# 2024 AIME II Solutions

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1. Among the $900$ residents of Aimeville, there are $195$ who own a diamond ring, $367$ who own a set of golf clubs, and $562$ who own a garden spade. In addition, each of the $900$ residents owns a bag of candy hearts. There are $437$ residents who own exactly two of these things, and $234$ residents who own exactly three of these things. Find the number of residents of Aimeville who own all four of these things.

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## Solution:

Adding the four ownership counts gives $195 + 367 + 562 + 900 = 2024$ item ownerships among the $900$ residents. Since everyone owns a bag of candy hearts, every resident owns at least one item, and a resident owning exactly $k$ items is counted $k - 1$ times beyond the first.

If $n_4$ residents own all four things, the extra counts total

$$2024 - 900 = 437 \cdot 1 + 234 \cdot 2 + n_4 \cdot 3,$$

so $1124 = 905 + 3n_4$, giving $n_4 = \frac{219}{3} = 73.$

**2.** A list of positive integers has the following properties:

- The sum of the items in the list is $30$.
- The unique mode of the list is $9$.
- The median of the list is a positive integer that does not appear in the list itself.

Find the sum of the squares of all the items in the list.

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## Solution:

The median is an integer that is not in the list, so the list cannot have odd length (then the median would be a member). The unique mode $9$ appears at least twice. Two items $9, 9$ sum to $18,$ not $30,$ so try four items $a < b < 9$ together with $9, 9,$ where $a$ and $b$ are distinct (a repeat would tie the mode) and $a + b = 12$. The median $\frac{b+9}{2}$ must be an integer, so $b$ is odd, and $a = 12 - b < b$ forces $b > 6$. Thus $b = 7$ and $a = 5$ : the list $5, 7, 9, 9$ has median $8,$ which indeed does not appear.

No longer list works: with two $9$s, six items would need four distinct other values summing to $12,$ namely $\{1, 2, 3, 6\}$ or $\{1, 2, 4, 5\},$ but both give median $4.5$. With three $9$s the remaining items sum to $3,$ and every option either puts $9$ at the median or ties the mode.

The sum of squares is $25 + 49 + 81 + 81 = 236.$

**3.** Find the number of ways to place a digit in each cell of a $2 \times 3$ grid so that the sum of the two numbers formed by reading left to right is $999$, and the sum of the three numbers formed by reading top to bottom is $99$. The grid below is an example of such an arrangement because $8 + 991 = 999$ and $9 + 9 + 81 = 99$.

| | | |
|---|---|---|
| 0 | 0 | 8 |
| 9 | 9 | 1 |

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## Solution:

Let the top row hold digits $a, b, c$ and the bottom row $d, e, f$. In the sum of the two row numbers, the units digits satisfy $c + f \equiv 9 \pmod{10}$, and since $c + f \leq 18$ in fact $c + f = 9$ with no carry. Repeating the argument in the tens and hundreds places gives $b + e = 9$ and $a + d = 9$.

The three column numbers add to $10(a + b + c) + (d + e + f) = 99$. Writing $S = a + b + c$, the bottom digits sum to $27 - S$, so $10S + 27 - S = 99$ and $S = 8$.

Conversely, any digits with $a + b + c = 8$ determine the bottom row by $d = 9 - a, e = 9 - b, f = 9 - c$, and both conditions hold. The number of solutions of $a + b + c = 8$ in nonnegative digits is $\binom{10}{2} = 45$.

