2025 AIME I Solutions

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1. Find the sum of all integer bases $b > 9$ for which 17_b is a divisor of 97_b.

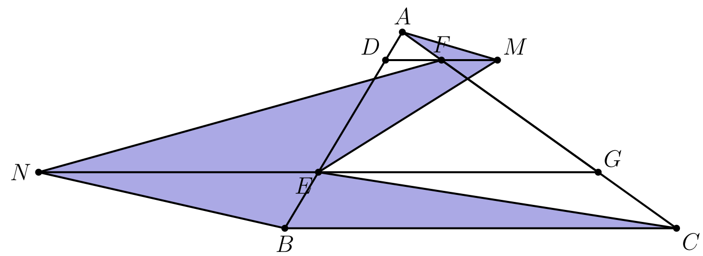
Solution:

In base b the two numbers are $17_b = b + 7$ and $97_b = 9b + 7$. We need $b + 7 \mid 9b + 7$, and since $b + 7$ certainly divides $9(b + 7) = 9b + 63$, this is equivalent to

$$b + 7 \mid (9b + 63) - (9b + 7) = 56.$$

For $b > 9$ we have $b + 7 > 16$, so $b + 7$ must be 28 or 56, giving $b = 21$ or $b = 49$. The sum is $21 + 49 = 70$.

2. On $\triangle ABC$ points $A, D, E,$ and B lie in that order on side $\overline{AB}$ with $AD = 4, DE = 16,$ and $EB = 8$. Points $A, F,$ $G,$ and C lie in that order on side $\overline{AC}$ with $AF = 13, FG = 52,$ and $GC = 26$. Let M be the reflection of D through F, and let N be the reflection of G through E. Quadrilateral $DEGF$ has area 288. Find the area of heptagon $AFNBCEM$.



Solution:

Here $AB = 4 + 16 + 8 = 28$ and $AC = 13 + 52 + 26 = 91$, so D and F lie $\frac{1}{7}$ of the way from A along their sides while E and G lie $\frac{5}{7}$ of the way. Triangles sharing angle A have areas proportional to the products of the adjacent sides, so $[ADF] = \frac{1}{49}[ABC]$ and $[AEG] = \frac{25}{49}[ABC]$. Therefore

$$[DEGF] = [AEG] - [ADF] = \frac{24}{49}[ABC] = 288,$$

which gives $[ABC] = 588$.

Now set $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{AC}$, so that $D = \frac{1}{7}\mathbf{b}, E = \frac{5}{7}\mathbf{b}, F = \frac{1}{7}\mathbf{c}, G = \frac{5}{7}\mathbf{c}$, and the reflections are $M = 2F - D = \frac{1}{7}(2\mathbf{c} - \mathbf{b})$ and $N = 2E - G = \frac{1}{7}(10\mathbf{b} - 5\mathbf{c})$. The shoelace formula for $AFNBCEM$ sums cross products of consecutive vertices: the two terms at A vanish, and

$$F \times N = -\tfrac{10}{49}\,\mathbf{b} \times \mathbf{c}, \quad N \times B = \tfrac{5}{7}\,\mathbf{b} \times \mathbf{c}, \quad B \times C = \mathbf{b} \times \mathbf{c}, \quad C \times E = -\tfrac{5}{7}\,\mathbf{b} \times \mathbf{c}, \quad E \times M = \tfrac{10}{49}\,\mathbf{b} \times \mathbf{c}.$$

Everything cancels except the single term $\mathbf{b} \times \mathbf{c}$, so the heptagon's area is $\frac{1}{2}\,|\mathbf{b} \times \mathbf{c}| = [ABC] = 588$.

3. The 9 members of a baseball team went to an ice-cream parlor after their game. Each player had a single scoop cone of chocolate, vanilla, or strawberry ice cream. At least one player chose each flavor, and the number of players who chose chocolate was greater than the number of players who chose vanilla, which was greater than the number of players who chose strawberry. Let N be the number of different assignments of flavors to players that meet these conditions. Find the remainder when N is divided by 1000.

Solution:

Let $c > v > s \geq 1$ be the numbers of players choosing chocolate, vanilla, and strawberry, with $c + v + s = 9$. Checking small values of s shows the only possibilities are $(6, 2, 1)$, $(5, 3, 1)$, and $(4, 3, 2)$.

