2000 AMC 10 Solutions

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1. In the year 2001, the United States will host the International Mathematical Olympiad. Let I, M, and O be distinct positive integers such that the product $I \cdot M \cdot O = 2001$. What is the largest possible value of the sum $I + M + O$?

A 23

B 55

C 99

D 111

E 671

Solution:

Factoring gives $2001 = 3 \cdot 23 \cdot 29$.

To maximize the sum with the product fixed, spread the factors as much as possible: take $I = 1$ and combine the two largest primes, $M = 3$ and $O = 23 \cdot 29 = 667$.

The sum is $1 + 3 + 667 = 671$.

Thus, the correct answer is **E**.

2. Which of the following is equal to $2000 \cdot 2000^{2000}$?

A 2000^{2001}

B 4000^{2000}

C 2000^{4000}

D $4{,}000{,}000^{2000}$

E $2000^{4{,}000{,}000}$

Solution:

Write the factor 2000 as 2000^1. Then

$$2000^1 \cdot 2000^{2000} = 2000^{1+2000} = 2000^{2001}.$$

Each of the other options is larger than 2000^{2001}.

Thus, the correct answer is **A**.

3. Each day, Jenny ate 20% of the jellybeans that were in her jar at the beginning of that day. At the end of the second day, 32 remained. How many jellybeans were in the jar originally?

A 40

B 50

C 55

D 60

E 75

Solution:

Since Jenny eats 20% each day, 80% remain at the end of each day.

If x is the original number, then

$$(0.8)^2x = 0.64x = 32,$$

so $x = 50.$

Thus, the correct answer is **B**.

4. Chandra pays an on-line service provider a fixed monthly fee plus an hourly charge for connect time. Her December bill was $12.48, but in January her bill was $17.54 because she used twice as much connect time as in December. What is the fixed monthly fee?

A $2.53

B $5.06

C $6.24

D $7.42

E $8.77

Solution:

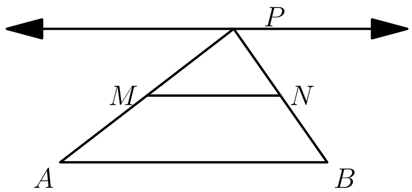
January doubled only the connect time, so the increase $\$17.54 - \$12.48 = \$5.06$ equals December's connect-time cost.

The fixed monthly fee is therefore $\$12.48 - \$5.06 = \$7.42.$

Thus, the correct answer is **D**.

5. Points M and N are the midpoints of sides PA and PB of $\triangle PAB$. As P moves along a line that is parallel to side AB, how many of the four quantities listed below change?

(a) the length of the segment MN; (b) the perimeter of $\triangle PAB$; (c) the area of $\triangle PAB$; (d) the area of trapezoid $ABNM$.



- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

Since MN is a midsegment, $MN = \frac{1}{2}AB$, which is fixed.

The base AB and the height from P to AB are both constant as P slides along the parallel line, so the area of $\triangle PAB$ does not change. The trapezoid $ABNM$ is the triangle minus $\triangle PMN$, both of whose areas are constant, so its area does not change either.

Only the perimeter changes, since PA and PB vary as P moves. So exactly one quantity changes.

Thus, the correct answer is **B**.

6. The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

A 0

B 4

C 6

D 7

E 9

Solution:

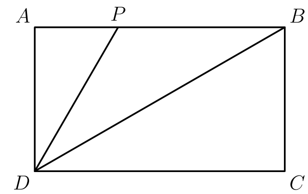
Recording only the units digits gives the sequence

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, \ldots$$

Scanning for the first appearance of each digit, the digit 6 is the last of the ten digits to show up.

Thus, the correct answer is **C**.

7. In rectangle $ABCD, AD = 1, P$ is on $\overline{AB}$, and $\overline{DB}$ and $\overline{DP}$ trisect $\angle ADC$. What is the perimeter of $\triangle BDP$?



A $3+\frac{\sqrt{3}}{3}$

B $2+\frac{4\sqrt{3}}{3}$

C $2+2\sqrt{2}$

D $\frac{3+3\sqrt{5}}{2}$

E $2+\frac{5\sqrt{3}}{3}$

Solution:

The right angle $\angle ADC = 90^\circ$ is trisected into three 30° angles, so $\angle ADP = 30^\circ$ and $\angle ADB = 60^\circ$.

In right triangle ADP, with $AD = 1$, we get $DP = \frac{1}{\cos 30^\circ} = \frac{2\sqrt{3}}{3}$ and $AP = \tan 30^\circ = \frac{\sqrt{3}}{3}$.