**4.** Let $x, y$, and $z$ be positive real numbers that satisfy the following system of equations:

$$\log_2\left(\frac{x}{yz}\right) = \frac{1}{2}$$
$$\log_2\left(\frac{y}{xz}\right) = \frac{1}{3}$$
$$\log_2\left(\frac{z}{xy}\right) = \frac{1}{4}$$

Then the value of $\left|\log_2(x^4y^3z^2)\right|$ is $\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

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## Solution:

Let $a = \log_2 x, b = \log_2 y, c = \log_2 z$. The equations become $a - b - c = \frac{1}{2}, b - a - c = \frac{1}{3}, c - a - b = \frac{1}{4}$. Adding all three gives $-(a+b+c) = \frac{13}{12}$. Since $a - b - c = 2a - (a + b + c)$, we get

$$2a = \frac{1}{2} - \frac{13}{12} = -\frac{7}{12},$$

so $a = -\frac{7}{24}$, and similarly $b = -\frac{3}{8}$ and $c = -\frac{5}{12}$.

Therefore $4a + 3b + 2c = -\frac{28}{24} - \frac{27}{24} - \frac{20}{24} = -\frac{75}{24} = -\frac{25}{8}$, so $\left|\log_2(x^4y^3z^2)\right| = \frac{25}{8}$ and $m + n = 25 + 8 = 33.$

**5.** Let $ABCDEF$ be a convex equilateral hexagon in which all pairs of opposite sides are parallel. The triangle whose sides are extensions of segments $\overline{AB}, \overline{CD}$, and $\overline{EF}$ has side lengths $200, 240$, and $300$. Find the side length of the hexagon.

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## Solution:

Let $s$ be the hexagon's side length, and let the triangle formed by lines $AB, CD, EF$ have sides of lengths $P, Q, R$ along those three lines, respectively. The corner triangle cut off at the vertex $X$ where lines $AB$ and $CD$ meet has third side $BC$, and since $BC \parallel EF$, all three of its sides are parallel to sides of the big triangle. So it is similar to the big triangle with ratio $\frac{BC}{R} = \frac{s}{R}$, and its side along line $AB$ has length $P \cdot \frac{s}{R}$. Likewise the corner at $AB \cap EF$ contains $FA \parallel CD$ and cuts off $P \cdot \frac{s}{Q}$ from the $P$-side.

The $P$-side therefore decomposes as corner piece, $AB$, corner piece:

$$P = P \cdot \frac{s}{R} + s + P \cdot \frac{s}{Q},$$

and dividing by $P$ gives $1 = s\left(\frac{1}{P} + \frac{1}{Q} + \frac{1}{R}\right)$, symmetric in the three sides.

Hence $s = \frac{1}{\frac{1}{200}+\frac{1}{240}+\frac{1}{300}} = \frac{1200}{6+5+4} = 80.$

**6.** Alice chooses a set $A$ of positive integers. Then Bob lists all finite nonempty sets $B$ of positive integers with the property that the maximum element of $B$ belongs to $A$. Bob's list has $2024$ sets. Find the sum of the elements of $A$.

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## Solution:

For a fixed $a \in A$, the sets $B$ with maximum element $a$ consist of $a$ together with an arbitrary subset of $\{1, \ldots, a-1\}$, so there are $2^{a-1}$ of them, and every set on Bob's list is counted exactly once by its maximum. Hence

$$\sum_{a \in A} 2^{a-1} = 2024.$$

Since $2024 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$ and binary representations are unique, $A = \{11, 10, 9, 8, 7, 6, 4\}$. The sum of the elements of $A$ is $11 + 10 + 9 + 8 + 7 + 6 + 4 = 55.$

**7.** Let $N$ be the greatest four-digit integer with the property that whenever one of its digits is changed to $1$, the resulting number is divisible by $7$. Let $Q$ and $R$ be the quotient and remainder, respectively, when $N$ is divided by $1000$. Find $Q + R$.