Since the players are distinct, each triple of counts contributes a multinomial coefficient:

$$\frac{9!}{6!\,2!\,1!} = 252, \qquad \frac{9!}{5!\,3!\,1!} = 504, \qquad \frac{9!}{4!\,3!\,2!} = 1260.$$

Thus $N = 252 + 504 + 1260 = 2016$, and the remainder modulo 1000 is 16.

4. Find the number of ordered pairs (x, y), where both x and y are integers between -100 and 100, inclusive, such that $12x^2 - xy - 6y^2 = 0$.

Solution:

The equation factors as

$$12x^2 - xy - 6y^2 = (3x + 2y)(4x - 3y) = 0,$$

so every solution has $4x = 3y$ or $3x = -2y$.

Integer solutions of $4x = 3y$ are $(x, y) = (3t, 4t)$; the constraint $|4t| \leq 100$ gives $-25 \leq t \leq 25$, or 51 pairs. Integer solutions of $3x = -2y$ are $(x, y) = (2t, -3t)$; the constraint $|3t| \leq 100$ gives $-33 \leq t \leq 33$, or 67 pairs. The families overlap only at $(0, 0)$, so the count is $51 + 67 - 1 = 117$.

5. There are $8! = 40320$ eight-digit positive integers that use each of the digits $1, 2, 3, 4, 5, 6, 7, 8$ exactly once. Let N be the number of these integers that are divisible by 22. Find the difference between N and 2025.

Solution:

The digits sum to 36. Divisibility by 11 requires the alternating sum of digits to be a multiple of 11, so if the four digits in odd positions sum to a, then $a - (36 - a) = 2a - 36$ must be a multiple of 11. Since $10 \leq a \leq 26$, the only possibility is $a = 18$: each block of four positions carries digit sum 18. The four-element subsets of $\{1, \ldots, 8\}$ with sum 18 are

$$\{1,2,7,8\},\ \{1,3,6,8\},\ \{1,4,5,8\},\ \{1,4,6,7\},\ \{2,3,5,8\},\ \{2,3,6,7\},\ \{2,4,5,7\},\ \{3,4,5,6\},$$

eight in all, and they come in complementary pairs.

Choose which of the 8 subsets occupies the even positions (which include the units place); the complement fills the odd positions. If that subset contains k of the even digits, then the units digit can be chosen in k ways, the rest of the even positions in $3!$ ways, and the odd positions in $4!$ ways, for $144k$ numbers. Complementary subsets have k-values summing to 4, so over all 8 choices $\sum k = 16$. Hence $N = 144 \cdot 16 = 2304$, and $N - 2025 = 279$.

6. An isosceles trapezoid has an inscribed circle tangent to each of its four sides. The radius of the circle is 3, and the area of the trapezoid is 72. Let the parallel sides of the trapezoid have lengths r and s, with $r \neq s$. Find $r^2 + s^2$.

Solution:

The circle is tangent to both parallel sides, so the height of the trapezoid is $2 \cdot 3 = 6$. From the area, $\frac{r+s}{2} \cdot 6 = 72$, so $r + s = 24$. By the Pitot theorem the legs together also sum to 24, and since the trapezoid is isosceles each leg is 12.

Dropping a perpendicular from an endpoint of the shorter base, the leg is the hypotenuse of a right triangle with legs 6 and $\frac{|r-s|}{2}$:

$$144 = 36 + \left(\frac{r-s}{2}\right)^2,$$

so $(r - s)^2 = 432$. Therefore $r^2 + s^2 = \frac{(r+s)^2+(r-s)^2}{2} = \frac{576+432}{2} = 504$.

7. The twelve letters $A, B, C, D, E, F, G, H, I, J, K$, and L are randomly grouped into six pairs of letters. The two letters in each pair are placed next to each other in alphabetical order to form six two-letter words, and then those six words are listed alphabetically. For example, a possible result is AB, CJ, DG, EK, FL, HI. The probability that the last word listed contains G is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

There are $11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 10395$ ways to pair the letters. Each word begins with the smaller letter of its pair, so the last word alphabetically is the pair whose smaller letter is largest.