In right triangle ADB, with $AD = 1$, we get $DB = \frac{1}{\cos 60^\circ} = 2$ and $AB = \tan 60^\circ = \sqrt{3}$.

Then $PB = AB - AP = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$.

The perimeter of $\triangle BDP$ is

$$DP + PB + DB = \frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{3} + 2 = 2 + \frac{4\sqrt{3}}{3}.$$

Thus, the correct answer is **B**.

8. At Olympic High School, $\frac{2}{5}$ of the freshmen and $\frac{4}{5}$ of the sophomores took the AMC 10. Given that the number of freshmen and sophomore contestants was the same, which of the following must be true?

- A There are five times as many sophomores as freshmen.
- B There are twice as many sophomores as freshmen.
- C There are as many freshmen as sophomores.
- D There are twice as many freshmen as sophomores.
- E There are five times as many freshmen as sophomores.

Solution:

Let f and s be the numbers of freshmen and sophomores. The contestant counts are equal, so

$$\tfrac{2}{5}f = \tfrac{4}{5}s.$$

Multiplying by 5 gives $2f = 4s$, so $f = 2s$. There are twice as many freshmen as sophomores.

Thus, the correct answer is **D**.

9. If $|x-2| = p$, where $x < 2$, then $x - p =$

- A -2
- B 2
- C $2-2p$
- D $2p-2$
- E $|2p-2|$

Solution:

Because $x < 2$, we have $|x-2| = 2 - x = p$, so $x = 2 - p$.

Then

$$x - p = (2-p) - p = 2 - 2p.$$

Thus, the correct answer is **C**.

10. The sides of a triangle with positive area have lengths $4, 6,$ and x. The sides of a second triangle with positive area have lengths $4, 6,$ and y. What is the smallest positive number that is not a possible value of $|x - y|$?

A 2

B 4

C 6

D 8

E 10

Solution:

By the triangle inequality, each of x and y can be any number strictly between $6 - 4 = 2$ and $6 + 4 = 10.$

Then $|x - y|$ can take any value with $0 \leq |x - y| < 8.$

The smallest positive number not attainable is $10 - 2 = 8.$

Thus, the correct answer is **D**.

11. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

A 21

B 60

C 119

D 180

E 231

Solution:

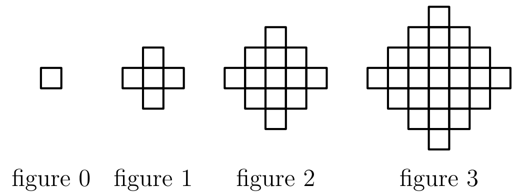
The primes between 4 and 18 are $5, 7, 11, 13,$ and $17.$ The product of two of them is odd and the sum is even, so $xy - (x + y)$ is odd.

Since $xy - (x + y) = (x - 1)(y - 1) - 1$ increases as either prime increases, the result ranges from $5 \cdot 7 - 12 = 23$ up to $13 \cdot 17 - 30 = 191.$

The only odd option in $[23, 191]$ is $119 = 11 \cdot 13 - (11 + 13).$

Thus, the correct answer is **C**.

12. Figures $0, 1, 2,$ and 3 consist of $1, 5, 13,$ and 25 nonoverlapping unit squares, respectively. If the pattern were continued, how many nonoverlapping unit squares would there be in figure 100?



- A 10401
- B 19801
- C 20201
- D 39801
- E 40801

Solution:

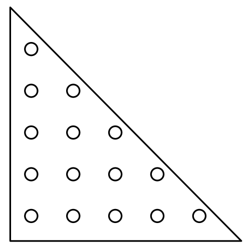
Figure n can be split into the sum of the first n odd numbers and the first $n + 1$ odd numbers, giving $n^2 + (n + 1)^2$ unit squares.

For figure 100, this is

$$100^2 + 101^2 = 10000 + 10201 = 20201.$$

Thus, the correct answer is **C**.

13. There are 5 yellow pegs, 4 red pegs, 3 green pegs, 2 blue pegs, and 1 orange peg to be placed on a triangular peg board. In how many ways can the pegs be placed so that no (horizontal) row or (vertical) column contains two pegs of the same color?



A 0

B 1

C $5! \cdot 4! \cdot 3! \cdot 2! \cdot 1!$

D $15!/(5! \cdot 4! \cdot 3! \cdot 2! \cdot 1!)$

E $15!$

Solution:

The board has five rows and five columns. To avoid two yellow pegs in a row or column, there must be exactly one yellow peg in each row, forcing the yellow pegs onto the long diagonal.

