2025 AIME II Solutions

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1. Six points $A, B, C, D, E,$ and F lie in a straight line in that order. Suppose that G is a point not on the line and that $AC = 26, BD = 22, CE = 31, DF = 33, AF = 73, CG = 40,$ and $DG = 30$. Find the area of $\triangle BGE$.

Solution:

Place the line on a number line with $A = 0$. Then $C = 26, E = 26 + 31 = 57, F = 73, D = 73 - 33 = 40$, and $B = 40 - 22 = 18$.

Write $G = (x, y)$. From $CG = 40$ and $DG = 30$,

$$(x-26)^2 + y^2 = 1600, \qquad (x-40)^2 + y^2 = 900.$$

Subtracting gives $14(2x - 66) = 700$, so $x = 58$, and then $y^2 = 1600 - 32^2 = 576$, so G is at height 24 above the line.

Since B and E both lie on the line, $BE = 57 - 18 = 39$ is a base with height 24, so the area is $\frac{1}{2} \cdot 39 \cdot 24 = 468$.

2. Find the sum of all positive integers n such that $n + 2$ divides the product $3(n+3)(n^2+9)$.

Solution:

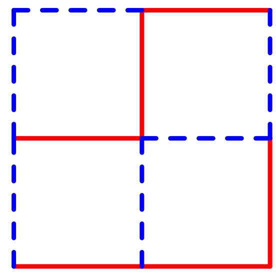
Work modulo $n + 2$, where $n \equiv -2$. Then

$$3(n+3)(n^2+9) \equiv 3 \cdot 1 \cdot (4+9) = 39 \pmod{n+2},$$

so $n + 2$ divides $3(n+3)(n^2+9)$ exactly when $n + 2$ divides 39.

The divisors of 39 that are at least 3 are $3, 13,$ and 39, giving $n = 1, 11,$ and 37. The sum is $1 + 11 + 37 = 49$.

3. Four unit squares form a 2×2 grid. Each of the 12 unit line segments forming the sides of the squares is colored either red or blue in such a way that each unit square has 2 red sides and 2 blue sides. One example is shown below (red is solid, blue is dashed). Find the number of such colorings.



Solution:

The 12 segments split into the 4 interior segments forming the central cross and 8 boundary segments, and each unit square has exactly two interior sides (its two cross arms) and two boundary sides. Color the cross first. A square that already has j red interior sides needs $2-j$ red boundary sides, which can be chosen in $\binom{2}{2-j}$ ways: 1 way if $j=0$ or $j=2$, and 2 ways if $j=1$.

Group the $2^4=16$ cross colorings by the set of red arms. If all four arms have the same color (2 colorings), every square has $j=0$ or $j=2$, contributing 1 each: total 2. If exactly one arm is red or exactly one is blue (8 colorings), the two squares touching the odd arm have $j=1$ and the others do not, contributing $2 \cdot 2 = 4$ each: total 32. If two adjacent arms are red (4 colorings), the squares have $j=2,1,1,0$, contributing 4 each: total 16. If two opposite arms are red (2 colorings), all four squares have $j=1$, contributing $2^4=16$ each: total 32.

The number of colorings is $2+32+16+32=82$.

4. The product

$$\prod_{k=4}^{63} \frac{\log_k(5^{k^2-1})}{\log_{k+1}(5^{k^2-4})} = \frac{\log_4(5^{15})}{\log_5(5^{12})} \cdot \frac{\log_5(5^{24})}{\log_6(5^{21})} \cdot \frac{\log_6(5^{35})}{\log_7(5^{32})} \cdots \frac{\log_{63}(5^{3968})}{\log_{64}(5^{3965})}$$

is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

By the change-of-base formula, $\log_k(5^{k^2-1}) = \frac{(k^2-1)\log 5}{\log k}$, so each factor of the product equals

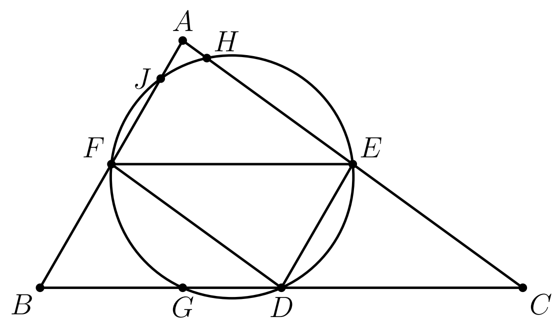
$$\frac{(k^2-1)/\log k}{(k^2-4)/\log(k+1)} = \frac{(k-1)(k+1)}{(k-2)(k+2)} \cdot \frac{\log(k+1)}{\log k}.$$

