2002 AMC 10A Solutions

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1. The ratio

$$\frac{10^{2000}+10^{2002}}{10^{2001}+10^{2001}}$$

is closest to which of the following numbers?

A 0.1

B 0.2

C 1

D 5

E 10

Solution:

Factoring gives

$$\frac{10^{2000}(1+100)}{2\cdot 10^{2001}}=\frac{101}{20}=5.05,$$

which is closest to 5.

Thus, the correct answer is **D**.

2. For the nonzero numbers $a, b,$ and c, define

$$(a, b, c) = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Find $(2, 12, 9)$.

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

$$(2, 12, 9) = \frac{2}{12} + \frac{12}{9} + \frac{9}{2} = \frac{1}{6} + \frac{4}{3} + \frac{9}{2}.$$

Over a denominator of 6, this is $\dfrac{1 + 8 + 27}{6} = \dfrac{36}{6} = 6.$

Thus, the correct answer is **C**.

3. According to the standard convention for exponentiation,

$$2^{2^{2^{2}}} = 2^{\left(2^{\left(2^{2}\right)}\right)} = 2^{16} = 65{,}536.$$

If the order in which the exponentiations are performed is changed, how many *other* values are possible?

- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

There are five ways to parenthesize the tower. Three of them, $(2^2)^{(2^2)}$, $\left(2^{(2^2)}\right)^2$, and $\left((2^2)^2\right)^2$, all equal $2^8 = 256$. The other two both give the standard value $2^{16} = 65{,}536$.

So exactly one other value, 256, is possible.

Thus, the correct answer is **B**.

4. For how many positive integers m does there exist at least one positive integer n such that $m \cdot n \leq m + n$?

A 4

B 6

C 9

D 12

E infinitely many

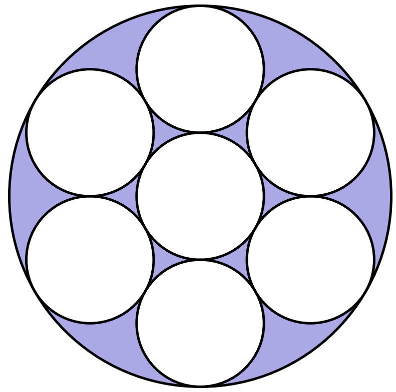
Solution:

Take $n = 1$. Then $m \cdot 1 \leq m + 1$ becomes $m \leq m + 1$, which holds for every positive integer m.

So every positive integer m works, giving infinitely many.

Thus, the correct answer is **E**.

5. Each of the small circles in the figure has radius one. The innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors. Find the area of the shaded region.



A π

B 1.5π

C 2π

D 3π

E 3.5π

Solution:

The center of a surrounding circle is 2 from the center (two radii), and adding its own radius 1 gives a large radius of 3.

The large circle has area 9π, and the seven unit circles have total area 7π, so the shaded region is $9\pi - 7\pi = 2\pi$.

Thus, the correct answer is **C**.

6. Cindy was asked by her teacher to subtract 3 from a certain number and then divide the result by $9.$ Instead, she subtracted 9 and then divided the result by $3,$ giving an answer of $43.$ What would her answer have been had she worked the problem correctly?

- A 15
- B 34
- C 43
- D 51
- E 138

Solution:

Let x be the number. Cindy computed $\dfrac{x-9}{3} = 43,$ so $x - 9 = 129$ and $x = 138.$

The correct computation is $\dfrac{138-3}{9} = \dfrac{135}{9} = 15.$

Thus, the correct answer is **A**.

