2023 AIME II Solutions

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1. The numbers of apples growing on each of six apple trees form an arithmetic sequence where the greatest number of apples growing on any of the six trees is double the least number of apples growing on any of the six trees. The total number of apples growing on all six trees is 990. Find the greatest number of apples growing on any of the six trees.

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Solution:

Let the six counts be $a, a+d, \ldots, a+5d$ with common difference $d \geq 0$. The greatest count is double the least, so $a+5d=2a$, which gives $a=5d$. The total is

$$6a+15d=30d+15d=45d=990,$$

so $d=22$.

The greatest number of apples is $a+5d=10d=220$.

2. Recall that a palindrome is a number that reads the same forward and backward. Find the greatest integer less than 1000 that is a palindrome both when written in base ten and when written in base eight, such as $292 = 444_{\text{eight}}$.

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Solution:

A four-digit base-eight number lies between 512 and 4095, so a base-eight palindrome less than 1000 with four digits must have leading (and trailing) digit 1 : it has the form $\overline{1bb1}_{\text{eight}} = 512 + 64b + 8b + 1 = 513 + 72b$. Keeping this below 1000 requires $b \leq 6$, giving the candidates $513, 585, 657, 729, 801, 873, 945$.

Checking from the top, the only one of these that is also a palindrome in base ten is $585 = 1111_{\text{eight}}$. Every base-eight palindrome with at most three digits is at most $777_{\text{eight}} = 511 < 585$, so the answer is 585.

3. Let $\triangle ABC$ be an isosceles triangle with $\angle A = 90^\circ$. There exists a point P inside $\triangle ABC$ such that $\angle PAB = \angle PBC = \angle PCA$ and $AP = 10$. Find the area of $\triangle ABC$.

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Solution:

Let ω denote the common angle and $L = AB = AC$. Since $\angle PAB = \omega$, we have $\angle PAC = 90^\circ - \omega$, and with $\angle PCA = \omega$ the angles of triangle APC give $\angle APC = 90^\circ$. Hence in right triangle APC,

$$L = AC = \frac{AP}{\sin\omega} = \frac{10}{\sin\omega}.$$

In triangle ABP, the angle at A is ω and the angle at B is $45^\circ - \omega$, so $\angle APB = 135^\circ$. The law of sines gives $\frac{AP}{\sin(45^\circ - \omega)} = \frac{AB}{\sin 135^\circ}$, that is, $10 \sin 135^\circ = L \sin(45^\circ - \omega)$. Substituting $L = \frac{10}{\sin\omega}$ and expanding yields

$$\sin\omega = \sqrt{2}\,\sin(45^\circ - \omega) = \cos\omega - \sin\omega,$$

so $\tan\omega = \frac{1}{2}$ and $\sin^2\omega = \frac{1}{5}$.

Therefore $L^2 = \frac{100}{\sin^2\omega} = 500$, and the area is $\frac{1}{2}L^2 = 250$.

4. Let x, y, and z be real numbers satisfying the system of equations

$$xy + 4z = 60, \qquad yz + 4x = 60, \qquad zx + 4y = 60.$$

Let S be the set of possible values of x. Find the sum of the squares of the elements of S.

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Solution:

Subtracting the second equation from the first gives $xy - yz + 4z - 4x = 0$, which factors as $(y-4)(x-z) = 0$. So $y = 4$ or $x = z$.

If $y = 4:$ the first equation becomes $4x + 4z = 60$, so $x + z = 15$, and the second becomes $4z + 4x = 60$ again while the third gives $zx = 44$. Then x and z are roots of $t^2 - 15t + 44 = (t-4)(t-11)$, so $x \in \{4, 11\}$.