All three pieces telescope over $k = 4, \ldots, 63:$

$$\prod_{k=4}^{63} \frac{k-1}{k-2} = \frac{62}{2} = 31, \qquad \prod_{k=4}^{63} \frac{k+1}{k+2} = \frac{5}{65} = \frac{1}{13}, \qquad \prod_{k=4}^{63} \frac{\log(k+1)}{\log k} = \frac{\log 64}{\log 4} = 3.$$

The product is $31 \cdot \frac{1}{13} \cdot 3 = \frac{93}{13}$, which is in lowest terms, so $m+n = 93+13 = 106.$

5. Suppose $\triangle ABC$ has angles $\angle BAC = 84^\circ$, $\angle ABC = 60^\circ$, and $\angle ACB = 36^\circ$. Let $D, E,$ and F be the midpoints of sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. The circumcircle of $\triangle DEF$ intersects $\overline{BD}$, $\overline{AE}$, and $\overline{AF}$ at points $G, H,$ and J, respectively. The points $G, D, E, H, J,$ and F divide the circumcircle of $\triangle DEF$ into six minor arcs, as shown. Find $\widehat{DE} + 2 \cdot \widehat{HJ} + 3 \cdot \widehat{FG}$, where the arcs are measured in degrees.



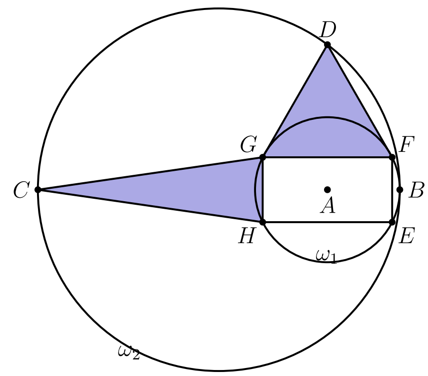
Solution:

The medial triangle DEF has sides parallel to those of ABC, so $\angle FDE = 84^\circ$, $\angle DEF = 60^\circ$, and $\angle DFE = 36^\circ$. Its circumcircle is the nine-point circle, whose second intersections with the sides of ABC are the feet of the altitudes: G is the foot from A, H the foot from B, and J the foot from C. By the inscribed angle theorem, $\widehat{DE} = 2\angle DFE = 72^\circ$.

For $\widehat{FG}$: since $\overline{DF} \parallel \overline{CA}$ and G lies on ray DB, the angle $\angle FDG$ equals the angle between lines CA and CB, which is 36°, so $\widehat{FG} = 2 \cdot 36^\circ = 72^\circ$. For $\widehat{HJ}$: because $\angle BJC = \angle BHC = 90^\circ$, both H and J lie on the circle with diameter $\overline{BC}$ centered at D, so $DJ = DB$ and $DH = DC$. Isosceles triangle BDJ gives $\angle JDB = 180^\circ - 2 \cdot 60^\circ = 60^\circ$, and isosceles triangle CDH gives $\angle HDC = 180^\circ - 2 \cdot 36^\circ = 108^\circ$. Hence $\angle JDH = 180^\circ - 60^\circ - 108^\circ = 12^\circ$ and $\widehat{HJ} = 24^\circ$.

Therefore $\widehat{DE} + 2 \cdot \widehat{HJ} + 3 \cdot \widehat{FG} = 72 + 48 + 216 = 336.$

6. Circle ω_1 with radius 6 centered at point A is internally tangent at point B to circle ω_2 with radius 15. Points C and D lie on ω_2 such that $\overline{BC}$ is a diameter of ω_2 and $\overline{BC} \perp \overline{AD}$. The rectangle $EFGH$ is inscribed in ω_1 such that $\overline{EF} \perp \overline{BC}$, C is closer to $\overline{GH}$ than to $\overline{EF}$, and D is closer to $\overline{FG}$ than to $\overline{EH}$, as shown. Triangles $\triangle DGF$ and $\triangle CHG$ have equal areas. The area of rectangle $EFGH$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution:

Center ω_2 at the origin with $B=(15,0)$. Internal tangency at B puts $A=(9,0)$, and $C=(-15,0)$. Since $\overline{AD} \perp \overline{BC}$ and D is on ω_2, we get $D=(9,12)$ (taking D above the line). Because $\overline{EF} \perp \overline{BC}$, the rectangle has vertical sides, so its vertices are $(9 \pm a, \pm b)$ with $a^2+b^2=36$. The conditions on C and D make $\overline{GH}$ the left side and $\overline{FG}$ the top side: $F=(9+a,b)$, $G=(9-a,b)$, $H=(9-a,-b)$, $E=(9+a,-b)$.