Case 1: G is the smaller letter of the last word. Then G pairs with one of H, I, J, K, L (5 ways), and no two of the remaining four late letters may pair together (such a pair would start with a letter after G). Those four letters must take distinct partners from $\{A, \ldots, F\}$, in $6 \cdot 5 \cdot 4 \cdot 3 = 360$ ways, and the two leftover early letters pair with each other. That gives $5 \cdot 360 = 1800$ pairings. Case 2: G is the larger letter, paired with some x before G. Then none of $H, \ldots, L$ may pair together, so all five take partners among the other five early letters; the six smaller letters are then exactly A through F, and the largest is F. So the last word is FG, and $H, \ldots, L$ match with $A, \ldots, E$ in $5! = 120$ ways.

The probability is $\frac{1800+120}{10395} = \frac{1920}{10395} = \frac{128}{693}$, so $m+n = 128 + 693 = 821$.

8. Let k be a real number such that the system

$$|25 + 20i - z| = 5$$

$$|z - 4 - k| = |z - 3i - k|$$

has exactly one complex solution z. The sum of all possible values of k can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$. Here $i = \sqrt{-1}$.

Solution:

The first equation says z lies on the circle of radius 5 centered at $(25, 20)$. The second says z is equidistant from $P_1 = (k+4, 0)$ and $P_2 = (k, 3)$, i.e. it lies on the perpendicular bisector of $\overline{P_1 P_2}$. The system has exactly one solution precisely when this line is tangent to the circle.

The midpoint is $\left(k+2, \frac{3}{2}\right)$ and $\overline{P_1 P_2}$ has slope $-\frac{3}{4}$, so the bisector has slope $\frac{4}{3}$: in standard form $8x - 6y - (8k+7) = 0$. Tangency requires

$$\frac{|8 \cdot 25 - 6 \cdot 20 - 8k - 7|}{\sqrt{8^2 + 6^2}} = \frac{|73 - 8k|}{10} = 5,$$

so $8k = 73 \pm 50$, giving $k = \frac{123}{8}$ or $k = \frac{23}{8}$.

The sum is $\frac{146}{8} = \frac{73}{4}$, so $m+n = 73 + 4 = 77$.

9. The parabola with equation $y = x^2 - 4$ is rotated $60°$ counterclockwise around the origin. The unique point in the fourth quadrant where the original parabola and its image intersect has y-coordinate $\frac{a-\sqrt{b}}{c}$, where $a, b,$ and c are positive integers, and a and c are relatively prime. Find $a + b + c$.

Solution:

A point P lies on the image parabola exactly when its rotation by $-60°$, namely

$$Q = \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y,\ -\frac{\sqrt{3}}{2}x + \frac{y}{2}\right),$$

lies on the original parabola. So we need P and Q both on $y = x^2 - 4$. The parabola is symmetric in $x \mapsto -x$, so we look for $P = (x, y)$ whose rotated image is the mirror point $Q = (-x, y)$.

Matching y-coordinates gives $-\frac{\sqrt{3}}{2}x + \frac{y}{2} = y$, i.e. $y = -\sqrt{3}\,x$, and then the x-coordinate works automatically: $\frac{x}{2} + \frac{\sqrt{3}}{2}(-\sqrt{3}x) = -x$. Substituting $y = -\sqrt{3}\,x$ into $y = x^2 - 4$ gives $x^2 + \sqrt{3}\,x - 4 = 0$, whose positive root is $x = \frac{-\sqrt{3}+\sqrt{19}}{2}$. Then

$$y = -\sqrt{3}\,x = \frac{3 - \sqrt{57}}{2} < 0,$$

so this point is in the fourth quadrant, on both curves.

The problem guarantees the fourth-quadrant intersection is unique, so its y-coordinate is $\frac{3-\sqrt{57}}{2}$, giving $a + b + c = 3 + 57 + 2 = 62$.

10. The 27 cells of a 3×9 grid are filled in using the numbers 1 through 9 so that each row contains 9 different numbers, and each of the three 3×3 blocks heavily outlined in the example below contains 9 different numbers, as in the first three rows of a Sudoku puzzle.

4	2	8	9	6	3	1	7	5
3	7	9	5	2	1	6	8	4
5	6	1	8	4	7	9	2	3

The number of different ways to fill such a grid can be written as $p^a \cdot q^b \cdot r^c \cdot s^d$, where $p, q, r,$ and s are distinct prime numbers and a, b, c, d are positive integers. Find $p \cdot a + q \cdot b + r \cdot c + s \cdot d$.

