2024 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/aime/2024I/solutions

1. Every morning Aya goes for a 9-kilometer-long walk and stops at a coffee shop afterwards. When she walks at a constant speed of s kilometers per hour, the walk takes her 4 hours, including t minutes spent in the coffee shop. When she walks $s+2$ kilometers per hour, the walk takes her 2 hours and 24 minutes, including t minutes spent in the coffee shop. Suppose Aya walks at $s+\frac{1}{2}$ kilometers per hour. Find the number of minutes the walk takes her, including the t minutes spent in the coffee shop.

Solution:

Measuring time in hours, the two scenarios say

$$\frac{9}{s}+\frac{t}{60}=4 \qquad \text{and} \qquad \frac{9}{s+2}+\frac{t}{60}=\frac{12}{5}.$$

Subtracting, $\frac{9}{s}-\frac{9}{s+2}=\frac{8}{5}$, so $\frac{18}{s(s+2)}=\frac{8}{5}$, giving $s(s+2)=\frac{45}{4}$. The positive root of $s^2+2s-\frac{45}{4}=0$ is $s=\frac{5}{2}$.

Then $\frac{t}{60}=4-\frac{9}{5/2}=\frac{2}{5}$, so $t=24$ minutes. Walking at $s+\frac{1}{2}=3$ kilometers per hour takes $\frac{9}{3}=3$ hours, so the total is $180+24=204$ minutes.

2. There exist real numbers x and y, both greater than 1, such that $\log_x\left(y^x\right) = \log_y\left(x^{4y}\right) = 10$. Find xy.

Solution:

Pulling the exponents out of the logarithms, the conditions become

$$x \log_x y = 10 \qquad \text{and} \qquad 4y \log_y x = 10.$$

Multiplying these equations and using $\log_x y \cdot \log_y x = 1$ gives $4xy = 100$, so $xy = 25$.

Such x and y do exist: the system solves to $\log_x y = \frac{10}{x}$ with $y = \frac{25}{x}$, which has a solution with $x, y > 1$, so the answer is 25.

3. Alice and Bob play the following game. A stack of n tokens lies before them. The players take turns with Alice going first. On each turn, the player removes 1 token or 4 tokens from the stack. The player who removes the last token wins. Find the number of positive integers n less than or equal to 2024 such that there is a strategy that guarantees that Bob wins, regardless of Alice's moves.

Solution:

Call n a losing position if the player about to move loses with best play. We claim the losing positions are exactly $n \equiv 0$ or $2 \pmod 5$. From such an n, removing 1 or 4 tokens leaves $n \equiv 4, 1, 3 \pmod 5$ — never again 0 or 2 — while from any $n \equiv 1, 3, 4 \pmod 5$ one move reaches a position $\equiv 0$ or $2 \pmod 5$ (remove $1, 1, 4$ tokens respectively). Since $n = 0$ is a loss for the player to move, induction confirms the pattern.

Bob wins exactly when n is a losing position for Alice. Among $1 \leq n \leq 2024$ there are 404 multiples of 5 and 405 values $n \equiv 2 \pmod 5$ (from 2 to 2022), for a total of $404 + 405 = 809$.

4. Jen enters a lottery by selecting 4 distinct elements of $S = \{1, 2, 3, \ldots, 9, 10\}$. Then four elements of S are drawn at random. Jen wins a prize if at least two of her numbers were drawn, and wins the grand prize if all four of her numbers were drawn. The probability that Jen wins the grand prize given that Jen wins a prize is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Solution:

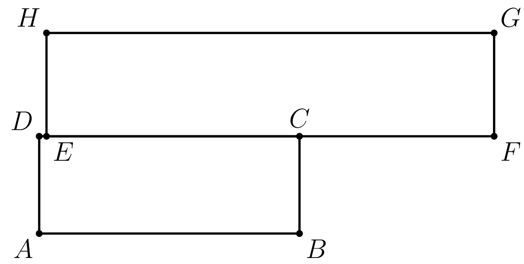
All $\binom{10}{4} = 210$ draws are equally likely. The number of draws sharing exactly k numbers with Jen's ticket is $\binom{4}{k}\binom{6}{4-k}$, so the number winning a prize is

$$\binom{4}{2}\binom{6}{2} + \binom{4}{3}\binom{6}{1} + \binom{4}{4}\binom{6}{0} = 90 + 24 + 1 = 115,$$

and exactly 1 of these wins the grand prize.