Triangle DGF has base $GF=2a$ and height $12-b$, so its area is $a(12-b)$. Triangle CHG has base $GH=2b$ and height $(9-a)-(-15)=24-a$, so its area is $b(24-a)$. Setting these equal, $12a-ab=24b-ab$, so $a=2b$, and then $a^2+b^2=5b^2=36$.

The area of the rectangle is $2a \cdot 2b = 8b^2 = \frac{288}{5}$, so $m+n=288+5=293$.

7. Let A be the set of positive integer divisors of $2025.$ Let B be a randomly selected subset of $A.$ The probability that B is a nonempty set with the property that the least common multiple of its elements is 2025 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n.$

Solution:

Since $2025 = 3^4 \cdot 5^2,$ the set A has $5 \cdot 3 = 15$ elements, and there are 2^{15} subsets. A subset has least common multiple 2025 exactly when it contains at least one divisor divisible by $3^4 = 81$ and at least one divisible by $5^2 = 25$ (such a subset is automatically nonempty). There are 12 divisors not divisible by $81, 10$ not divisible by $25,$ and 8 divisible by neither.

By inclusion-exclusion, the number of good subsets is

$$2^{15} - 2^{12} - 2^{10} + 2^8 = 32768 - 4096 - 1024 + 256 = 27904.$$

Since $27904 = 2^8 \cdot 109,$ the probability is $\frac{27904}{32768} = \frac{109}{128}$, and $m+n = 109 + 128 = 237.$

8. From an unlimited supply of 1-cent coins, 10-cent coins, and 25-cent coins, Silas wants to find a collection of coins that has a total value of N cents, where N is a positive integer. He uses the so-called *greedy algorithm*, successively choosing the coin of greatest value that does not cause the value of his collection to exceed N. For example, to get 42 cents, Silas will choose a 25-cent coin, then a 10-cent coin, then 7 1-cent coins. However, this collection of 9 coins uses more coins than necessary to get a total of 42 cents; indeed, choosing 4 10-cent coins and 2 1-cent coins achieves the same total value with only 6 coins.

In general, the greedy algorithm succeeds for a given N if no other collection of 1-cent, 10-cent, and 25-cent coins gives a total value of N cents using strictly fewer coins than the collection given by the greedy algorithm. Find the number of values of N between 1 and 1000 inclusive for which the greedy algorithm succeeds.

Solution:

In any optimal collection there are at most 9 pennies (ten pennies could become a dime) and at most 4 dimes (five dimes could become two quarters), so its dimes and pennies are worth at most 49 cents. Hence an optimal collection uses either $q = \lfloor N/25 \rfloor$ quarters, like greedy, or $q-1$ quarters. For an amount v made only of dimes and pennies, the best count is $f(v) = \lfloor v/10 \rfloor + (v \bmod 10)$, which is what greedy does on the remainder.

Let $r = N \bmod 25$. Greedy uses $q + f(r)$ coins, and the only rival uses $(q-1) + f(r+25)$ coins (possible when $q \geq 1$), so greedy fails exactly when $f(r+25) \leq f(r)$. Tabulating: for $r = 0, \ldots, 4$, $f(r+25) = r+7 > f(r) = r$; for $r = 5, \ldots, 9$, $f(r+25) = r-2 \leq r$; for $r = 10, \ldots, 14$, $f(r+25) = r-2 > r-9$; for $r = 15, \ldots, 19$, $f(r+25) = r-11 \leq r-9$; for $r = 20, \ldots, 24$, $f(r+25) = r-11 > r-18$. So greedy fails exactly when $N \geq 25$ and $r \in \{5, \ldots, 9\} \cup \{15, \ldots, 19\}$.

Each residue class mod 25 contains 40 values of N in $1, \ldots, 1000$, so these 10 residues give 400 values, of which the 10 values less than 25 do not count (there $q = 0$). Greedy fails for 390 values and succeeds for $1000 - 390 = 610$.