Solution:

Fill the left block arbitrarily: $9!$ ways. Let R_1, R_2, R_3 be the sets of three digits in its rows. In the middle block, row i must avoid R_i (those digits already appear in row i), and the block's three rows must partition $\{1, \ldots, 9\}$. Say its top row takes j digits from R_2 and $3 - j$ from R_3. Balancing the three rows then forces the middle row to take $3 - j$ digits from R_1 together with all j remaining digits of R_3, and the bottom row is determined. The number of content choices is

$$\sum_{j=0}^{3} \binom{3}{j} \binom{3}{3-j}^2 = 1 + 27 + 27 + 1 = 56.$$

The right block's row contents are then forced (row i takes whatever is missing from row i), and each of the six rows of the middle and right blocks can be ordered internally in $3!$ ways. The total is

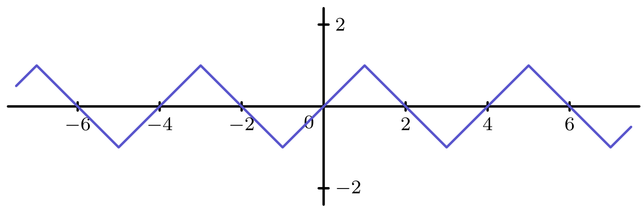
$$9! \cdot 56 \cdot 6^6 = (2^7 \cdot 3^4 \cdot 5 \cdot 7)(2^3 \cdot 7)(2^6 \cdot 3^6) = 2^{16} \cdot 3^{10} \cdot 5^1 \cdot 7^2.$$

Therefore $p \cdot a + q \cdot b + r \cdot c + s \cdot d = 2 \cdot 16 + 3 \cdot 10 + 5 \cdot 1 + 7 \cdot 2 = 81.$

11. A piecewise linear function is defined by

$$f(x) = \begin{cases} x & \text{if } -1 \leq x < 1 \\ 2 - x & \text{if } 1 \leq x < 3 \end{cases}$$

and $f(x+4) = f(x)$ for all real numbers x. The graph of $f(x)$ has the sawtooth pattern depicted below.



The parabola $x = 34y^2$ intersects the graph of $f(x)$ at finitely many points. The sum of the y-coordinates of all these intersection points can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a, b, d have greatest common divisor equal to 1, and c is not divisible by the square of any prime. Find $a + b + c + d$.

Solution:

Since f only takes values in $[-1, 1]$, any intersection has $-1 \leq y \leq 1$ and hence $x = 34y^2 \in [0, 34]$. On the rising pieces, $x \in [4k-1, 4k+1)$ with $f(x) = x - 4k$, so $y = f(34y^2)$ becomes $34y^2 - y - 4k = 0$; on the falling pieces, $x \in [4k+1, 4k+3)$ with $f(x) = 4k + 2 - x$, giving $34y^2 + y - (4k+2) = 0$. In each case a root is valid exactly when it lies in $[-1, 1)$ (rising) or $(-1, 1]$ (falling), since then $x = 34y^2$ automatically falls in the correct interval.

For the rising pieces the roots are $\frac{1 \pm \sqrt{1+544k}}{68}$, and both are valid exactly when $\sqrt{1 + 544k} \leq 67$, i.e. for $k = 0, 1, \ldots, 8$: nine quadratics, each contributing root sum $\frac{1}{34}$ by Vieta. For the falling pieces the roots are $\frac{-1 \pm \sqrt{544k+273}}{68}$. The root with the minus sign requires $\sqrt{544k + 273} < 67$, which holds for $k = 0, \ldots, 7$; those eight quadratics each contribute $-\frac{1}{34}$. For $k = 8$ only the positive root $\frac{-1+\sqrt{4625}}{68} = \frac{-1+5\sqrt{185}}{68}$ is valid.

The total is

$$\frac{9}{34} - \frac{8}{34} + \frac{-1 + 5\sqrt{185}}{68} = \frac{1 + 5\sqrt{185}}{68},$$

and $185 = 5 \cdot 37$ is squarefree, so $a + b + c + d = 1 + 5 + 185 + 68 = 259$.

12. The set of points in 3-dimensional coordinate space that lie in the plane $x+y+z=75$ whose coordinates satisfy the inequalities

$$x - yz < y - zx < z - xy$$

forms three disjoint convex regions. Exactly one of those regions has finite area. The area of this finite region can be expressed in the form $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime. Find $a+b$.

