2002 AMC 10B Solutions

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1. What is the value of the ratio

$$\frac{2^{2001} \cdot 3^{2003}}{6^{2002}}?$$

A $\frac{1}{6}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{2}$

Solution:

Since $6^{2002} = 2^{2002} \cdot 3^{2002}$, the ratio becomes

$$\frac{2^{2001} \cdot 3^{2003}}{2^{2002} \cdot 3^{2002}} = \frac{3}{2}.$$

Thus, the correct answer is **E**.

2. For the nonzero numbers $a, b,$ and c, define

$$(a, b, c) = \frac{abc}{a+b+c}.$$

What is $(2, 4, 6)$?

- A 1
- B 2
- C 4
- D 6
- E 24

Solution:

Substituting directly,

$$(2, 4, 6) = \frac{2 \cdot 4 \cdot 6}{2+4+6} = \frac{48}{12} = 4.$$

Thus, the correct answer is **C**.

3. The arithmetic mean of the nine numbers in the set $\{9, 99, 999, 9999, \ldots, 999999999\}$ is a 9-digit number M, all of whose digits are distinct. Which digit does the number M not contain?

A 0

B 2

C 4

D 6

E 8

Solution:

The mean is

$$\frac{1}{9}(9 + 99 + 999 + \cdots + 999999999) = 1 + 11 + 111 + \cdots + 111111111.$$

Adding these nine repunits column by column gives $M = 123456789.$

The only digit missing from M is $0.$

Thus, the correct answer is **A**.

4. What is the value of

$$(3x-2)(4x+1)-(3x-2)4x+1$$

when $x=4$?

A 0

B 1

C 10

D 11

E 12

Solution:

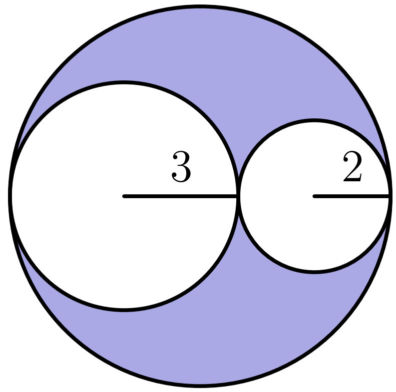
Factoring $3x-2$ from the first two terms,

$$(3x-2)\big[(4x+1)-4x\big]+1=(3x-2)(1)+1=3x-1.$$

At $x=4$, this equals $3\cdot 4-1=11.$

Thus, the correct answer is **D**.

5. Circles of radius 2 and 3 are externally tangent and are circumscribed by a third circle, as shown in the figure. What is the area of the shaded region?



A 3π

B 4π

C 6π

D 9π

E 12π

Solution:

The two small circles line up along a diameter of the big circle, so that diameter is $2 \cdot 3 + 2 \cdot 2 = 10$ and the large radius is 5.

The shaded region is the large disk with the two small disks removed:

$$\pi(5^2) - \pi(3^2) - \pi(2^2) = \pi(25 - 9 - 4) = 12\pi.$$

Thus, the correct answer is **E**.

6. For how many positive integers n is $n^2 - 3n + 2$ a prime number?

- A none
- B one
- C two
- D more than two, but finitely many
- E infinitely many

Solution:

Factor as $n^2 - 3n + 2 = (n-1)(n-2)$.

For $n \geq 4$, both factors exceed 1, so the product is composite. For $n = 1$ and $n = 2$ the value is 0, and for $n = 3$ the value is $(2)(1) = 2$, which is prime.

So exactly one value of n works.

Thus, the correct answer is **B**.

7. Let n be a positive integer such that $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}$ is an integer. Which of the following statements is **not** true?

- A 2 divides n
- B 3 divides n
- C 6 divides n
- D 7 divides n
- E $n > 84$

Solution:

The sum $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}$ is greater than 0 and less than $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+1<2$, so as an integer it must equal 1.

Since $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}=\frac{41}{42}$, we need $\frac{1}{n}=\frac{1}{42}$, so $n=42$.

Then $2, 3, 6,$ and 7 all divide 42, but $n=42$ is not greater than 84. So the false statement is $n>84$.

Thus, the correct answer is **E**.