7. If an arc of 45° on circle A has the same length as an arc of 30° on circle B, then the ratio of the area of circle A to the area of circle B is

A $\frac{4}{9}$

B $\frac{2}{3}$

C $\frac{5}{6}$

D $\frac{3}{2}$

E $\frac{9}{4}$

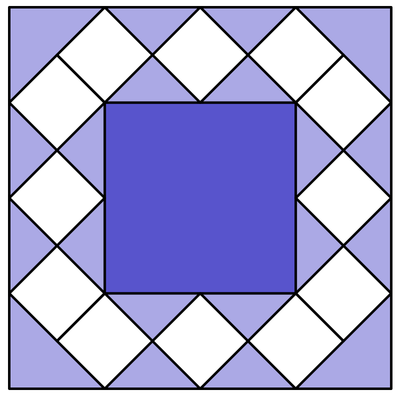
Solution:

Equal arc lengths give $\frac{45}{360} \cdot 2\pi r_A = \frac{30}{360} \cdot 2\pi r_B$, so $45r_A = 30r_B$ and $\frac{r_A}{r_B} = \frac{2}{3}$.

The ratio of areas is $\left(\frac{r_A}{r_B}\right)^2 = \frac{4}{9}$.

Thus, the correct answer is **A**.

8. Betsy designed a flag using blue triangles, small white squares, and a red center square, as shown. Let B be the total area of the blue triangles, W the total area of the white squares, and R the area of the red square. Which of the following is correct?



A $B = W$

B $W = R$

C $B = R$

D $3B = 2R$

E $2R = W$

Solution:

Divide the flag into congruent right triangles by drawing the grid lines and diagonals. Counting gives 24 triangles in the blue region, 24 in the white region, and 16 in the red region.

Hence $B = W$.

Thus, the correct answer is **A**.

9. Suppose $A, B,$ and C are three numbers for which $1001C - 2002A = 4004$ and $1001B + 3003A = 5005.$ The average of the three numbers $A, B,$ and C is

- A 1
- B 3
- C 6
- D 9
- E not uniquely determined

Solution:

Adding the equations, $1001C - 2002A + 1001B + 3003A = 1001A + 1001B + 1001C = 9009.$

So $A + B + C = 9$ and the average is $\frac{9}{3} = 3.$

Thus, the correct answer is **B**.

10. Compute the sum of all the roots of $(2x+3)(x-4)+(2x+3)(x-6)=0$.

A $\frac{7}{2}$

B 4

C 5

D 7

E 13

Solution:

Factoring, $(2x+3)\left[(x-4)+(x-6)\right]=(2x+3)(2x-10)=0$.

The roots are $-\frac{3}{2}$ and 5, which sum to $\frac{7}{2}$.

Thus, the correct answer is **A**.

11. Jamal wants to store 30 computer files on floppy disks, each of which has a capacity of 1.44 megabytes (mb). Three of his files require 0.8 mb of memory each, 12 more require 0.7 mb each, and the remaining 15 require 0.4 mb each. No file can be split between floppy disks. What is the minimal number of floppy disks that will hold all the files?

A 12

B 13

C 14

D 15

E 16

Solution:

The files need $3(0.8) + 12(0.7) + 15(0.4) = 16.8$ mb. On any disk holding a 0.8 mb file, only one 0.4 mb file fits alongside it (since $0.8 + 0.7 > 1.44$), leaving at least 0.24 mb wasted. Across the three such disks that is at least 0.72 mb, so the effective demand is at least $16.8 + 0.72 = 17.52$ mb, requiring at least $\left\lceil \frac{17.52}{1.44} \right\rceil = 13$ disks.

This is achievable: 3 disks each hold one 0.8 file and one 0.4 file, 6 disks each hold two 0.7 files, and 4 disks each hold three 0.4 files.

Thus, the correct answer is **B**.

12. Mr. Earl E. Bird leaves his house for work at exactly 8:00 A.M. every morning. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early. At what average speed, in miles per hour, should Mr. Bird drive to arrive at his workplace precisely on time?

- A 45
- B 48
- C 50
- D 55
- E 58

Solution:

Let t hours be the on-time travel time. Since 3 minutes is 0.05 hours, $40(t+0.05) = 60(t-0.05)$. Then $40t+2=60t-3$, so $t=0.25$.

The distance is $40(0.30)=12$ miles, so the required speed is $\dfrac{12}{0.25}=48$ mph.

Thus, the correct answer is **B**.