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## Solution:

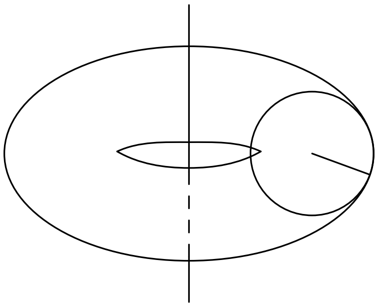
Write $N$ with digits $a, b, c, d$. Changing the thousands digit to $1$ produces $N - (a-1) \cdot 1000$, so $N \equiv 1000(a-1) \pmod 7$, and similarly for the other digits. Since $1000 \equiv 6$, $100 \equiv 2$, and $10 \equiv 3 \pmod 7$,

$$N \equiv 6(a-1) \equiv 2(b-1) \equiv 3(c-1) \equiv d-1 \pmod 7.$$

Let $k = N \bmod 7$. Using $6 \equiv -1$ and the inverses $2^{-1} \equiv 4, 3^{-1} \equiv 5$, the digits satisfy $a \equiv 1-k, b \equiv 1+4k, c \equiv 1+5k, d \equiv 1+k \pmod 7$. But also $k \equiv N \equiv 6a + 2b + 3c + d \pmod 7$; substituting gives $k \equiv 12 + 18k \equiv 5 + 4k$, so $3k \equiv 2$ and $k \equiv 3 \pmod 7$.

Then $a \equiv 5, b \equiv 6, c \equiv 2, d \equiv 4 \pmod 7$, and taking the largest digit in each class gives $a = 5$ (the class $\{5, 12, \ldots\}$ has no larger digit), $b = 6, c = 9, d = 4 : N = 5694$. Indeed $1694, 5194, 5614, 5691$ are all multiples of $7$. Finally $Q = 5, R = 694$, and $Q + R = 699$.

**8.** Torus $T$ is the surface produced by revolving a circle with radius $3$ around an axis in the plane of the circle that is a distance $6$ from the center of the circle (so like a donut).



Let $S$ be a sphere with a radius $11$. When $T$ rests on the inside of $S$, it is internally tangent to $S$ along a circle with radius $r_i$, and when $T$ rests on the outside of $S$, it is externally tangent to $S$ along a circle with radius $r_o$. The difference $r_i - r_o$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

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## Solution:

By symmetry the axis of the torus passes through the center $O$ of the sphere. Work in a plane through the axis: there the torus appears as a circle of radius $3$ (the tube) whose center sits at distance $6$ from the axis, and the sphere appears as a circle of radius $11$ centered at $O$. The two surfaces are tangent along the circle swept by the tangency point of these cross-sections, which lies on the ray from $O$ through the tube's center. For internal tangency the tube's center is at distance $11 - 3 = 8$ from $O$; for external tangency, $11 + 3 = 14$.

The tangency point lies at distance $11$ from $O$ along that ray, so it is the tube center scaled by $\frac{11}{8}$ (resp. $\frac{11}{14}$) from $O$, and its distance from the axis is the same multiple of the tube center's distance $6$ :

$$r_i = \frac{11}{8} \cdot 6 = \frac{33}{4}, \qquad r_o = \frac{11}{14} \cdot 6 = \frac{33}{7}.$$

Then $r_i - r_o = \frac{33 \cdot 3}{28} = \frac{99}{28}$, which is in lowest terms, so $m + n = 99 + 28 = 127.$

**9.** There is a collection of $25$ indistinguishable white chips and $25$ indistinguishable black chips. Find the number of ways to place some of these chips in a $5 \times 5$ grid such that:

- each cell contains at most one chip
- all chips in the same row and all chips in the same column have the same color, and
- any additional chip placed on the grid would violate one or more of the previous two conditions.

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## Solution:

In a valid placement, each nonempty row has a single color, and likewise each column. If some row were empty, choose any cell of it: a chip of the color of that cell's column (either color if the column is also empty) could legally be added, contradicting the third condition. So every row and every column is nonempty, and we may speak of its color.

A chip at a cell forces its row and column colors to agree; conversely, if a row and a column share a color but their common cell is empty, a chip of that color could be added. Hence chips occupy exactly the cells whose row color equals the column color. For every row to be nonempty, each row's color must appear among the column colors, and vice versa — the rows and the columns use the same set of colors. Any such coloring conversely yields a valid maximal placement (at most $25$ cells hold chips of each color, so the supply suffices), and distinct colorings give distinct placements.