Since the grand prize implies a prize, the conditional probability is $\frac{1/210}{115/210} = \frac{1}{115}$, so $m + n = 1 + 115 = 116.$

5. Rectangle $ABCD$ has dimensions $AB = 107$ and $BC = 16$, and rectangle $EFGH$ has dimensions $EF = 184$ and $FG = 17$. Points D, E, C, and F lie on line DF in that order, and A and H lie on opposite sides of line DF, as shown. Points A, D, H, and G lie on a common circle. Find CE.



Solution:

Put line DF on the x-axis with $D = (0, 0)$ and $C = (107, 0)$, so $A = (0, -16)$. Let $DE = e$. Then $E = (e, 0)$, $F = (e + 184, 0)$, and the second rectangle sits above the line: $H = (e, 17)$ and $G = (e + 184, 17)$.

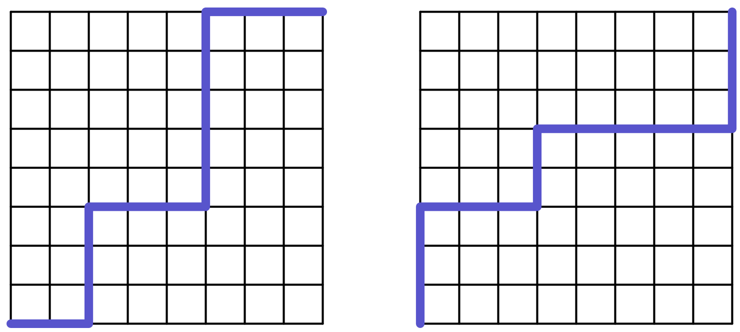
The center of the circle through A, D, H, G lies on the perpendicular bisector of the vertical segment $\overline{AD}$, the line $y = -8$, and on the perpendicular bisector of the horizontal segment $\overline{HG}$, the line $x = e + 92$. Equating the center's squared distances to D and to H,

$$(e + 92)^2 + 8^2 = 92^2 + 25^2 = 9089,$$

so $(e + 92)^2 = 9025$ and $e + 92 = 95$, giving $e = 3$.

Therefore $CE = DC - DE = 107 - 3 = 104$.

6. Consider the paths of length 16 that follow the lines from the lower left corner to the upper right corner on an 8×8 grid. Find the number of such paths that change direction exactly four times, as in the examples shown below.



Solution:

A path that changes direction exactly four times consists of five maximal straight runs, alternating between rightward and upward moves. If the first run is rightward, the pattern is $R, U, R, U, R:$ three rightward runs with positive lengths summing to $8,$ and two upward runs with positive lengths summing to $8.$ The counts of such compositions are $\binom{7}{2} = 21$ and $\binom{7}{1} = 7,$ giving $21 \cdot 7 = 147$ paths.

Paths starting upward are counted symmetrically, another $147.$ The total is $147 + 147 = 294.$

7. Find the largest possible real part of

$$(75+117i)z+\frac{96+144i}{z}$$

where z is a complex number with $|z|=4$. Here $i=\sqrt{-1}$.

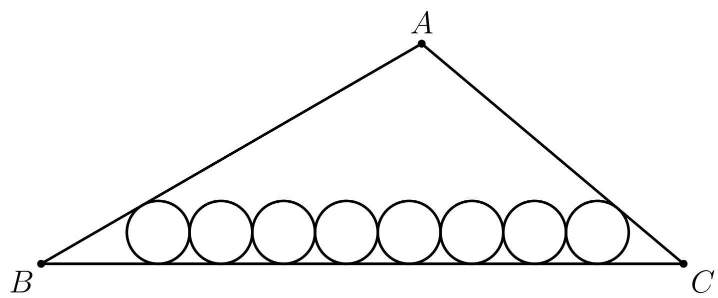
Solution:

Write $z=4(\cos\theta+i\sin\theta)$, so $\frac{1}{z}=\frac{1}{4}(\cos\theta-i\sin\theta)$. The real part of $(75+117i)z$ is $4(75\cos\theta-117\sin\theta)=300\cos\theta-468\sin\theta$, and the real part of $(96+144i)\cdot\frac{1}{4}(\cos\theta-i\sin\theta)$ is $24\cos\theta+36\sin\theta$.

