation by x gives $ax^2 + x + 1 = 0$.

If the roots are r and s, then by Vieta's formulas $r + s = -\frac{1}{a}$ and $rs = \frac{1}{a}$.

The sum of reciprocals is $\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} = \frac{-1/a}{1/a} = -1$.

Thus, the correct answer is **B**.

19. A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a *lune*. Determine the area of this lune.



A $\frac{1}{6}\pi - \frac{\sqrt{3}}{4}$

B $\frac{\sqrt{3}}{4} - \frac{1}{12}\pi$

C $\frac{\sqrt{3}}{4} - \frac{1}{24}\pi$

D $\frac{\sqrt{3}}{4} + \frac{1}{24}\pi$

E $\frac{\sqrt{3}}{4} + \frac{1}{12}\pi$

Solution:

The small semicircle's diameter is a chord of length 1 in the large circle. Joining its endpoints to the large circle's center gives an equilateral triangle of side 1 and area $\frac{\sqrt{3}}{4}$.

The region between the chord and the small arc, taken together with that triangle, has area $\frac{\sqrt{3}}{4} + \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\pi}{8}$.

Subtracting the 60° sector of the large circle, of area $\frac{1}{6}\pi(1)^2 = \frac{\pi}{6}$, leaves the lune: $\frac{\sqrt{3}}{4} + \frac{\pi}{8} - \frac{\pi}{6} = \frac{\sqrt{3}}{4} - \frac{\pi}{24}$.

Thus, the correct answer is **C**.

20. A base-10 three-digit number n is selected at random. Which of the following is closest to the probability that the base-9 representation and the base-11 representation of n are both three-digit numerals?

A 0.3

B 0.4

C 0.5

D 0.6

E 0.7

Solution:

The largest three-digit base-9 number is $9^3 - 1 = 728$, and the smallest three-digit base-11 number is $11^2 = 121$.

So both conditions hold exactly when $121 \le n \le 728$, giving 608 integers.

Out of 900 three-digit numbers, the probability is $\frac{608}{900} \approx 0.7$.

Thus, the correct answer is **E**.

21. Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

- A 22
- B 25
- C 27
- D 28
- E 729

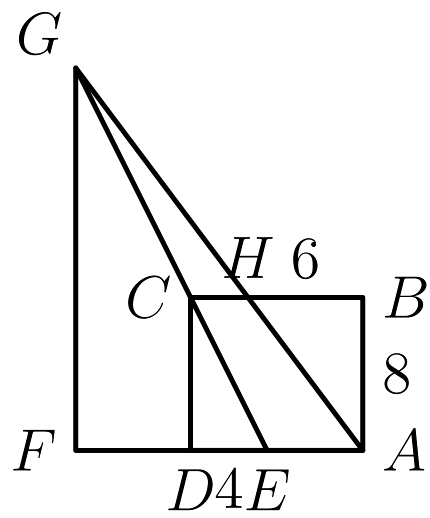
Solution:

An assortment is determined by how many of each type are chosen, so we count nonnegative integer solutions to $a + b + c = 6$.

By stars and bars, placing 2 dividers among 8 slots gives $\binom{8}{2} = 28$ assortments.

Thus, the correct answer is **D**.

22. In rectangle $ABCD$, we have $AB = 8, BC = 9, H$ is on $\overline{BC}$ with $BH = 6, E$ is on AD with $DE = 4,$ line EC intersects line AH at $G,$ and F is on line AD with $\overline{GF} \perp \overline{AF}$. Find the length $\overline{GF}$.



A 16

B 20

C 24

D 28

E 30

Solution:

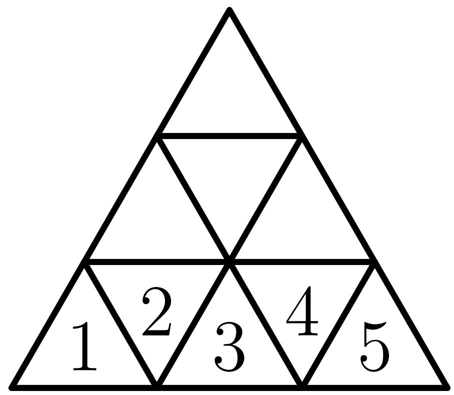
Place $D = (0,0), A = (9,0), B = (9,8), C = (0,8), H = (3,8),$ and $E = (4,0)$.

Line AH has equation $y = -\frac{4}{3}x + 12,$ and line EC has equation $y = -2x + 8.$

Setting them equal gives $x = -6$ and $y = 20,$ so $G = (-6,20)$. Since $\overline{GF}$ is perpendicular to line AD (the x-axis), its length is the height $20.$

Thus, the correct answer is **B**.

23. A large equilateral triangle is constructed by using toothpicks to create rows of small equilateral triangles. For example, in the figure we have 3 rows of small congruent equilateral triangles, with 5 small triangles in the base row. How many toothpicks would be needed to construct a large equilateral triangle if the base row of the triangle consists of 2003 small equilateral triangles?



A $1{,}004{,}004$

B $1{,}005{,}006$

C $1{,}507{,}509$

D $3{,}015{,}018$

E $6{,}021{,}018$

Solution:

A triangle with n rows has $2n-1$ small triangles in its base row, so $2n-1=2003$ gives $n=1002$.

Each row k requires $3k$ toothpicks, so the total is $3(1+2+\cdots+1002)$.

This equals $3\cdot\dfrac{1002\cdot 1003}{2}=1{,}507{,}509.$

Thus, the correct answer is **C**.

24. Sally has five red cards numbered 1 through 5 and four blue cards numbered 3 through 6. She stacks the cards so that the colors alternate and so that the number on each red card divides evenly into the number on each neighboring blue card. What is the sum of the numbers on the middle three cards?

A 8

B 9

C 10

D 11

E 12

Solution:

Among blue cards $3, 4, 5, 6$, red 5 divides only 5 and red 4 divides only 4, so those pairs must sit at the ends.

Red 2 divides only 4 and 6, and red 3 divides only 3 and 6. Chaining these forces the stack $R4, B4, R2, B6, R3, B3, R1, B5, R5.$

The middle three cards are $B6, R3, B3,$ summing to $6 + 3 + 3 = 12.$

Thus, the correct answer is **E**.

25. Let n be a 5-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 100. For how many values of n is $q+r$ divisible by 11?

A 8180

B 8181

C 8182

D 9000

E 9090

Solution:

Write $n = 100q + r = (q+r) + 99q.$

Since $99q$ is a multiple of $11, q+r$ is divisible by 11 if and only if n is.

The 5-digit multiples of 11 satisfy $10000 \leq n \leq 99999,$ and there are $\left\lfloor \frac{99999}{11} \right\rfloor - \left\lfloor \frac{9999}{11} \right\rfloor = 9090 - 909 = 8181.$

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc10/2003A