Counting the colorings: all rows and columns white, all black, or both colors used by the rows and by the columns: $1 + 1 + (2^5 - 2)^2 = 2 + 900 = 902.$

**10.** Let $\triangle ABC$ have incenter $I$, circumcenter $O$, inradius $6$, and circumradius $13$. Suppose that $\overline{IA} \perp \overline{OI}$. Find $AB \cdot AC$.

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## Solution:

Since $\angle OIA = 90^\circ$, the Pythagorean theorem in triangle $OIA$ gives $IA^2 = OA^2 - OI^2 = R^2 - OI^2$, and Euler's formula $OI^2 = R^2 - 2Rr$ yields

$$IA^2 = 2Rr = 2 \cdot 13 \cdot 6 = 156.$$

Combining with $IA = \frac{r}{\sin(A/2)}$ gives $\sin^2 \frac{A}{2} = \frac{36}{156} = \frac{3}{13}$, so $\cos^2 \frac{A}{2} = \frac{10}{13}$.

Then $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} = \frac{2\sqrt{30}}{13}$, so $a = BC = 2R \sin A = 4\sqrt{30}$, while $s - a = r \cot \frac{A}{2} = 6\sqrt{\frac{10}{3}} = 2\sqrt{30}$. Hence the semiperimeter is $s = 6\sqrt{30}$.

Equating the two area formulas $[ABC] = rs = \frac{1}{2}\, bc \sin A$,

$$bc = \frac{2rs}{\sin A} = \frac{2 \cdot 6 \cdot 6\sqrt{30}}{2\sqrt{30}/13} = 36 \cdot 13 = 468.$$

**11.** Find the number of triples of nonnegative integers $(a, b, c)$ satisfying $a + b + c = 300$ and

$$a^2b + a^2c + b^2a + b^2c + c^2a + c^2b = 6{,}000{,}000.$$

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## Solution:

The left side is the symmetric sum $(a + b + c)(ab + bc + ca) - 3abc = 300q - 3p$, where $q = ab + bc + ca$ and $p = abc$. So the condition is $100q - p = 2{,}000{,}000$. Now expand

$$(100 - a)(100 - b)(100 - c) = 10^6 - 10^4(a + b + c) + 100q - p = (100q - p) - 2 \cdot 10^6,$$

using $a + b + c = 300$. The condition holds exactly when this product is $0$, that is, when at least one of $a, b, c$ equals $100$.

If $a = 100$, then $b + c = 200$, giving $201$ triples, and likewise for $b$ and $c : 3 \cdot 201 = 603$. A triple counted more than once has two variables equal to $100$, which forces the third to be $100$ as well; the triple $(100, 100, 100)$ is counted three times, so the total is $603 - 2 = 601$.

**12.** Let $O = (0,0)$, $A = \left(\frac{1}{2}, 0\right)$, and $B = \left(0, \frac{\sqrt{3}}{2}\right)$ be points in the coordinate plane. Let $\mathcal{F}$ be the family of segments $\overline{PQ}$ of unit length lying in the first quadrant with $P$ on the $x$-axis and $Q$ on the $y$-axis. There is a unique point $C$ on $\overline{AB}$, distinct from $A$ and $B$, that does not belong to any segment from $\mathcal{F}$ other than $\overline{AB}$. Then $OC^2 = \frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.