The total real part is $324\cos\theta-432\sin\theta$, whose maximum over θ is

$$\sqrt{324^2+432^2}=108\sqrt{3^2+4^2}=108\cdot 5=540.$$

8. Eight circles of radius 34 can be placed tangent to $\overline{BC}$ of $\triangle ABC$ so that the circles are sequentially tangent to each other, with the first circle being tangent to $\overline{AB}$ and the last circle being tangent to $\overline{AC}$, as shown. Similarly, 2024 circles of radius 1 can be placed tangent to $\overline{BC}$ in the same manner. The inradius of $\triangle ABC$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution:

For a chain of n circles of radius ρ tangent to $\overline{BC}$, the centers lie at height ρ with consecutive centers 2ρ apart. The first circle is tangent to $\overline{AB}$ and $\overline{BC}$, so its center lies on the bisector from B, at horizontal distance $\rho\cot\frac{B}{2}$ from B; similarly the last center is $\rho\cot\frac{C}{2}$ from C. Hence with $k=\cot\frac{B}{2}+\cot\frac{C}{2}$,

$$BC=\rho k+2\rho(n-1).$$

The two chains give $34k+34\cdot 14=BC=k+2\cdot 2023$, so $33k=3570$ and $k=\frac{1190}{11}$, whence $BC=k+4046=\frac{45696}{11}$.

The incircle is a chain of one circle of radius $r : BC=rk$. Therefore

$$r=\frac{BC}{k}=\frac{45696}{1190}=\frac{192}{5},$$

and $m+n=192+5=197$.

9. Let A, B, C, and D be points on the hyperbola $\frac{x^2}{20} - \frac{y^2}{24} = 1$ such that $ABCD$ is a rhombus whose diagonals intersect at the origin. Find the largest number less than BD^2 for all rhombuses $ABCD$.

Solution:

The diagonals of a rhombus are perpendicular bisectors of each other, so $C = -A$, $D = -B$, and $OA \perp OB$. Let line BD have slope m, so $B = (x, mx)$ with

$$x^2\left(\frac{1}{20} - \frac{m^2}{24}\right) = 1, \qquad \text{i.e.} \qquad x^2 = \frac{120}{6 - 5m^2},$$

which requires $m^2 < \frac{6}{5}$. Then $BD^2 = 4(x^2 + m^2x^2) = \frac{480(1+m^2)}{6-5m^2}$. Line AC has slope $-\frac{1}{m}$, so it meets the hyperbola only when $\frac{1}{m^2} < \frac{6}{5}$, that is $m^2 > \frac{5}{6}$.

On the interval $\frac{5}{6} < m^2 < \frac{6}{5}$, the quantity $\frac{480(1+m^2)}{6-5m^2}$ is strictly increasing in m^2 : as $m^2 \to \frac{5}{6}$ it tends to $480 \cdot \frac{11/6}{11/6} = 480$, and as $m^2 \to \frac{6}{5}$ it grows without bound. Hence BD^2 takes exactly the values in $(480, \infty)$ and never equals 480.

The largest number that is less than BD^2 for every such rhombus is therefore 480.

10. Let $\triangle ABC$ have side lengths $AB = 5, BC = 9,$ and $CA = 10.$ The tangents to the circumcircle of $\triangle ABC$ at B and C intersect at point D, and $\overline{AD}$ intersects the circumcircle at $P \neq A.$ The length of AP is equal to $\frac{m}{n}$, where m and n are relatively prime integers. Find $m + n.$

Solution:

By the tangent-chord angle, $\angle DBC = \angle DCB = \angle A$, so triangle DBC is isosceles with $DB = \frac{BC/2}{\cos A}$. The law of cosines gives $\cos A = \frac{10^2+5^2-9^2}{2\cdot 10\cdot 5} = \frac{11}{25}$ and $\cos B = \frac{9^2+5^2-10^2}{2\cdot 9\cdot 5} = \frac{1}{15}$, so $DB = \frac{9/2}{11/25} = \frac{225}{22}$.

Since D lies on the opposite side of BC from A, $\angle ABD = A + B$, and

$$\cos(A+B) = \frac{11}{25}\cdot\frac{1}{15} - \frac{6\sqrt{14}}{25}\cdot\frac{4\sqrt{14}}{15} = \frac{11-336}{375} = -\frac{13}{15}.$$

The law of cosines in triangle ABD then gives

$$DA^2 = 5^2 + \left(\tfrac{225}{22}\right)^2 + 2\cdot 5\cdot \tfrac{225}{22}\cdot\tfrac{13}{15} = \frac{105625}{484}, \qquad DA = \frac{325}{22}.$$