13. The sides of a triangle have lengths of $15, 20,$ and $25.$ Find the length of the shortest altitude.

A 6

B 12

C 12.5

D 13

E 15

Solution:

Since $15^2 + 20^2 = 225 + 400 = 625 = 25^2$, the triangle is right with legs 15 and $20,$ and area $\frac{1}{2}(15)(20) = 150.$

The shortest altitude falls to the longest side $25,$ and equals $\frac{2 \cdot 150}{25} = 12.$

Thus, the correct answer is **B**.

14. Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is

- A 0
- B 1
- C 2
- D 4
- E more than four

Solution:

If the roots are primes p and q, then $p + q = 63$ and $pq = k$. Because 63 is odd, one prime must be 2, making the other 61, which is prime.

So $k = 2 \cdot 61 = 122$ is the only possible value.

Thus, the correct answer is **B**.

15. The digits $1, 2, 3, 4, 5, 6, 7,$ and 9 are used to form four two-digit prime numbers, with each digit used exactly once. What is the sum of these four primes?

A 150

B 160

C 170

D 180

E 190

Solution:

A two-digit prime cannot end in $2, 4, 5,$ or $6,$ so these four are the tens digits and $1, 3, 7, 9$ are the units digits.

The sum is $10(2+4+5+6)+(1+3+7+9)=170+20=190.$ One valid set is $\{23, 47, 59, 61\}.$

Thus, the correct answer is **E**.

16. If $a+1=b+2=c+3=d+4=a+b+c+d+5$, then $a+b+c+d$ is

A -5

B $-\frac{10}{3}$

C $-\frac{7}{3}$

D $\frac{5}{3}$

E 5

Solution:

Let the common value be k. Then $a=k-1, b=k-2, c=k-3, d=k-4$, so $a+b+c+d=4k-10$.

Since $a+b+c+d+5=k$, we get $4k-10+5=k$, so $3k=5$ and $k=\frac{5}{3}$. Then $a+b+c+d=k-5=\frac{5}{3}-5=-\frac{10}{3}$.

Thus, the correct answer is **B**.

17. Sarah pours four ounces of coffee into an eight-ounce cup and four ounces of cream into a second cup of the same size. She then transfers half the coffee from the first cup to the second and, after stirring thoroughly, transfers half the liquid in the second cup back to the first. What fraction of the liquid in the first cup is now cream?

- A $\frac{1}{4}$
- B $\frac{1}{3}$
- C $\frac{3}{8}$
- D $\frac{2}{5}$
- E $\frac{1}{2}$

Solution:

After transferring 2 oz of coffee, cup 1 has 2 oz coffee and cup 2 has 2 oz coffee plus 4 oz cream, a total of 6 oz.

Transferring back half of cup 2 (that is 3 oz, consisting of 1 oz coffee and 2 oz cream) leaves cup 1 with 3 oz coffee and 2 oz cream. The fraction that is cream is $\frac{2}{2+3} = \frac{2}{5}$.

Thus, the correct answer is **D**.

18. A $3 \times 3 \times 3$ cube is formed by gluing together 27 standard cubical dice. (On a standard die, the sum of the numbers on any pair of opposite faces is 7.) The smallest possible sum of all the numbers showing on the surface of the $3 \times 3 \times 3$ cube is

- A 60
- B 72
- C 84
- D 90
- E 96

Solution:

The 8 corner dice show 3 faces each, minimized at $1+2+3=6$, contributing $8 \cdot 6 = 48$. The 12 edge dice show 2 faces, minimized at $1+2=3$, contributing $12 \cdot 3 = 36$.

The 6 face-center dice show 1 face, minimized at 1, contributing 6, and the hidden interior die contributes 0. The total is $48+36+6=90$.

Thus, the correct answer is **D**.

19. Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the area, in square yards, of the region outside the doghouse that Spot can reach?