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## Solution:

The members of $\mathcal{F}$ are the segments from $(\cos\theta, 0)$ to $(0, \sin\theta)$ for $0 < \theta < 90^\circ$, lying on the lines $\frac{x}{\cos\theta} + \frac{y}{\sin\theta} = 1$; the segment $\overline{AB}$ is the member with $\theta = 60^\circ$. For a point $(x, y)$ of $\overline{AB}$ with $x, y > 0$, let

$$g(\theta) = \frac{x}{\cos\theta} + \frac{y}{\sin\theta} - 1,$$

so the point lies on the member for angle $\theta$ exactly when $g(\theta) = 0$. Note $g \to +\infty$ at both endpoints of $(0^\circ, 90^\circ)$ and $g(60^\circ) = 0$. If $g'(60^\circ) \neq 0$, then $g$ is negative on one side of $60^\circ$, and the intermediate value theorem produces another zero on that side — the point is covered by another segment. So $C$ must satisfy $g'(60^\circ) = 0$, and for that point $60^\circ$ is the strict global minimum of $g$, so no other segment contains it.

Now $g'(\theta) = \frac{x\sin\theta}{\cos^2\theta} - \frac{y\cos\theta}{\sin^2\theta}$, and $g'(60^\circ) = 0$ gives $x\sin^3 60^\circ = y\cos^3 60^\circ$, i.e. $y = 3\sqrt{3}\,x$. Intersecting with $\overline{AB}: y = \frac{\sqrt{3}}{2} - \sqrt{3}\,x$ gives $3x = \frac{1}{2} - x$, so $x = \frac{1}{8}$ and $y = \frac{3\sqrt{3}}{8}$, an interior point of $\overline{AB}$.

Therefore $OC^2 = \frac{1}{64} + \frac{27}{64} = \frac{28}{64} = \frac{7}{16}$, and $p + q = 7 + 16 = 23$.

**13.** Let $\omega \neq 1$ be a $13$th root of unity. Find the remainder when

$$\prod_{k=0}^{12}\left(2-2\omega^k+\omega^{2k}\right)$$

is divided by $1000$.

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## Solution:

Since $2-2x+x^2=(x-1)^2+1=(x-(1+i))(x-(1-i))$, each factor of the product splits, and as $k$ runs from $0$ to $12$, $\omega^k$ runs over all $13$th roots of unity. Because $\prod_k(x-\omega^k)=x^{13}-1$, for any $\alpha$ we get $\prod_k(\omega^k-\alpha)=(-1)^{13}(\alpha^{13}-1)=1-\alpha^{13}$. Hence the product equals

$$\left(1-(1+i)^{13}\right)\left(1-(1-i)^{13}\right).$$

Since $(1+i)^2=2i$, we get $(1+i)^{13}=(1+i)(2i)^6=-64(1+i)=-64-64i$, and by conjugation $(1-i)^{13}=-64+64i$. So the product is

$$(65+64i)(65-64i)=65^2+64^2=4225+4096=8321,$$

whose remainder upon division by $1000$ is $321$.

**14.** Let $b \geq 2$ be an integer. Call a positive integer $n$ *$b$-eautiful* if it has exactly two digits when expressed in base $b$ and these two digits sum to $\sqrt{n}$. For example, $81$ is $13$-eautiful because $81 = \underline{6}\,\underline{3}_{13}$ and $6 + 3 = \sqrt{81}$. Find the least integer $b \geq 2$ for which there are more than ten $b$-eautiful integers.

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## Solution:

A two-digit number in base $b$ is $n = xb + y$ with $1 \leq x \leq b - 1$ and $0 \leq y \leq b - 1$, and the condition says $n = s^2$ where $s = x + y$. Then $s^2 = xb + y = x(b - 1) + s$, so