The power of D gives $DP \cdot DA = DB^2$, so

$$AP = DA - \frac{DB^2}{DA} = \frac{DA^2 - DB^2}{DA} = \frac{(105625-50625)/484}{325/22} = \frac{55000}{22\cdot 325} = \frac{100}{13},$$

and $m + n = 100 + 13 = 113.$

11. Each vertex of a regular octagon is independently colored either red or blue with equal probability. The probability that the octagon can then be rotated so that all of the blue vertices end up at positions where there had been red vertices is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

Label the vertices $0, \ldots, 7$ and let B be the blue set, $b=|B|$. Rotation by k works exactly when $(B+k) \cap B=\varnothing$. Since $B+k$ must fit inside the $8-b$ red positions, $b \leq 4$. Summing $|B \cap(B+k)|$ over all eight rotations counts all pairs $(i, j) \in B \times B$ once (via $k=i-j$), a total of b^2, and the $k=0$ term contributes b. So for $b \leq 3$ the seven nonzero rotations share only $b^2-b \leq 6$ overlaps, and some rotation has none: all $1+8+28+56=93$ colorings with $b \leq 3$ succeed.

For $b=4$, disjointness forces $B+k$ to be exactly the complement of B. If k is odd, the cycle $0, k, 2k, \ldots$ visits all vertices and must alternate between B and its complement, so B is the evens or the odds: 2 sets. If $k \equiv 2 \pmod 4$, then B meets each of the 4-cycles $\{0,2,4,6\}$ and $\{1,3,5,7\}$ in an antipodal pair: $2 \cdot 2=4$ sets, such as $\{0,1,4,5\}$. If $k=4$, then B contains exactly one of each pair $\{i, i+4\}:$ $2^4=16$ sets. The first two families contain both members of some antipodal pair while the third never does, and the evens/odds take both their antipodal pairs from one 4-cycle, so the three families are disjoint: $2+4+16=22$ sets.

In total $93+22=115$ of the $2^8=256$ colorings work, so the probability is $\frac{115}{256}$ and $m+n=115+256=371.$

12. Define $f(x) = \left||x| - \frac{1}{2}\right|$ and $g(x) = \left||x| - \frac{1}{4}\right|$. Find the number of intersections of the graphs of

$$y = 4g(f(\sin(2\pi x))) \quad \text{and} \quad x = 4g(f(\cos(3\pi y))).$$

Solution:

Both right-hand sides take values in $[0, 1]$, so every intersection lies in the unit square, and there we may write both curves using $\varphi(u) = 4\left||u - \frac{1}{2}| - \frac{1}{4}\right|$: the first is $y = \varphi(|\sin 2\pi x|)$ and the second is $x = \varphi(|\cos 3\pi y|)$. As u increases from 0 to 1, $\varphi(u)$ runs linearly $1 \to 0 \to 1 \to 0 \to 1$ with corners at $u = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. For $x \in [0, 1]$, $|\sin 2\pi x|$ sweeps $[0, 1]$ monotonically 4 times, so the first graph consists of $4 \cdot 4 = 16$ monotone arcs, each climbing or descending through the full range $0 \le y \le 1$ within a narrow vertical strip. Likewise $|\cos 3\pi y|$ sweeps $[0, 1]$ monotonically 6 times for $y \in [0, 1]$, so the second graph consists of 24 monotone arcs, each crossing the full range $0 \le x \le 1$ within a narrow horizontal strip.

Take one arc of each graph, living in the vertical strip $[a, b]$ and the horizontal strip $[c, d]$. Inside the rectangle $[a, b] \times [c, d]$, the first arc joins the bottom edge to the top edge and the second joins the left edge to the right edge, and each is monotone, so the two arcs cross exactly once. This yields $16 \cdot 24 = 384$ intersection points.

One further point hides at the corner $(1, 1)$, which lies on both graphs: $\varphi(|\sin 2\pi|) = \varphi(0) = 1$ and $\varphi(|\cos 3\pi|) = \varphi(1) = 1$. Near it the first graph is $y \approx 1 - 8\pi(1 - x)$ while the second satisfies $x \approx 1 - 18\pi^2(1 - y)^2$, so the two final arcs meet at their shared endpoint $(1, 1)$ in addition to the transversal crossing already counted. The total is $384 + 1 = 385$.

13. Let p be the least prime number for which there exists an integer n such that n^4+1 is divisible by p^2. Find the least positive integer m such that m^4+1 is divisible by p^2.