8. Suppose July of year N has five Mondays. Which of the following must occur five times in August of year N? (Note: both months have 31 days.)

A Monday

B Tuesday

C Wednesday

D Thursday

E Friday

Solution:

A 31-day month is 4 weeks plus 3 extra days, so exactly the weekdays of the 1st, 2nd, and 3rd of the month occur five times.

For July to have five Mondays, Monday must be one of July $1, 2,$ or $3.$ In all three cases August 1 lands on a Wednesday, Thursday, or Friday, and the common weekday among the resulting five-time days is Thursday.

More directly, since July has 31 days, August 1 is the same weekday as July $4.$ With Monday on July $1, 2,$ or $3,$ the three five-time weekdays of August always include Thursday.

Thus, the correct answer is **D**.

9. Using the letters $A, M, O, S,$ and U, we can form 120 five-letter "words." If these "words" are arranged in alphabetical order, then the "word" $USAMO$ occupies which position?

- A 112
- B 113
- C 114
- D 115
- E 116

Solution:

The alphabetical order of the letters is A, M, O, S, U. Words beginning with $A, M,$ $O,$ or S fill positions 1 through 96 (four choices of first letter, 24 each).

Words beginning with U occupy positions 97–120. Listing them alphabetically, $USAMO$ is the 19th such word, so it occupies position $96 + 19 = 115$.

Thus, the correct answer is **D**.

10. Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. What is the pair (a, b)?

- A $(-2, 1)$
- B $(-1, 2)$
- C $(1, -2)$
- D $(2, -1)$
- E $(4, 4)$

Solution:

Since the roots are a and b, Vieta's formulas give $a + b = -a$ and $ab = b$.

From $ab = b$ with $b \neq 0$, we get $a = 1$. Then $a + b = -a$ gives $1 + b = -1$, so $b = -2$.

Thus $(a, b) = (1, -2)$, and the correct answer is **C**.

11. The product of three consecutive positive integers is 8 times their sum. What is the sum of their squares?

A 50

B 77

C 110

D 149

E 194

Solution:

Let the integers be $n-1, n, n+1$. Their product is $n(n^2-1)$ and their sum is $3n$, so

$$n(n^2-1) = 8(3n) = 24n.$$

Since $n \neq 0$, we get $n^2 - 1 = 24$, so $n^2 = 25$ and $n = 5$.

The three integers are $4, 5, 6$, and $4^2 + 5^2 + 6^2 = 16 + 25 + 36 = 77$.

Thus, the correct answer is **B**.

12. For which of the following values of k does the equation

$$\frac{x-1}{x-2} = \frac{x-k}{x-6}$$

have no solution for x?

A 1

B 2

C 3

D 4

E 5

Solution:

Cross multiplying gives $(x-1)(x-6) = (x-2)(x-k)$, which expands to

$$x^2 - 7x + 6 = x^2 - (2+k)x + 2k.$$

Cancelling x^2 leaves $(k-5)x = 2k - 6$, so $x = \dfrac{2k-6}{k-5}$. This has no solution exactly when $k = 5$, since then the coefficient of x is 0 while the right side is nonzero.

Thus, the correct answer is **E**.

13. What value of x makes $8xy - 12y + 2x - 3 = 0$ true for all values of y?

A $\frac{2}{3}$

B $\frac{3}{2}$ or $-\frac{1}{4}$

C $-\frac{2}{3}$ or $-\frac{1}{4}$

D $\frac{3}{2}$

E $-\frac{3}{2}$ or $-\frac{1}{4}$

Solution:

Grouping and factoring,

$$8xy - 12y + 2x - 3 = 4y(2x - 3) + (2x - 3) = (4y + 1)(2x - 3).$$

For this to equal 0 for all y, the factor that depends on y cannot be forced to zero, so we need $2x - 3 = 0$, giving $x = \frac{3}{2}$.

Thus, the correct answer is **D**.

14. The number $25^{64} \cdot 64^{25}$ is the square of a positive integer N. In decimal representation, what is the sum of the digits of N?