9. There are n values of x in the interval $0 < x < 2\pi$ where $f(x) = \sin(7\pi \cdot \sin(5x)) = 0$. For t of these n values of x, the graph of $y = f(x)$ is tangent to the x-axis. Find $n + t$.

Solution:

$f(x) = 0$ exactly when $7\pi \sin(5x)$ is a multiple of π, that is, $\sin(5x) = \frac{k}{7}$ for an integer k with $|k| \leq 7$. As x runs over $(0, 2\pi)$, the quantity $5x$ runs over $(0, 10\pi)$, five full periods. For $k = 0$, the solutions are $5x = \pi, 2\pi, \ldots, 9\pi : 9$ values. For each of the 12 values $k = \pm 1, \ldots, \pm 6$, each period contributes 2 solutions: 10 values each. For $k = \pm 7$, we need $\sin(5x) = \pm 1$, which happens 5 times each. So $n = 9 + 120 + 10 = 139$.

The graph is tangent to the x-axis at a zero exactly when $f'(x) = 35\pi \cos(7\pi \sin(5x)) \cos(5x) = 0$ there. At any zero, $\cos(7\pi \sin(5x)) = \cos(k\pi) = \pm 1 \neq 0$, so tangency requires $\cos(5x) = 0$, which means $\sin(5x) = \pm 1$: exactly the 10 zeros with $k = \pm 7$ (there $\sin(5x)$ has an extremum, so f touches without crossing). Thus $t = 10$ and $n + t = 149$.

10. Sixteen chairs are arranged in a row. Eight people each select a chair in which to sit so that no person sits next to two other people. Let N be the number of subsets of 16 chairs that could be selected. Find the remainder when N is divided by 1000.

Solution:

A person sits next to two others exactly when three consecutive chairs are all occupied, so we count 8-element subsets of the 16 chairs with no three consecutive chairs chosen. The occupied chairs then form maximal blocks of size 1 or 2. If there are m blocks, then $8 - m$ of them are pairs and $2m - 8$ are singles, so $4 \leq m \leq 8$, and the pair positions can be chosen in $\binom{m}{8-m}$ ways. The 8 empty chairs create 9 gaps (including the ends), and the m blocks occupy m distinct gaps: $\binom{9}{m}$ ways.

Therefore

$$N = \sum_{m=4}^{8} \binom{m}{8-m} \binom{9}{m} = 1 \cdot 126 + 10 \cdot 126 + 15 \cdot 84 + 7 \cdot 36 + 1 \cdot 9 = 2907.$$

The remainder when $N = 2907$ is divided by 1000 is 907.

11. Let S be the set of vertices of a regular 24-gon. Find the number of ways to draw 12 segments of equal lengths so that each vertex in S is an endpoint of exactly one of the 12 segments.

Solution:

Two chords of a circle through equally spaced points have equal length exactly when they skip the same number of vertices, so all 12 segments join pairs of vertices exactly k apart for one common $k \in \{1, \ldots, 12\}$. For fixed k, form the graph on the 24 vertices joining each i to $i \pm k \pmod{24}$: we need a perfect matching in this graph. For $k < 12$ the graph is a disjoint union of $\gcd(24, k)$ cycles of length $24/\gcd(24, k)$, while for $k = 12$ it is 12 disjoint diameters.

A cycle of even length has exactly 2 perfect matchings (alternate edges), and a cycle of odd length has none. So each $k < 12$ with even cycle length contributes $2^{\gcd(24,k)}$: $k = 1, 5, 7, 11$ give 2 each; $k = 2, 10$ give 4 each; $k = 3, 9$ give 8 each; $k = 4$ gives 16; $k = 6$ gives 64. For $k = 8$ the cycles have odd length 3, giving 0. For $k = 12$ the matching is forced: 1 way.

The total is $4 \cdot 2 + 2 \cdot 4 + 2 \cdot 8 + 16 + 64 + 0 + 1 = 113.$

12. Let $A_1A_2\ldots A_{11}$ be an 11-sided non-convex simple polygon with the following properties:

- For every integer $2 \le i \le 10$, the area of $\triangle A_iA_1A_{i+1}$ is 1.
- For every integer $2 \le i \le 10$, $\cos(\angle A_iA_1A_{i+1}) = \frac{12}{13}$.
- The perimeter of the 11-gon $A_1A_2\ldots A_{11}$ is equal to 20.