Solution:

Since $x - yz - (y - zx) = (x-y) + z(x-y) = (x-y)(1+z)$, and similarly $y - zx - (z - xy) = (y - z)(1+x)$, the conditions are

$$(x-y)(1+z) < 0 \qquad \text{and} \qquad (y-z)(1+x) < 0.$$

Each condition offers two sign patterns, giving four combinations. The combination $x > y, z < -1, y > z, x < -1$ is impossible on the plane: $x, z < -1$ forces $y > 77$, contradicting $x > y$. Two of the remaining combinations allow a coordinate to run off to infinity, producing the two unbounded regions.

The bounded region is $x < y, y < z, x > -1$ (the fourth constraint $z > -1$ is then automatic): the set $-1 < x < y < z$ on the plane. Its closure is the triangle whose vertices come from intersecting the boundary lines pairwise: $x = -1, x = y$ gives $(-1, -1, 77)$; $x = -1, y = z$ gives $(-1, 38, 38)$; and $x = y = z$ gives $(25, 25, 25)$.

With $A = (-1, -1, 77)$, the edge vectors are $B - A = (0, 39, -39)$ and $C - A = (26, 26, -52)$, whose cross product is $-1014\,(1,1,1)$, of length $1014\sqrt{3}$. The area is $\frac{1014\sqrt{3}}{2} = 507\sqrt{3}$, so $a + b = 507 + 3 = 510$.

13. Alex divides a disk into four quadrants with two perpendicular diameters intersecting at the center of the disk. He draws 25 more line segments through the disk, drawing each segment by selecting two points at random on the perimeter of the disk in different quadrants and connecting these two points. Find the expected number of regions into which these 27 line segments divide the disk.

Solution:

Adding chords one at a time, each new chord increases the region count by 1 plus the number of existing chords it crosses inside the disk. Starting from one region, the expected total is $1 + 27 + E$, where E is the expected number of interior crossing pairs. The two diameters cross once. A random chord's endpoints land in one of the 6 quadrant pairs, each with probability $\frac{1}{6}$. The chord crosses the vertical diameter exactly when its endpoints have opposite x-signs, which happens for 4 of the 6 pairs, so it meets each diameter with probability $\frac{2}{3}$ and both diameters together $\frac{4}{3}$ times on average: the 25 chords contribute $\frac{100}{3}$ expected crossings with the diameters.

For two random chords, condition on their quadrant pairs (36 equally likely ordered combinations). If one uses quadrants $1, 3$ and the other $2, 4$, the endpoints always alternate, so they always cross: 2 combinations. If the two pairs are adjacent and disjoint, such as $\{1, 2\}$ and $\{3, 4\}$, the chords never cross: 4 combinations. In each of the other 30 combinations, whether the endpoints alternate around the circle reduces to comparing independent uniform points inside shared quadrants — for example, a $\{1, 3\}$ chord and a $\{1, 2\}$ chord cross exactly when the two quadrant-1 points come in one specific order — and the probability is $\frac{1}{2}$ by symmetry. So two random chords cross with probability

$$\frac{2 \cdot 1 + 4 \cdot 0 + 30 \cdot \frac{1}{2}}{36} = \frac{17}{36}.$$

The $\binom{25}{2} = 300$ chord pairs contribute $300 \cdot \frac{17}{36} = \frac{425}{3}$ expected crossings, so $E = 1 + \frac{100}{3} + \frac{425}{3} = 176$ and the expected number of regions is $1 + 27 + 176 = 204.$

14. Let $ABCDE$ be a convex pentagon with $AB = 14, BC = 7, CD = 24, DE = 13, EA = 26$, and $\angle B = \angle E = 60^\circ$. For each point X in the plane, define $f(X) = AX + BX + CX + DX + EX$. The least possible value of $f(X)$ can be expressed as $m + n\sqrt{p}$, where m and n are positive integers and p is not divisible by the square of any prime. Find $m + n + p$.