If $x = z:$ the first equation reads $x(y+4) = 60$, so $y = \frac{60}{x} - 4$, and the third reads $x^2 + 4y = 60$. Substituting,

$$x^2 + \frac{240}{x} - 16 = 60 \quad \Longrightarrow \quad x^3 - 76x + 240 = 0 = (x-4)(x-6)(x+10),$$

so $x \in \{4, 6, -10\}$, each with real y and z. Hence $S = \{-10, 4, 6, 11\}$ and the sum of squares is $100 + 16 + 36 + 121 = 273$.

5. Let S be the set of all positive rational numbers r such that when the two numbers r and $55r$ are written as fractions in lowest terms, the sum of the numerator and denominator of one fraction is the same as the sum of the numerator and denominator of the other fraction. The sum of all the elements of S can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

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Solution:

Write $r = \frac{a}{b}$ in lowest terms and let $g = \gcd(b, 55)$. Then $55r$ in lowest terms is $\frac{55a/g}{b/g}$ (no further cancellation is possible since $\gcd(a, b) = 1$ and 55 is squarefree). The condition is

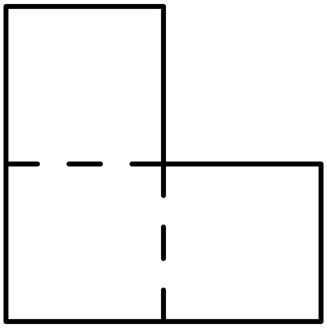
$$a + b = \frac{55a}{g} + \frac{b}{g}.$$

If $g = 1$ this forces $a = 55a$, impossible; if $g = 55$ it forces $b = \frac{b}{55}$, impossible.

If $g = 5 : a + b = 11a + \frac{b}{5}$ gives $\frac{4b}{5} = 10a$, so $2b = 25a$. Since $\gcd(a, b) = 1$, we need $a = 2$ and $b = 25$, so $r = \frac{2}{25}$ (indeed $2 + 25 = 27 = 22 + 5$ from $55r = \frac{22}{5}$). If $g = 11 : a + b = 5a + \frac{b}{11}$ gives $\frac{10b}{11} = 4a$, so $5b = 22a$, forcing $a = 5, b = 22$ and $r = \frac{5}{22}$ (with $5 + 22 = 27 = 25 + 2$ from $55r = \frac{25}{2}$).

Hence $S = \left\{\frac{2}{25}, \frac{5}{22}\right\}$ and the sum is $\frac{2}{25} + \frac{5}{22} = \frac{44+125}{550} = \frac{169}{550}$, already in lowest terms. The answer is $169 + 550 = 719$.

6. Consider the L-shaped region formed by three unit squares joined at their sides, as shown below. Two points A and B are chosen independently and uniformly at random from inside the region. The probability that the midpoint of $\overline{AB}$ also lies inside this L-shaped region can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



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Solution:

Place the region as $[0,1]^2 \cup \left([0,1] \times [1,2]\right) \cup \left([1,2] \times [0,1]\right)$, so it is the 2×2 square with the top-right unit square removed. Both coordinates of the midpoint are averages of numbers in $[0,2]$, so the midpoint always lies in the 2×2 square; it fails to lie in the region exactly when it lands in the missing square, i.e. when $x_A + x_B > 2$ and $y_A + y_B > 2$.

If neither point is in the right square, then $x_A + x_B \leq 2$; if neither is in the top square, then $y_A + y_B \leq 2$. So failure requires one point in the top square and the other in the right square, which happens with probability $2 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$. In that case, one x-coordinate is uniform on $[0,1]$ and the other on $[1,2]$, so $x_A + x_B > 2$ with probability $\frac{1}{2}$, and independently $y_A + y_B > 2$ with probability $\frac{1}{2}$.

The failure probability is $\frac{2}{9} \cdot \frac{1}{4} = \frac{1}{18}$, so the desired probability is $\frac{17}{18}$ and $m + n = 17 + 18 = 35$.

7. Each vertex of a regular dodecagon (12-gon) is to be colored either red or blue, and thus there are 2^{12} possible colorings. Find the number of these colorings with the property that no four vertices colored the same color are the four vertices of a rectangle.