Then $A_1A_2 + A_1A_{11}$ can be expressed as $\frac{m\sqrt{n}-p}{q}$ where m, n, p, and q are positive integers, n is not divisible by the square of any prime, and no prime divides all of m, p, and q. Find $m+n+p+q$.

Solution:

Let $r_i = A_1A_i$ for $2 \le i \le 11$, and let θ be the common angle, with $\cos\theta = \frac{12}{13}$ and $\sin\theta = \frac{5}{13}$. Each area condition says $\frac{1}{2}r_ir_{i+1} \cdot \frac{5}{13} = 1$, so $r_ir_{i+1} = \frac{26}{5}$ for $i = 2, \ldots, 10$. Consecutive products being equal forces the r_i to alternate between two values $a = r_2 = r_4 = \cdots$ and $b = r_3 = r_5 = \cdots$, with $ab = \frac{26}{5}$; in particular $r_{11} = b$.

By the law of cosines, every side A_iA_{i+1} with $2 \le i \le 10$ has the same length s, where

$$s^2 = a^2 + b^2 - 2ab \cdot \frac{12}{13} = (a+b)^2 - 2ab - \frac{48}{5} = (a+b)^2 - 20.$$

Writing $u = a + b$, the perimeter condition is $9\sqrt{u^2-20} + u = 20$. Squaring $9\sqrt{u^2-20} = 20 - u$ gives $81u^2 - 1620 = 400 - 40u + u^2$, which simplifies to $4u^2 + 2u - 101 = 0$, so $u = \frac{-1+9\sqrt{5}}{4}$ (the positive root; then $20 - u > 0$ as required).

Thus $A_1A_2 + A_1A_{11} = a + b = \frac{9\sqrt{5}-1}{4}$, with 5 squarefree and no prime dividing all of $9, 1, 4$. The answer is $9 + 5 + 1 + 4 = 19$.

13. Let the sequence of rationals $x_1, x_2, \ldots$ be defined such that $x_1 = \frac{25}{11}$ and

$$x_{k+1} = \frac{1}{3}\left(x_k + \frac{1}{x_k} - 1\right)$$

for all $k \geq 1$. Then x_{2025} can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find the remainder when $m + n$ is divided by 1000.

Solution:

Let $y_k = \frac{2x_k - 1}{x_k + 1}$. From the recurrence, $2x_{k+1} - 1 = \frac{(2x_k-1)(x_k-2)}{3x_k}$ and $x_{k+1} + 1 = \frac{(x_k+1)^2}{3x_k}$, so

$$y_{k+1} = \frac{(2x_k - 1)(x_k - 2)}{(x_k + 1)^2} = y_k(y_k - 1) = y_k^2 - y_k,$$

since $y_k - 1 = \frac{x_k - 2}{x_k + 1}$. Here $y_1 = \frac{39/11}{36/11} = \frac{13}{12}$. By induction $y_k = \frac{c_k}{12^{2^{k-1}}}$ where $c_1 = 13$ and $c_{k+1} = c_k\left(c_k - 12^{2^{k-1}}\right)$; since $12^{2^{k-1}}$ is divisible by 6, every c_k stays coprime to 6.

Inverting the substitution, $x_k = \frac{1+y_k}{2-y_k} = \frac{d+c}{2d-c}$ with $d = 12^{2^{k-1}}$ and $c = c_k$. All x_k are positive (for $x > 0, x + \frac{1}{x} - 1 \geq 1$), so $y_k = \frac{2x_k - 1}{x_k + 1} \in (-1, 2)$, making both $d + c$ and $2d - c$ positive. Any common divisor of $d + c$ and $2d - c$ divides their combinations $3d$ and $3c$; as $\gcd(c, d) = 1$, it divides 3, but $3 \mid d$ and $3 \nmid c$, so $3 \nmid d + c$. Hence the fraction is in lowest terms and $m + n = 3d = 3 \cdot 12^{2^{2024}}$.

Modulo 8, $12^{2^{2024}} \equiv 0$. Modulo 125, the multiplicative order of 12 divides $\lambda(125) = 100$, and $2^{2024} \equiv 16 \pmod{100}$ (it is 0 mod 4, and $2^{20} \equiv 1$ mod 25 with $2024 \equiv 4$ mod 20), so $12^{2^{2024}} \equiv 12^{16} \equiv 41 \pmod{125}$. The Chinese remainder theorem gives $12^{2^{2024}} \equiv 416 \pmod{1000}$, so $m + n \equiv 3 \cdot 416 = 1248 \equiv 248 \pmod{1000}$.

