2001 AMC 10 Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc10/2001/solutions

1. The median of the list

$$n,\ n+3,\ n+4,\ n+5,\ n+6,\ n+8,\ n+10,\ n+12,\ n+15$$

is 10. What is the mean?

A 4

B 6

C 7

D 10

E 11

Solution:

The list has 9 numbers in increasing order, so the median is the 5th term, $n+6$. Setting $n+6=10$ gives $n=4$.

The sum of the terms is $9n+(3+4+5+6+8+10+12+15)=9n+63=99$, so the mean is $99/9=11$.

Thus, the correct answer is **E**.

2. A number x is 2 more than the product of its reciprocal and its additive inverse. In which interval does the number lie?

A $-4 \leq x \leq -2$

B $-2 < x \leq 0$

C $0 < x \leq 2$

D $2 < x \leq 4$

E $4 < x \leq 6$

Solution:

The reciprocal of x is $\frac{1}{x}$ and its additive inverse is $-x$. Their product is $\frac{1}{x} \cdot (-x) = -1.$

So $x = 2 + (-1) = 1,$ which lies in the interval $0 < x \leq 2.$

Thus, the correct answer is **C**.

3. The sum of two numbers is S. Suppose 3 is added to each number and then each of the resulting numbers is doubled. What is the sum of the final two numbers?

A $2S + 3$

B $3S + 2$

C $3S + 6$

D $2S + 6$

E $2S + 12$

Solution:

Let the numbers be a and b, so $a + b = S$. After adding 3 to each and doubling, the sum is $2(a + 3) + 2(b + 3) = 2(a + b) + 12 = 2S + 12.$

Thus, the correct answer is **E**.

4. What is the maximum number of possible points of intersection of a circle and a triangle?

A 2

B 3

C 4

D 5

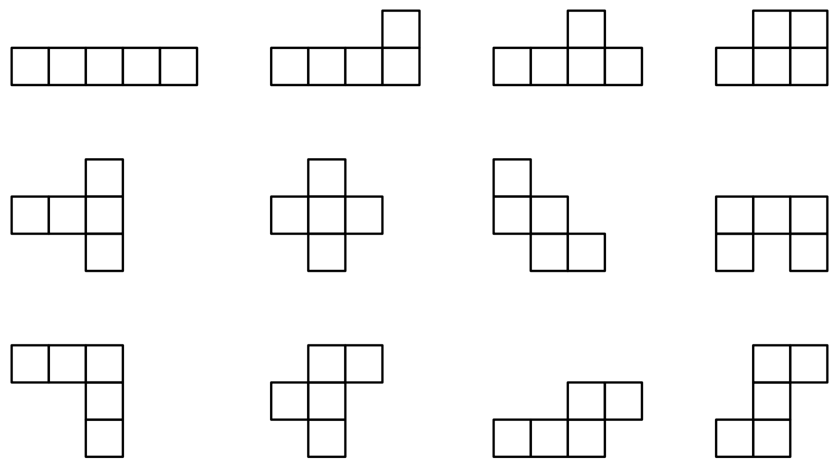
E 6

Solution:

Each side of the triangle is a segment, which can intersect a circle in at most 2 points. With 3 sides, the maximum is $3 \cdot 2 = 6$ points, and this is achievable.

Thus, the correct answer is **E**.

5. How many of the twelve pentominoes pictured below have at least one line of symmetry?



- A 3
- B 4
- C 5
- D 6
- E 7

Solution:

Checking each pentomino for a reflection line, exactly six have at least one: the straight bar, the plus, the T-shape, the U-shape, the V-shape, and the W-shape. The remaining six have no line of symmetry.

Thus, the correct answer is **D**.

6. Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the integer n. For example, $P(23) = 6$ and $S(23) = 5$. Suppose N is a two-digit number such that $N = P(N) + S(N)$. What is the units digit of N?

A 2

B 3

C 6

D 8

E 9

Solution:

Write $N = 10a + b$. Then $P(N) = ab$ and $S(N) = a + b$, so $10a + b = ab + a + b$. This reduces to $9a = ab$, and since $a \neq 0$, we get $b = 9$.

The units digit is 9. Thus, the correct answer is **E**.

7. When the decimal point of a certain positive decimal number is moved four places to the right, the new number is four times the reciprocal of the original number. What is the original number?

A 0.0002

B 0.002

C 0.02

D 0.2

E 2

Solution:

Moving the decimal four places right multiplies x by 10000. So $10000x = 4 \cdot \frac{1}{x}$, giving $x^2 = \frac{4}{10000}$.

Since $x > 0, x = \frac{2}{100} = 0.02.$

Thus, the correct answer is **C**.