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Solution:

The twelve vertices lie on a circle, and a rectangle inscribed in a circle must have its diagonals pass through the center. So the rectangles with vertices among the twelve are exactly the pairs of distinct diameters, where the diameters join the 6 antipodal pairs of vertices. A monochromatic rectangle appears exactly when two antipodal pairs are each colored solidly in the same color.

Each antipodal pair is independently both red (1 way), both blue (1 way), or mixed (2 ways). A coloring is valid exactly when at most one pair is both red and at most one pair is both blue. Counting by the numbers of solid red and solid blue pairs:

$$2^6 + 6 \cdot 2^5 + 6 \cdot 2^5 + 6 \cdot 5 \cdot 2^4 = 64 + 192 + 192 + 480 = 928.$$

8. Let $\omega = \cos\frac{2\pi}{7} + i \cdot \sin\frac{2\pi}{7}$, where $i = \sqrt{-1}$. Find the value of the product

$$\prod_{k=0}^{6}\left(\omega^{3k} + \omega^{k} + 1\right).$$

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Solution:

Let $P(x) = x^3 + x + 1$, so the product is $\prod_{k=0}^{6} P(\omega^k)$, where $\omega^0, \ldots, \omega^6$ are all seventh roots of unity. Since $x^7 - 1 = \prod_k (x - \omega^k)$, writing $P(x) = (x - \beta_1)(x - \beta_2)(x - \beta_3)$ and swapping the order of the double product gives

$$\prod_{k=0}^{6} P(\omega^k) = \prod_{j=1}^{3}\prod_{k=0}^{6}(\omega^k - \beta_j) = \prod_{j=1}^{3}\left(-(\beta_j^7 - 1)\right) = \prod_{j=1}^{3}(1 - \beta_j^7).$$

For a root β of P, repeatedly using $\beta^3 = -\beta - 1$ gives $\beta^4 = -\beta^2 - \beta$, $\beta^5 = -\beta^2 + \beta + 1$, $\beta^6 = \beta^2 + 2\beta + 1$, and $\beta^7 = 2\beta^2 - 1$. Hence $1 - \beta^7 = 2(1 - \beta)(1 + \beta)$, and

$$\prod_j (1 - \beta_j^7) = 2^3 \prod_j (1 - \beta_j) \prod_j (1 + \beta_j) = 8 \cdot P(1) \cdot \left(-P(-1)\right) = 8 \cdot 3 \cdot 1 = 24.$$

So the requested product equals 24.

9. Circles ω_1 and ω_2 intersect at two points P and Q, and their common tangent line closer to P intersects ω_1 and ω_2 at points A and B, respectively. The line parallel to $\overline{AB}$ that passes through P intersects ω_1 and ω_2 for the second time at points X and Y, respectively. Suppose $PX = 10, PY = 14,$ and $PQ = 5$. Then the area of trapezoid $XABY$ is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

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Solution:

Since the tangent to ω_1 at A is parallel to the chord XP, the point A is the midpoint of arc XP, so the perpendicular from A to line XY lands at the midpoint of $\overline{XP}$; similarly the perpendicular from B lands at the midpoint of $\overline{PY}$. As X and Y are on opposite sides of P, the parallel sides of the trapezoid are $XY = 10 + 14 = 24$ and $AB = \frac{10}{2} + \frac{14}{2} = 12$.

Line PQ is the radical axis, so its intersection M with the tangent line satisfies $MA^2 = MP \cdot MQ = MB^2 :$ M is the midpoint of $\overline{AB}$, and with $MA = 6$ and $MQ = MP + 5$,

$$36 = MP(MP + 5), \qquad MP = 4.$$

Set up coordinates along $AB :$ the feet of A and B are the midpoints of XP and PY, so P lies 5 units from the first foot, while M lies 6 units from A. Hence the horizontal offset between M and P is $6 - 5 = 1$, and the height h of the trapezoid satisfies $h^2 = MP^2 - 1 = 15$. The area is

$$\frac{24 + 12}{2}\sqrt{15} = 18\sqrt{15},$$

so $m + n = 18 + 15 = 33$.