14. Let $\triangle ABC$ be a right triangle with $\angle A = 90^\circ$ and $BC = 38$. There exist points K and L inside the triangle such that

$$AK = AL = BK = CL = KL = 14.$$

The area of the quadrilateral $BKLC$ can be expressed as $n\sqrt{3}$ for some positive integer n. Find n.

Solution:

Since $AK = AL = KL = 14$, triangle AKL is equilateral and $\angle KAL = 60^\circ$. Let $\alpha = \angle BAK$ and $\beta = \angle LAC$, so $\alpha + \beta = 30^\circ$. Because $AK = KB$, point K lies on the perpendicular bisector of $\overline{AB}$, so $AB = 2 \cdot 14 \cos\alpha = 28\cos\alpha$; similarly $AC = 28\cos\beta$. Then $AB^2 + AC^2 = 38^2$ gives $\cos^2\alpha + \cos^2\beta = \frac{361}{196}$, i.e. $\cos 2\alpha + \cos 2\beta = \frac{165}{98}$. By sum-to-product, $2\cos(\alpha+\beta)\cos(\alpha-\beta) = \sqrt{3}\cos(\alpha-\beta) = \frac{165}{98}$, so $\cos(\alpha-\beta) = \frac{55\sqrt{3}}{98}$.

Decompose $[BKLC] = [ABC] - [ABK] - [ACL] - [AKL]$. First,

$$[ABC] = \tfrac{1}{2}AB \cdot AC = 392\cos\alpha\cos\beta = 196\big(\cos(\alpha-\beta) + \cos(\alpha+\beta)\big) = 110\sqrt{3} + 98\sqrt{3} = 208\sqrt{3}.$$

Next, K has height $14\sin\alpha$ over $\overline{AB}$, so $[ABK] = \frac{1}{2} \cdot 28\cos\alpha \cdot 14\sin\alpha = 98\sin 2\alpha$, and likewise $[ACL] = 98\sin 2\beta$; their sum is $196\sin(\alpha+\beta)\cos(\alpha-\beta) = 98 \cdot \frac{55\sqrt{3}}{98} = 55\sqrt{3}$. Finally $[AKL] = \frac{\sqrt{3}}{4} \cdot 14^2 = 49\sqrt{3}$.

Therefore $[BKLC] = 208\sqrt{3} - 55\sqrt{3} - 49\sqrt{3} = 104\sqrt{3}$, so $n = 104$.

15. There are exactly three positive real numbers k such that the function

$$f(x) = \frac{(x-18)(x-72)(x-98)(x-k)}{x}$$

defined over the positive real numbers achieves its minimum value at exactly two positive real numbers x. Find the sum of these three values of k.

Solution:

For $x > 0, f(x) \to +\infty$ both as $x \to 0^+$ (the numerator tends to $18 \cdot 72 \cdot 98 \cdot k > 0$) and as $x \to \infty$, so f attains a global minimum value c on $(0, \infty)$. It is attained at exactly two points precisely when $f(x) - c \geq 0$ with two distinct positive double roots, i.e.

$$(x-18)(x-72)(x-98)(x-k) - cx = (x^2 - Sx + P)^2$$

where the roots of $x^2 - Sx + P$ are positive and distinct (so $S, P > 0$).

Matching coefficients of x^3, x^2, and the constant (the x-coefficient just determines c):

$$2S = 188 + k, \qquad S^2 + 2P = 10116 + 188k, \qquad P^2 = 18 \cdot 72 \cdot 98 \cdot k = 127008k.$$

Substitute $k = 2t^2$ with $t > 0$: then $S = 94 + t^2$ and $P = 504t$. The middle equation becomes $(94 + t^2)^2 + 1008t = 10116 + 376t^2$, i.e.

$$t^4 - 188t^2 + 1008t - 1280 = 0,$$

which factors as $(t-2)(t-4)(t+16)(t-10) = 0.$

The positive roots $t = 2, 4, 10$ give $k = 2t^2 = 8, 32, 200$ (each indeed yields $S^2 > 4P$, matching the problem's promise of exactly three values). The sum is $8 + 32 + 200 = 240.$

Problems: https://live.poshenloh.com/past-contests/aime/2025II