8. Wanda, Darren, Beatrice, and Chi are tutors in the school math lab. Their schedule is as follows: Darren works every third school day, Wanda works every fourth school day, Beatrice works every sixth school day, and Chi works every seventh school day. Today they are all working in the math lab. In how many school days from today will they next be together tutoring in the lab?

- A 42
- B 84
- C 126
- D 178
- E 252

Solution:

They meet again after $\text{lcm}(3, 4, 6, 7)$ days. Since $4 = 2^2$ and $6 = 2 \cdot 3$, the least common multiple is $2^2 \cdot 3 \cdot 7 = 84$.

Thus, the correct answer is **B**.

9. The state income tax where Kristin lives is levied at the rate of $p\%$ of the first $\$28000$ of annual income plus $(p+2)\%$ of any amount above $\$28000$. Kristin noticed that the state income tax she paid amounted to $(p+0.25)\%$ of her annual income. What was her annual income?

- A $\$28000$
- B $\$32000$
- C $\$35000$
- D $\$42000$
- E $\$56000$

Solution:

Let her income be $x \geq 28000$. Then

$$\frac{p}{100}(28000) + \frac{p+2}{100}(x - 28000) = \frac{p+0.25}{100}x.$$

Multiplying by 100 and expanding, all the p terms cancel, leaving $2x - 56000 = 0.25x$. So $1.75x = 56000$ and $x = 32000$.

Thus, the correct answer is **B**.

10. If $x, y,$ and z are positive with $xy = 24, xz = 48,$ and $yz = 72,$ then $x + y + z$ is

A 18

B 19

C 20

D 22

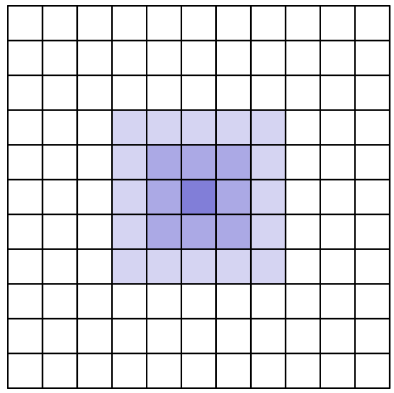
E 24

Solution:

Dividing $xz = 48$ by $xy = 24$ gives $z = 2y.$ Then $yz = 2y^2 = 72,$ so $y = 6, z = 12,$ and $x = 24/y = 4.$

Hence $x + y + z = 22.$ Thus, the correct answer is **D**.

11. Consider the dark square in an array of unit squares, part of which is shown. The first ring of squares around this center square contains 8 unit squares. The second ring contains 16 unit squares. If we continue this process, the number of unit squares in the 100th ring is



A 396

B 404

C 800

D $10{,}000$

E $10{,}404$

Solution:

The nth ring is the border of a $(2n+1) \times (2n+1)$ square surrounding a $(2n-1) \times (2n-1)$ square, so it contains $(2n+1)^2 - (2n-1)^2 = 8n$ unit squares.

For $n = 100$, that is 800. Thus, the correct answer is **C**.

12. Suppose that n is the product of three consecutive integers and that n is divisible by 7. Which of the following is not necessarily a divisor of n?

A 6

B 14

C 21

D 28

E 42

Solution:

Among three consecutive integers, at least one is even and one is a multiple of $3,$ so n is divisible by $6.$ With the given factor of $7,$ it is divisible by $6, 14, 21,$ and $42.$

But $28 = 2^2 \cdot 7$ requires two factors of $2,$ which is not guaranteed: $5 \cdot 6 \cdot 7 = 210$ is divisible by 7 but not by $28.$

Thus, the correct answer is **D**.

13. A telephone number has the form $ABC - DEF - GHIJ$, where each letter represents a different digit. The digits in each part of the number are in decreasing order; that is, $A > B > C, D > E > F$, and $G > H > I > J$. Furthermore, D, E, and F are consecutive even digits; G, H, I, and J are consecutive odd digits; and $A + B + C = 9$. Find A.

A 4

B 5

C 6

D 7

E 8

Solution:

The consecutive odd digits $GHIJ$ are 9753 or 7531, leaving one odd digit (1 or 9) for A, B, C. Since $A + B + C = 9$, the odd digit there must be 1, so the two even digits in ABC sum to 8.

The consecutive even digits DEF are $864, 642$, or 420, leaving even-digit pairs $\{2, 0\}, \{8, 0\}$, or $\{8, 6\}$ for ABC. Only $\{8, 0\}$ sums to 8, so $ABC = 810$ and $A = 8$.

Thus, the correct answer is **E**.