The four red pegs must then each go in rows 2 through 5, and the only positions left force them into a single diagonal as well. Continuing with green, blue, and orange, every color is forced into a unique position.

Hence there is exactly one valid arrangement.

Thus, the correct answer is **B**.

14. Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were $71, 76, 80, 82,$ and $91.$ What was the last score Mrs. Walter entered?

A 71

B 76

C 80

D 82

E 91

Solution:

The residues of $71, 76, 80, 82, 91$ modulo 3 are $2, 1, 2, 1, 1.$ The sum of the first three scores must be divisible by $3,$ and the only such triple is $76 + 82 + 91 = 249,$ so the third score entered is 91 and the first two are 76 and $82.$

Since 249 is one more than a multiple of $4,$ the fourth score must be three more than a multiple of $4,$ which only 71 satisfies. That leaves 80 as the fifth and last score.

Indeed $76, 158, 249, 320, 400$ are divisible by $1, 2, 3, 4, 5.$

Thus, the correct answer is **C**.

15. Two non-zero real numbers, a and b, satisfy $ab = a - b$. Find a possible value of

$$\frac{a}{b} + \frac{b}{a} - ab.$$

A -2

B $-\frac{1}{2}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E 2

Solution:

Over the common denominator ab,

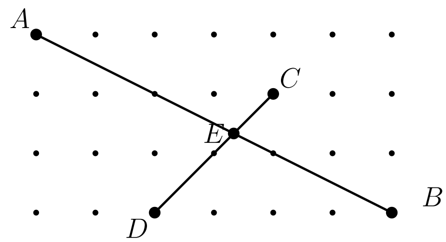
$$\frac{a}{b} + \frac{b}{a} - ab = \frac{a^2 + b^2 - (ab)^2}{ab}.$$

Substituting $ab = a - b$ gives

$$\frac{a^2 + b^2 - (a - b)^2}{ab} = \frac{2ab}{ab} = 2.$$

Thus, the correct answer is **E**.

16. The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE.



A $\dfrac{4\sqrt{5}}{3}$

B $\dfrac{5\sqrt{5}}{3}$

C $\dfrac{12\sqrt{5}}{7}$

D $2\sqrt{5}$

E $\dfrac{5\sqrt{65}}{9}$

Solution:

Place the points at $A = (0, 3), B = (6, 0), C = (4, 2), D = (2, 0)$.

Line AB is $x + 2y = 6$ and line CD is $x - y = 2$. Solving simultaneously gives $E = \left(\frac{10}{3}, \frac{4}{3}\right)$.

Then

$$AE = \sqrt{\left(\frac{10}{3}\right)^2 + \left(\frac{4}{3} - 3\right)^2} = \sqrt{\frac{100}{9} + \frac{25}{9}} = \frac{5\sqrt{5}}{3}.$$

Thus, the correct answer is **B**.

17. Boris has an incredible coin changing machine. When he puts in a quarter, it returns five nickels; when he puts in a nickel, it returns five pennies; and when he puts in a penny, it returns five quarters. Boris starts with just one penny. Which of the following amounts could Boris have after using the machine repeatedly?

- A $3.63
- B $5.13
- C $6.30
- D $7.45
- E $9.07

Solution:

Trading a quarter for five nickels or a nickel for five pennies does not change the total value. Only trading a penny for five quarters changes it, adding $5 \cdot 25 - 1 = 124$ cents.

Starting from 1 cent, Boris always has $1 + 124n$ cents for some nonnegative integer n.

Only $\$7.45$ has this form, since $745 = 1 + 124 \cdot 6$.

Thus, the correct answer is **D**.

18. Charlyn walks completely around the boundary of a square whose sides are each 5 km long. From any point on her path she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk, expressed in square kilometers and rounded to the nearest whole number?

- A 24
- B 27
- C 39
- D 40
- E 42

Solution:

Inside the square, Charlyn sees everything except a central square of side $5 - 2 = 3$, an area of $25 - 9 = 16$ km^2.

Outside the square, the region is four rectangles each 5×1 plus four quarter circles of radius 1, an area of $4 \cdot 5 + \pi = 20 + \pi$ km^2.

The total area is $36 + \pi \approx 39$ km^2.

Thus, the correct answer is **C**.

19. Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. The ratio of the area of the other small right triangle to the area of the square is

- A $\frac{1}{2m+1}$
- B m
- C $1-m$
- D $\frac{1}{4m}$
- E $\frac{1}{8m^2}$

Solution:

Let the square have side 1. One small triangle has legs 1 and r, with area $\frac{1}{2}r = m$, so $r = 2m$.