Solution:

If $p \mid n^4+1$, then $n^8 \equiv 1$ and $n^4 \equiv -1 \pmod{p}$, so n has order 8 modulo p and $8 \mid p-1$ (and $p=2$ fails since $n^4+1 \equiv 2 \pmod{4}$). The smallest prime $p \equiv 1 \pmod{8}$ is 17, and indeed $2^4 = 16 \equiv -1 \pmod{17}$. Because the derivative $4n^3$ is not divisible by 17 at such an n, each root lifts to a root modulo $17^2 = 289$, so $p = 17$.

The fourth roots of -1 modulo 17 are ± 2 and ± 8. To lift $n=8$, set $n = 8+17t$: modulo 289,

$$n^4+1 \equiv 8^4+1+4\cdot 8^3\cdot 17t = 17(241+2048t),$$

so we need $241+2048t \equiv 3+8t \equiv 0 \pmod{17}$, giving $t \equiv 6$ and $n \equiv 8+102 = 110 \pmod{289}$.

The same computation lifts $2, 15,$ and 9 to $155, 134,$ and 179 respectively, so the least positive m is 110. Indeed $110^4+1 = 146410001 = 289 \cdot 506609$.

14. Let $ABCD$ be a tetrahedron such that $AB = CD = \sqrt{41}$, $AC = BD = \sqrt{80}$, and $BC = AD = \sqrt{89}$. There exists a point I inside the tetrahedron such that the distances from I to each of the faces of the tetrahedron are all equal. This distance can be written in the form $\frac{m\sqrt{n}}{p}$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime. Find $m + n + p$.

Solution:

A tetrahedron with equal opposite edges embeds in a rectangular box with the six edges as face diagonals. If the box has dimensions $a \times b \times c$, then $a^2 + b^2 = 41$, $a^2 + c^2 = 80$, and $b^2 + c^2 = 89$. Adding gives $a^2 + b^2 + c^2 = 105$, so $(a, b, c) = (4, 5, 8)$. The box minus four corner tetrahedra of volume $\frac{abc}{6}$ each leaves

$$V = abc - 4 \cdot \frac{abc}{6} = \frac{abc}{3} = \frac{160}{3}.$$

All four faces are congruent triangles with sides $\sqrt{41}, \sqrt{80}, \sqrt{89}$. By Heron's formula in the form $16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$ applied to the squared sides $41, 80, 89$, we get $16F^2 = 28098 - 16002 = 12096$, so $F = \sqrt{756} = 6\sqrt{21}$.

The point equidistant from all four faces is the insphere center, and decomposing the tetrahedron into four pyramids over the faces gives $V = \frac{1}{3}r \cdot 4F$. Hence

$$r = \frac{3V}{4F} = \frac{160}{24\sqrt{21}} = \frac{20}{3\sqrt{21}} = \frac{20\sqrt{21}}{63},$$

and $m + n + p = 20 + 21 + 63 = 104$.

15. Let $\mathcal{B}$ be the set of rectangular boxes with surface area 54 and volume 23. Let r be the radius of the smallest sphere that can contain each of the rectangular boxes that are elements of $\mathcal{B}$. The value of r^2 can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

For a box with dimensions a, b, c, the conditions are $2(ab+bc+ca)=54$ and $abc=23$, so $ab+bc+ca=27$. The smallest sphere containing a box has the box's space diagonal as a diameter, so

$$r^2=\max_{\mathcal{B}}\frac{a^2+b^2+c^2}{4}=\max_{\mathcal{B}}\frac{(a+b+c)^2-54}{4}.$$

With $ab+bc+ca$ and abc fixed, $s=a+b+c$ ranges over an interval, and at an endpoint the cubic $t^3-st^2+27t-23$ has a double root, meaning two dimensions coincide. Setting $b=c: 2ab+b^2=27$ and $ab^2=23$, so eliminating a gives $\frac{b(27-b^2)}{2}=23$, i.e. $b^3-27b+46=0$, which factors as $(b-2)(b^2+2b-23)=0$. The roots are $b=2$ and $b=2\sqrt{6}-1$.

For $b=2, a=\frac{23}{4}$ and $s=\frac{23}{4}+4=\frac{39}{4}=9.75$; for $b=2\sqrt{6}-1, s\approx 9.31$ is smaller. So the maximum of $a^2+b^2+c^2$ is $\left(\frac{39}{4}\right)^2-54=\frac{657}{16}$, giving $r^2=\frac{657}{64}$ and $p+q=657+64=721$.

Problems: https://live.poshenloh.com/past-contests/aime/2024I