14. A charity sells 140 benefit tickets for a total of $\$2001.$ Some tickets sell for full price (a whole dollar amount), and the rest sell for half price. How much money is raised by the full-price tickets?

A $\$782$

B $\$986$

C $\$1158$

D $\$1219$

E $\$1449$

Solution:

Let n full-price tickets sell at p dollars each. Then $np + (140 - n)\frac{p}{2} = 2001,$ so $p(n + 140) = 4002 = 2 \cdot 3 \cdot 23 \cdot 29.$

Since $140 \leq n + 140 \leq 280,$ the only factor of 4002 in range is $174 = 2 \cdot 3 \cdot 29.$ So $n + 140 = 174,$ giving $n = 34$ and $p = 23.$ The full-price tickets raise $34 \cdot 23 = 782$ dollars.

Thus, the correct answer is **A**.

15. A street has parallel curbs 40 feet apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet and each stripe is 50 feet long. Find the distance, in feet, between the stripes.

- A 9
- B 10
- C 12
- D 15
- E 25

Solution:

The crosswalk is a parallelogram. Using the curb (15 ft) as base and the street width (40 ft) as height, its area is $15 \cdot 40 = 600$ square feet.

Using a stripe (50 ft) as base, the area equals 50 times the distance d between the stripes, so $d = 600/50 = 12.$

Thus, the correct answer is **C**.

16. The mean of three numbers is 10 more than the least of the numbers and 15 less than the greatest. The median of the three numbers is 5. What is their sum?

A 5

B 20

C 25

D 30

E 36

Solution:

Let m be the mean. The least number is $m - 10$ and the greatest is $m + 15$, with median 5.

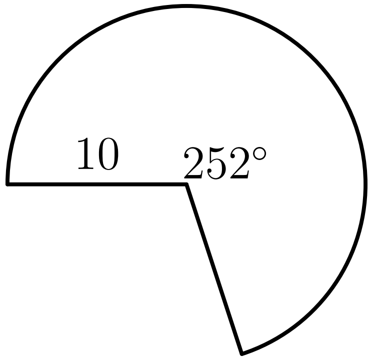
Since the mean of the three is m,

$$\frac{1}{3}\big((m - 10) + 5 + (m + 15)\big) = m,$$

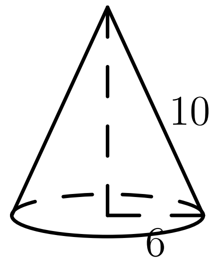
which gives $m = 10$.

The sum is $3m = 30$. Thus, the correct answer is **D**.

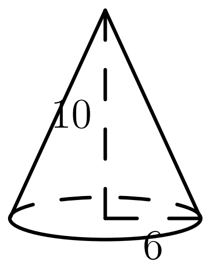
17. Which of the cones below can be formed from a 252° sector of a circle of radius 10 by aligning the two straight sides?



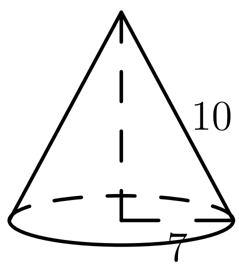
A



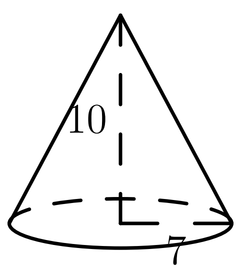
B



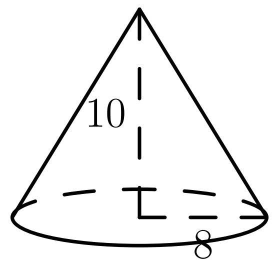
C



D



E



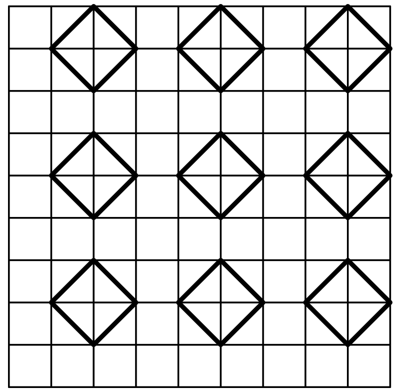
Solution:

When rolled into a cone, the sector's radius 10 becomes the slant height, and the arc length becomes the base circumference.

The arc length is $\frac{252}{360} \cdot 2\pi \cdot 10 = 14\pi$, so $2\pi r = 14\pi$ gives base radius $r = 7$. The cone has slant height 10 and base radius 7.

Thus, the correct answer is **C**.