The two small triangles are similar, so the other has legs 1 and $\frac{1}{r}$, with area $\frac{1}{2} \cdot \frac{1}{r} = \frac{1}{4m}$.

Since the square has area 1, the desired ratio is $\frac{1}{4m}$.

Thus, the correct answer is **D**.

20. Let A, M, and C be nonnegative integers such that $A + M + C = 10$. What is the maximum value of

$$A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A?$$

- A 49
- B 59
- C 69
- D 79
- E 89

Solution:

Notice that

$$A \cdot M \cdot C + AM + MC + CA = (A+1)(M+1)(C+1) - (A+M+C) - 1 = (A+1)(M+1)(C+1) - 11.$$

We maximize a product of three positive integers summing to 13. The most balanced split is $4, 4, 5$, giving $4 \cdot 4 \cdot 5 = 80$.

The maximum is $80 - 11 = 69$.

Thus, the correct answer is **C**.

21. If all alligators are ferocious creatures and some creepy crawlers are alligators, which statement(s) must be true?

I. All alligators are creepy crawlers.

II. Some ferocious creatures are creepy crawlers.

III. Some alligators are not creepy crawlers.

- A I only
- B II only
- C III only
- D II and III only
- E None must be true

Solution:

Some creepy crawlers are alligators, and all alligators are ferocious, so those creatures are both creepy crawlers and ferocious. Hence some ferocious creatures are creepy crawlers, making II true.

Statement I fails because not every alligator need be a creepy crawler, and III fails because it is possible that all alligators are creepy crawlers. Only II must hold.

Thus, the correct answer is **B**.

22. One morning each member of Angela's family drank an 8-ounce mixture of coffee with milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Angela drank a quarter of the total amount of milk and a sixth of the total amount of coffee. How many people are in the family?

- A 3
- B 4
- C 5
- D 6
- E 7

Solution:

Let there be n people, drinking $8n$ ounces total, split into milk M and coffee C. Angela drank one cup, so

$$\tfrac{1}{4}M + \tfrac{1}{6}C = \tfrac{1}{n}(M + C).$$

The left side is a weighted average of $\frac{1}{4}$ and $\frac{1}{6}$, so $\frac{1}{n}$ lies strictly between $\frac{1}{6}$ and $\frac{1}{4}$. That forces $4 < n < 6$, so $n = 5$.

Thus, the correct answer is **C**.

23. When the mean, median, and mode of the list

$$10, 2, 5, 2, 4, 2, x$$

are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

A 3

B 6

C 9

D 17

E 20

Solution:

The mode is always 2, and the mean is $\frac{25+x}{7}$. For the values to form a non-constant arithmetic progression we examine the median.

If $x = 3$, the sorted list is $2, 2, 2, 3, 4, 5, 10$, with median 3 and mean 4, giving the progression $2, 3, 4$.

If $x = 17$, the sorted list is $2, 2, 2, 4, 5, 10, 17$, with median 4 and mean 6, giving the progression $2, 4, 6$.

No other value of x works, so the sum of all possible values is $3 + 17 = 20$.

Thus, the correct answer is **E**.

24. Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

- A $-\frac{1}{3}$
- B $-\frac{1}{9}$
- C 0
- D $\frac{5}{9}$
- E $\frac{5}{3}$

Solution:

To evaluate $f(3z)$, set $\frac{x}{3} = 3z$, so $x = 9z$. Then

$$f(3z) = (9z)^2 + 9z + 1 = 81z^2 + 9z + 1.$$

Setting this equal to 7 gives $81z^2 + 9z - 6 = 0$.

By the sum-of-roots formula, the sum of the values of z is $-\frac{9}{81} = -\frac{1}{9}$.

Thus, the correct answer is **B**.

25. In year N, the 300th day of the year is a Tuesday. In year $N+1$, the 200th day is also a Tuesday. On what day of the week did the 100th day of year $N-1$ occur?

- A Thursday
- B Friday
- C Saturday
- D Sunday
- E Monday

Solution:

From day 300 of year N to day 200 of year $N+1$ is $(L-300)+200$ days, where L is the length of year N. If N were not a leap year, this is $265 \equiv 6 \pmod 7$, giving a Monday, not a Tuesday. So year N is a leap year, and the count is $266 = 7 \cdot 38$, consistent with Tuesday.

Then years $N-1$ and $N+1$ are not leap years.

The 100th day of year $N-1$ precedes the Tuesday (day 300 of year N) by $(365-100)+300 = 565$ days. Since $565 = 7 \cdot 80 + 5$, that day is 5 days earlier in the week than Tuesday, which is a Thursday.

Thus, the correct answer is **A**.

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