A $\frac{2}{3}\pi$

B 2π

C $\frac{5}{2}\pi$

D $\frac{8}{3}\pi$

E 3π

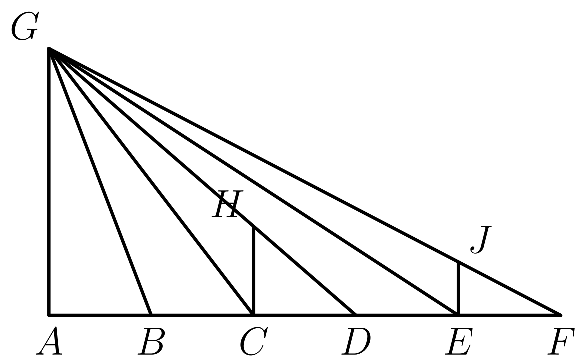
Solution:

At the tether vertex the hexagon blocks its 120° interior angle, leaving a 240° sector of radius 2 : area $\frac{240}{360}\pi(2)^2 = \frac{8\pi}{3}$.

Wrapping around each of the two adjacent vertices, 1 yard of rope remains and sweeps a 60° sector: $2 \cdot \frac{60}{360}\pi(1)^2 = \frac{\pi}{3}$. The total is $\frac{8\pi}{3} + \frac{\pi}{3} = 3\pi$.

Thus, the correct answer is **E**.

20. Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}, \overline{JE}$, and $\overline{AG}$ are parallel. Find HC/JE.



A $\frac{5}{4}$

B $\frac{4}{3}$

C $\frac{3}{2}$

D $\frac{5}{3}$

E 2

Solution:

Since $HC \parallel AG, \triangle DHC \sim \triangle DGA$, so $\frac{HC}{AG} = \frac{DC}{DA} = \frac{1}{3}$, giving $HC = \frac{AG}{3}$.

Since $JE \parallel AG, \triangle FJE \sim \triangle FGA$, so $\frac{JE}{AG} = \frac{FE}{FA} = \frac{1}{5}$, giving $JE = \frac{AG}{5}$.

Therefore $\frac{HC}{JE} = \frac{AG/3}{AG/5} = \frac{5}{3}$.

Thus, the correct answer is **D**.

21. The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. The largest integer that can be an element of this collection is

A 11

B 12

C 13

D 14

E 15

Solution:

The sum is $8 \cdot 8 = 64$. The collection $6, 6, 6, 8, 8, 8, 8, 14$ has mean, median, unique mode, and range all equal to 8, so 14 is attainable.

If the largest were 15, the range 8 forces the smallest to be 7, so all eight integers are at least 7. The other seven then sum to $64 - 15 = 49 = 7 \cdot 7$, forcing every one of them to equal 7. But then the median and mode would be 7, not 8, a contradiction.

Thus, the correct answer is **D**.

22. A set of tiles numbered 1 through 100 is modified repeatedly by the following operation: remove all tiles numbered with a perfect square, and renumber the remaining tiles consecutively starting with 1. How many times must the operation be performed to reduce the number of tiles in the set to one?

A 10

B 11

C 18

D 19

E 20

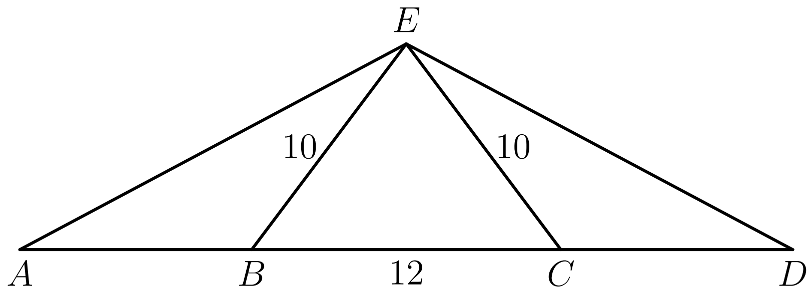
Solution:

Starting from n^2 tiles, one operation removes the n perfect squares, leaving $n^2 - n$. The next operation removes $n - 1$ perfect squares, leaving $n^2 - n - (n - 1) = (n - 1)^2$.