18. The plane is tiled by congruent squares and congruent pentagons as indicated. The percent of the plane that is enclosed by the pentagons is closest to



A 50

B 52

C 54

D 56

E 58

Solution:

Consider a repeating 3×3 block of nine small squares. Four of these nine squares are not part of the pentagons, so the pentagons cover $1 - \frac{4}{9} = \frac{5}{9} \approx 55.6\%$ of the area.

This is closest to 56. Thus, the correct answer is **D**.

19. Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

- A 6
- B 9
- C 12
- D 15
- E 18

Solution:

The number of selections is the number of nonnegative integer solutions of $g + c + p = 4$. By stars and bars, this is $\binom{4+2}{2} = \binom{6}{2} = 15.$

Thus, the correct answer is **D**.

20. A regular octagon is formed by cutting an isosceles right triangle from each of the corners of a square with sides of length 2000. What is the length of each side of the octagon?

A $\frac{1}{3}(2000)$

B $2000(\sqrt{2}-1)$

C $2000(2-\sqrt{2})$

D 1000

E $1000\sqrt{2}$

Solution:

Let each octagon side be x. It is the hypotenuse of each cut isosceles right triangle, whose legs are $\frac{x}{\sqrt{2}}$.

Along one side of the square, two legs and one octagon side give $2 \cdot \frac{x}{\sqrt{2}} + x = 2000$, so $x(\sqrt{2}+1) = 2000$ and $x = \frac{2000}{\sqrt{2}+1} = 2000(\sqrt{2}-1)$.

Thus, the correct answer is **B**.

21. A right circular cylinder with its diameter equal to its height is inscribed in a right circular cone. The cone has diameter 10 and altitude $12,$ and the axes of the cylinder and cone coincide. Find the radius of the cylinder.

A $\frac{8}{3}$

B $\frac{30}{11}$

C 3

D $\frac{25}{8}$

E $\frac{7}{2}$

Solution:

Take an axial cross-section. The cone has base radius 5 and height 12; the cylinder appears as a rectangle of width $2r$ and height $2r$.

By similar triangles, $\frac{12-2r}{r}=\frac{12}{5}$, so $5(12-2r)=12r$, giving $60=22r$ and $r=\frac{30}{11}$.

Thus, the correct answer is **B**.

22. In the magic square shown, the sums of the numbers in each row, column, and diagonal are the same. Five of these numbers are represented by $v, w, x, y,$ and z. Find $y + z$.

v	24	w
18	x	y
25	z	21

A 43

B 44

C 45

D 46

E 47

Solution:

Since v sits in the first row, first column, and main diagonal, the remaining two entries of each of those lines have equal sums: $25 + 18 = 24 + w = 21 + x$. So $w = 19$ and $x = 22$.

The anti-diagonal $25, 22, 19$ sums to 66, so the magic sum is 66. Then $v = 66 - 24 - 19 = 23$, $y = 66 - 18 - 22 = 26$, and $z = 66 - 25 - 21 = 20$.

Hence $y + z = 46$. Thus, the correct answer is **D**.

23. A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

A $\frac{3}{10}$

B $\frac{2}{5}$

C $\frac{1}{2}$

D $\frac{3}{5}$

E $\frac{7}{10}$

Solution:

Imagine drawing all five chips in a random order. The drawing stops on a white chip exactly when both white chips come out before all three reds, which happens precisely when the very last chip in the full ordering is red.

That probability is $\frac{3}{5}$. Thus, the correct answer is **D**.

24. In trapezoid $ABCD$, $\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$, with $AB + CD = BC$, $AB < CD$, and $AD = 7$. What is $AB \cdot CD$?

A 12

B 12.25

C 12.5

D 12.75

E 13

Solution:

Drop a perpendicular from B to CD, meeting it at E. Then $BE = AD = 7$ and $CE = CD - AB$. By the Pythagorean theorem, $BC^2 = BE^2 + CE^2$.

Since $BC = CD + AB$,

$$(CD + AB)^2 - (CD - AB)^2 = BE^2 = 49.$$

The left side equals $4 \cdot AB \cdot CD$, so $AB \cdot CD = \dfrac{49}{4} = 12.25.$

Thus, the correct answer is **B**.

25. How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

A 768

B 801

C 934

D 1067

E 1167

Solution:

Multiples of 3 or 4 up to 2001 : $\lfloor \frac{2001}{3} \rfloor + \lfloor \frac{2001}{4} \rfloor - \lfloor \frac{2001}{12} \rfloor = 667 + 500 - 166 = 1001.$

Among these, remove the multiples of 5 : multiples of 15 (133) and of 20 (100), re-adding multiples of 60 (33): $133 + 100 - 33 = 200.$

So the count is $1001 - 200 = 801.$ Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc10/2001