Solution:

In triangle ABC, the law of cosines with $\angle B = 60^\circ$ gives $AC^2 = 14^2 + 7^2 - 14 \cdot 7 = 147$, so $AC = 7\sqrt{3}$; since $7^2 + 147 = 14^2$, the angle at C is right and $\angle BAC = 30^\circ$. Likewise $AD = 13\sqrt{3}$, with a right angle at D and $\angle DAE = 30^\circ$. In triangle ACD with $CD = 24$,

$$\cos \angle CAD = \frac{147 + 507 - 576}{2 \cdot 7\sqrt{3} \cdot 13\sqrt{3}} = \frac{1}{7}, \qquad \sin \angle CAD = \frac{4\sqrt{3}}{7}.$$

Split $f(X) = (BX + EX) + (AX + CX + DX) \geq BE + T$, where T is the minimum of $AX + CX + DX$. Since $\angle BAE = 30^\circ + \angle CAD + 30^\circ$, we get $\cos \angle BAE = \frac{1}{2} \cdot \frac{1}{7} - \frac{\sqrt{3}}{2} \cdot \frac{4\sqrt{3}}{7} = -\frac{11}{14}$, so $BE^2 = 14^2 + 26^2 + 2 \cdot 14 \cdot 26 \cdot \frac{11}{14} = 1444$ and $BE = 38$. All angles of triangle ACD are less than 120°, so T is attained at its Fermat point; erecting an equilateral triangle ACP on side AC away from D, the standard rotation argument gives $T = PD$, and since $\angle PAD = 60^\circ + \angle CAD$ also has cosine $-\frac{11}{14}$,

$$T^2 = 147 + 507 + 2 \cdot 7\sqrt{3} \cdot 13\sqrt{3} \cdot \frac{11}{14} = 1083, \qquad T = 19\sqrt{3}.$$

Both bounds are tight simultaneously: let F be the Fermat point of ACD, so $\angle AFC = \angle AFD = 120^\circ$. Since $\angle AFC + \angle ABC = 180^\circ$, point F lies on the circumcircle of ABC, whence $\angle AFB = \angle ACB = 90^\circ$; similarly F lies on the circumcircle of AED and $\angle AFE = \angle ADE = 90^\circ$. Thus $\angle BFE = 180^\circ$, so F lies on segment BE and $f(F) = BE + T = 38 + 19\sqrt{3}$. The answer is $m + n + p = 38 + 19 + 3 = 60$.

15. Let N denote the number of ordered triples of positive integers (a, b, c) such that $a, b, c \leq 3^6$ and $a^3 + b^3 + c^3$ is a multiple of 3^7. Find the remainder when N is divided by 1000.

Solution:

Since $(a + 3^6 t)^3 \equiv a^3 \pmod{3^7}$, the cube of a modulo 3^7 depends only on $a \bmod 3^6$, and each residue occurs exactly once in $1 \leq a \leq 3^6$. Moreover, the only cube roots of 1 modulo 3^7 are $1 + 3^6 t$, which all agree modulo 3^6; hence cubing is a bijection from the 486 units modulo 3^6 onto the set of unit cubes modulo 3^7, which is exactly the set of units $\equiv \pm 1 \pmod{9}$. If all three of a, b, c are prime to 3 (or exactly one is), then modulo 9 the sum of cubes is $\pm 1 \pm 1 \pm 1$ or ± 1, never 0 : no solutions.

Exactly one multiple of 3, say $c = 3z$: for each of the 486 units a and 243 choices of c, the requirement $b^3 \equiv -a^3 - 27z^3 \pmod{3^7}$ has a right side that is a unit $\equiv \pm 1 \pmod{9}$, hence has exactly one solution b modulo 3^6. With 3 choices for which variable is the multiple of 3, this case gives $3 \cdot 486 \cdot 243 = 354294$ triples.

All three multiples of 3 : writing $a = 3x$ etc. with x, y, z ranging modulo 3^5, the condition becomes $x^3 + y^3 + z^3 \equiv 0 \pmod{3^4}$, which depends only on the residues modulo 3^3, so the count is 9^3 times the count modulo 27. Repeating the same analysis one level down: the two-unit case gives $3 \cdot 18 \cdot 9 = 486$, and the all-divisible case reduces to $u + v + w \equiv 0 \pmod{3}$ with u, v, w modulo 9, giving 243; that is $486 + 243 = 729$ triples modulo 27, hence $729 \cdot 729 = 531441$ here. In total $N = 354294 + 531441 = 885735$, whose remainder modulo 1000 is 735.

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