A 7

B 14

C 21

D 28

E 35

Solution:

Since $25 = 5^2$ and $64 = 2^6$, we have $25^{64} \cdot 64^{25} = 5^{128} \cdot 2^{150}$, so

$$N = \sqrt{5^{128} \cdot 2^{150}} = 5^{64} \cdot 2^{75}.$$

Writing $2^{75} = 2^{64} \cdot 2^{11}$, we get $N = (5 \cdot 2)^{64} \cdot 2^{11} = 10^{64} \cdot 2048.$

So N is 2048 followed by 64 zeros, and its digit sum is $2 + 0 + 4 + 8 = 14.$

Thus, the correct answer is **B**.

15. The positive integers $A, B, A-B$, and $A+B$ are all prime numbers. The sum of these four primes is

A even

B divisible by 3

C divisible by 5

D divisible by 7

E prime

Solution:

The numbers $A-B$ and $A+B$ differ by $2B$, so they have the same parity. Being prime, they must both be odd, which forces A and B to have opposite parity.

Since 2 is the only even prime, $B=2$. Then $A-2, A, A+2$ are three primes forming an arithmetic progression of odd numbers, which must be $3, 5, 7.$

The four primes are $2, 3, 5, 7,$ and their sum is $17,$ which is prime.

Thus, the correct answer is **E**.

16. For how many integers n is $\frac{n}{20-n}$ the square of an integer?

A 1

B 2

C 3

D 4

E 10

Solution:

Suppose $\frac{n}{20-n} = k^2$ for some integer $k \geq 0$. Solving,

$$n = \frac{20k^2}{k^2+1}.$$

Since k^2 and k^2+1 share no common factor, k^2+1 must divide 20. This happens only for $k = 0, 1, 2, 3$, giving $k^2+1 = 1, 2, 5, 10$.

The corresponding values $n = 0, 10, 16, 18$ are all integers, so there are 4 such n.

Thus, the correct answer is **D**.

17. A regular octagon $ABCDEFGH$ has sides of length two. What is the area of $\triangle ADG$?

A $4+2\sqrt{2}$

B $6+\sqrt{2}$

C $4+3\sqrt{2}$

D $3+4\sqrt{2}$

E $8+\sqrt{2}$

Solution:

Set the octagon on coordinate axes with the axis-aligned sides of length 2 and each slanted side spanning $\sqrt{2}$ horizontally and $\sqrt{2}$ vertically. Then

$$A=(\sqrt{2},0),\quad D=(2+2\sqrt{2},2+\sqrt{2}),\quad G=(0,2+\sqrt{2}).$$

Since D and G share the height $2+\sqrt{2}$, segment DG is horizontal with length $2+2\sqrt{2}$, and the height from A up to that level is $2+\sqrt{2}$.

Therefore

$$[\triangle ADG]=\tfrac{1}{2}(2+2\sqrt{2})(2+\sqrt{2})=(1+\sqrt{2})(2+\sqrt{2})=4+3\sqrt{2}.$$

Thus, the correct answer is **C**.

18. Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?

A 8

B 9

C 10

D 12

E 16

Solution:

Any two distinct circles intersect in at most 2 points. There are $\binom{4}{2} = 6$ pairs of circles, giving at most $6 \cdot 2 = 12$ intersection points.

This maximum is achievable by a configuration where every pair of circles crosses twice, so the answer is $12.$

Thus, the correct answer is **D**.

19. Suppose that $\{a_n\}$ is an arithmetic sequence with

$$a_1 + a_2 + \cdots + a_{100} = 100 \quad \text{and} \quad a_{101} + a_{102} + \cdots + a_{200} = 200.$$

What is the value of $a_2 - a_1$?

- A 0.0001
- B 0.001
- C 0.01
- D 0.1
- E 1

Solution:

Let $d = a_2 - a_1$. Then $a_{k+100} = a_k + 100d$, so the second block sum is the first plus $100 \cdot 100d$:

$$a_{101} + \cdots + a_{200} = (a_1 + \cdots + a_{100}) + 10000d.$$

Therefore $200 = 100 + 10000d$, giving $d = \dfrac{100}{10000} = 0.01.$

Thus, the correct answer is **C**.