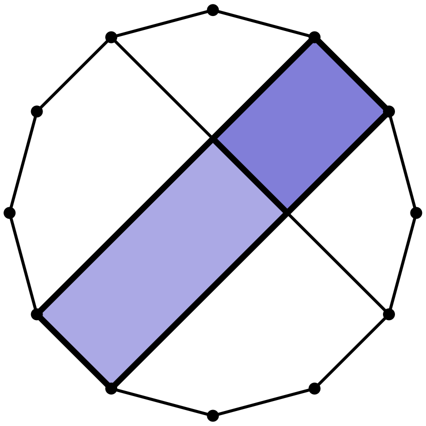
$$s(s - 1) = x(b - 1).$$

Note $s \leq \sqrt{b^2 - 1} < b$. Conversely, for any $s$ with $2 \leq s \leq b - 1$ and $(b - 1) \mid s(s - 1)$, setting $x = \frac{s(s-1)}{b-1}$ and $y = s - x = \frac{s(b-s)}{b-1}$ gives $1 \leq x \leq b - 1$ and $0 \leq y \leq b - 1$, hence exactly one $b$-eautiful integer $n = s^2$. So the count equals the number of $s \in \{2, \ldots, b - 1\}$ with $s(s - 1) \equiv 0 \pmod{b - 1}$.

Let $m = b - 1$. Since $s$ and $s - 1$ are coprime, each prime power dividing $m$ must divide $s$ or $s - 1$, so by the Chinese remainder theorem there are $2^{\omega(m)}$ solutions modulo $m$, where $\omega(m)$ is the number of distinct prime factors of $m$. Among the representatives $1, 2, \ldots, m$, only $s = 1$ falls outside our range (and $s = m$ qualifies), so the count is $2^{\omega(m)} - 1$.

We need $2^{\omega(m)} - 1 > 10$, i.e. $\omega(m) \geq 4$. The smallest positive integer with four distinct prime factors is $2 \cdot 3 \cdot 5 \cdot 7 = 210$, so the least base is $b = 211$ (which has $2^4 - 1 = 15$ $b$-eautiful integers).

**15.** Find the number of rectangles that can be formed inside a fixed regular dodecagon ($12$-gon) where each side of the rectangle lies on either a side or a diagonal of the dodecagon. The diagram below shows three of those rectangles.



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## Solution:

Put the vertices at angles $30k^\circ$ on a unit circle. The chord joining vertices $i$ and $j$ has direction $15(i+j)^\circ + 90^\circ$, so chords come in $12$ directions spaced $15^\circ$ apart, and a rectangle uses two chords from each of two perpendicular directions. The six perpendicular direction pairs split into two kinds, three of each, by rotation. When $i+j$ is even, a family of parallel chords has $5$ members, at distances $0, \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}$ from the center with half-lengths $1, \frac{\sqrt{3}}{2}, \frac{1}{2}$ respectively; when $i+j$ is odd, a family has $6$ members, at distances $\pm\cos 75^\circ, \pm\cos 45^\circ, \pm\cos 15^\circ$ with half-lengths $\sin 75^\circ, \sin 45^\circ, \sin 15^\circ$.

A corner is the intersection of one chord from each direction, and its offset along a chord equals the other chord's distance from the center. Since half-lengths shrink as distance grows, the four corners lie on all four chord segments exactly when, writing $D_1, D_2$ for the larger distances of the two chosen pairs, each $D$ is at most the half-length of the other pair's farther chord. For the $5$-chord families: pairs with $D = \frac{\sqrt{3}}{2}$ (there are $7$) have half-length bound $\frac{1}{2}$, and pairs with $D = \frac{1}{2}$ (there are $3$) have bound $\frac{\sqrt{3}}{2}$; the valid combinations give $7 \cdot 3 + 3 \cdot 7 + 3 \cdot 3 = 51$ rectangles. For the $6$-chord families: there are $1, 5, 9$ pairs with $D = \cos 75^\circ, \cos 45^\circ, \cos 15^\circ$, and the valid combinations are $(\cos 75^\circ, \cos 75^\circ)$, both orders of $(\cos 75^\circ, \cos 45^\circ)$ and $(\cos 75^\circ, \cos 15^\circ)$, and $(\cos 45^\circ, \cos 45^\circ)$, giving $1+5+5+9+9+25 = 54$.

Each kind of direction pair occurs three times, so the total is $3(51+54) = 315$.

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