So every two operations reduce n^2 to $(n - 1)^2$. Going from $10^2 = 100$ down to $1^2 = 1$ takes $2(10 - 1) = 18$ operations.

Thus, the correct answer is **C**.

23. Points $A, B, C,$ and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice the perimeter of $\triangle BEC$. Find AB.



A $\frac{15}{2}$

B 8

C $\frac{17}{2}$

D 9

E $\frac{19}{2}$

Solution:

Let M be the midpoint of BC. Since $BE = CE$, $EM \perp BC$ and $EM = \sqrt{10^2 - 6^2} = 8$. By symmetry $AE = ED$; write $AB = CD = x$ and $AE = ED = y$.

The perimeter condition gives $2y + (2x + 12) = 2(10 + 10 + 12) = 64$, so $x + y = 26$. Also $y^2 = EM^2 + (x + 6)^2 = 64 + (x + 6)^2$.

Substituting $y = 26 - x$, $(26 - x)^2 = 64 + (x + 6)^2$, which simplifies to $676 - 52x = 100 + 12x$, so $64x = 576$ and $x = 9$.

Thus, the correct answer is **D**.

24. Tina randomly selects two distinct numbers from the set $\{1, 2, 3, 4, 5\}$, and Sergio randomly selects a number from the set $\{1, 2, \ldots, 10\}$. The probability that Sergio's number is larger than the sum of the two numbers chosen by Tina is

A $\frac{2}{5}$

B $\frac{9}{20}$

C $\frac{1}{2}$

D $\frac{11}{20}$

E $\frac{24}{25}$

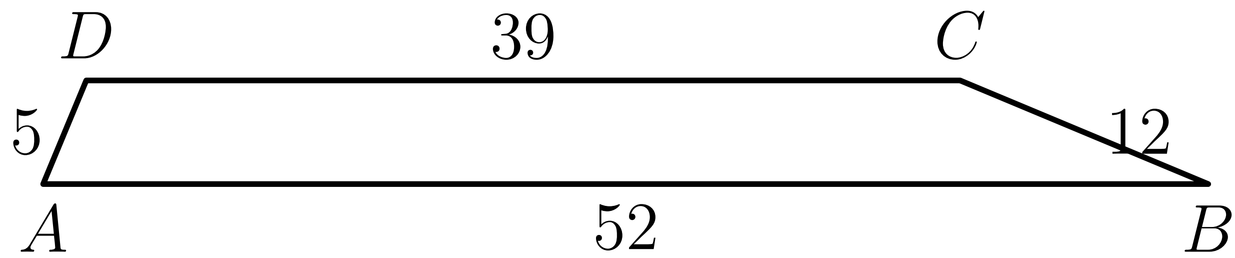
Solution:

Tina's 10 equally likely pairs give sums $3, 4, 5, 5, 6, 6, 7, 7, 8, 9$. For a sum s, Sergio's number exceeds it with probability $\frac{10 - s}{10}$.

Averaging the winning probability over the ten pairs, the total is
$$\frac{7+6+5+5+4+4+3+3+2+1}{100} = \frac{40}{100} = \frac{2}{5}.$$

Thus, the correct answer is **A**.

25. In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, we have $AB = 52, BC = 12, CD = 39$, and $DA = 5$. The area of $ABCD$ is



- A 182
- B 195
- C 210
- D 234
- E 260

Solution:

Extend DA and CB to meet at P. Since $DC \parallel AB$, $\triangle PDC \sim \triangle PAB$ with ratio $\frac{39}{52} = \frac{3}{4}$. From $\frac{PD}{PD+5} = \frac{3}{4}$ we get $PD = 15$, and similarly $PC = 36$.

Then $PD : PC : DC = 15 : 36 : 39 = 3 \cdot (5 : 12 : 13)$, so $\angle P$ is a right angle. The area of $ABCD$ is

$$\frac{1}{2}(PA)(PB) - \frac{1}{2}(PD)(PC) = \frac{1}{2}(20)(48) - \frac{1}{2}(15)(36) = 480 - 270 = 210.$$

Thus, the correct answer is **C**.

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