20. Let $a, b,$ and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. What is $a^2 - b^2 + c^2$?

A 0

B 1

C 4

D 7

E 8

Solution:

Rewrite the equations as $a + 8c = 4 + 7b$ and $8a - c = 7 - 4b$. Squaring both and adding,

$$(a + 8c)^2 + (8a - c)^2 = (4 + 7b)^2 + (7 - 4b)^2.$$

The left side expands to $65a^2 + 65c^2$ (the ac terms cancel), and the right side expands to $65 + 65b^2$ (the b terms cancel). So

$$65(a^2 + c^2) = 65(1 + b^2),$$

giving $a^2 - b^2 + c^2 = 1$.

Thus, the correct answer is **B**.

21. Andy's lawn has twice as much area as Beth's lawn and three times as much area as Carlos' lawn. Carlos' lawn mower cuts half as fast as Beth's mower and one third as fast as Andy's mower. If they all start to mow their lawns at the same time, who will finish first?

- A Andy
- B Beth
- C Carlos
- D Andy and Carlos tie for first.
- E All three tie.

Solution:

Let Andy's lawn have area A, so Beth's is $\frac{A}{2}$ and Carlos' is $\frac{A}{3}$. Let Carlos mow at rate R, so Beth mows at $2R$ and Andy at $3R$.

The times are

$$\text{Andy: } \frac{A}{3R}, \quad \text{Beth: } \frac{A/2}{2R} = \frac{A}{4R}, \quad \text{Carlos: } \frac{A/3}{R} = \frac{A}{3R}.$$

Since $\frac{A}{4R}$ is the smallest, Beth finishes first.

Thus, the correct answer is **B**.

22. Let $\triangle XOY$ be a right-angled triangle with $m\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, what is XY?

A 24

B 26

C 28

D 30

E 32

Solution:

Let $OM = a$ and $ON = b$, so $OX = 2a$ and $OY = 2b$. The right angle at O gives

$$19^2 = (2a)^2 + b^2 \quad \text{and} \quad 22^2 = a^2 + (2b)^2.$$

Adding these, $5(a^2 + b^2) = 19^2 + 22^2 = 845$, so $a^2 + b^2 = 169$ and $MN = \sqrt{a^2 + b^2} = 13$.

Since $\triangle XOY \sim \triangle MON$ with ratio 2, we have $XY = 2 \cdot MN = 26$.

Thus, the correct answer is **B**.

23. Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$ for all positive integers m and n. What is a_{12}?

- A 45
- B 56
- C 67
- D 78
- E 89

Solution:

Setting $n = 1$, we get $a_{m+1} = a_m + a_1 + m = a_m + (m+1)$, so $a_{m+1} - a_m = m + 1$.

Summing from $m = 1$ to 11,

$$a_{12} - a_1 = 2 + 3 + \cdots + 12 = \frac{12 \cdot 13}{2} - 1 = 77.$$

Therefore $a_{12} = 1 + 77 = 78$.

Thus, the correct answer is **D**.

24. Riders on a Ferris wheel travel in a circle in a vertical plane. A particular wheel has radius 20 feet and revolves at the constant rate of one revolution per minute. How many seconds does it take a rider to travel from the bottom of the wheel to a point 10 vertical feet above the bottom?

A 5

B 6

C 7.5

D 10

E 15

Solution:

Put the center O at height 20. The bottom A is at height 0, and the rider reaches height 10, which is 10 feet below the center.

The horizontal from the center down to the rider's level forms a right triangle where the vertical leg is 10 and the hypotenuse (the radius) is 20. That leg is half the hypotenuse, so the radius to the rider makes 60° with the downward vertical.

The wheel turns 360° in 60 seconds, so turning 60° takes $\frac{60}{360} \cdot 60 = 10$ seconds.

Thus, the correct answer is **D**.

25. When 15 is appended to a list of integers, the mean is increased by $2.$ When 1 is appended to the enlarged list, the mean of the enlarged list is decreased by $1.$ How many integers were in the original list?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

Let the original list have n integers with mean $m,$ so its sum is $mn.$ Appending 15 gives

$$(m+2)(n+1) = mn + 15 \implies m + 2n = 13.$$

Appending 1 to that enlarged list gives

$$(m+1)(n+2) = mn + 16 \implies 2m + n = 14.$$

Solving $m + 2n = 13$ and $2m + n = 14$ yields $m = 5$ and $n = 4.$

Thus, the correct answer is **A**.

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