10. Let N be the number of ways to place the integers 1 through 12 in the 12 cells of a 2×6 grid so that for any two cells sharing a side, the difference between the numbers in those cells is not divisible by 3. One way to do this is shown below. Find the number of positive integer divisors of N.

1	3	5	7	9	11
2	4	6	8	10	12

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Solution:

The condition says adjacent cells have different residues mod 3. Each residue class among $1, \ldots, 12$ has exactly 4 members, so $N = K \cdot (4!)^3$, where K is the number of ways to fill the grid with residues $0, 1, 2$, each used 4 times, with adjacent cells different.

A column is an ordered pair (a, b) of distinct residues. If the current column is (a, b) and e is the third residue, the next column must be one of $(b, a), (b, e), (e, a)$: each of the three unordered pairs $\{a, b\}, \{b, e\}, \{a, e\}$ occurs in exactly one allowed orientation. So a residue pattern is determined by the sequence of six unordered pairs together with the orientation of the first column. Since each residue must appear 4 times, each of the three pairs must be used exactly twice, giving $\frac{6!}{2!\,2!\,2!} = 90$ sequences and $K = 2 \cdot 90 = 180$.

Therefore $N = 180 \cdot 24^3 = 2{,}488{,}320 = 2^{11} \cdot 3^5 \cdot 5$, which has $12 \cdot 6 \cdot 2 = 144$ positive divisors.

11. Find the number of collections of 16 distinct subsets of $\{1, 2, 3, 4, 5\}$ with the property that for any two subsets X and Y in the collection, $X \cap Y \neq \emptyset$.

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Solution:

The 32 subsets split into 16 complementary pairs $\{X, X^{\mathrm{c}}\}$, and no collection can contain both members of a pair (they are disjoint). A collection of 16 pairwise-intersecting subsets must therefore contain exactly one member of every pair; in particular it contains $\{1, 2, 3, 4, 5\}$ and not $\emptyset$.

If some singleton $\{x\}$ is chosen, every member must meet $\{x\}$, i.e. contain x. Exactly one set in each complementary pair contains x, so the collection must be exactly the 16 subsets containing x : this gives 5 collections. Otherwise no singleton is chosen, so all five 4-element sets are in the collection. Any two 3-element subsets of a 5-element set intersect, a 4-element set is disjoint only from its complement, and a chosen 2-element set and a chosen 3-element set are disjoint only if they are complements, which cannot both be chosen. So the only remaining condition is that the chosen 2-element sets pairwise intersect.

Viewing 2-element sets as edges of K_5, a pairwise-intersecting collection of edges either has all edges through one common vertex or is a triangle. The number of such edge families is: the empty family (1), triangles ($\binom{5}{3} = 10$), and nonempty families within a star, $5(2^4 - 1) - 10 = 65$ (subtracting the 10 single edges counted at both endpoints). That is $1 + 10 + 65 = 76$ collections, for a total of $5 + 76 = 81$.

12. In $\triangle ABC$ with side lengths $AB = 13, BC = 14,$ and $CA = 15,$ let M be the midpoint of $\overline{BC}$. Let P be the point on the circumcircle of $\triangle ABC$ such that M is on $\overline{AP}$. There exists a unique point Q on segment $\overline{AM}$ such that $\angle PBQ = \angle PCQ$. Then AQ can be written as $\frac{m}{\sqrt{n}}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Place $B = (0,0), C = (14,0), A = (5,12)$, so $M = (7,0)$ and $AM = \sqrt{4+144} = 2\sqrt{37}$. By power of the point M in the circumcircle, $MA \cdot MP = MB \cdot MC = 49$, so $MP = \frac{49}{2\sqrt{37}}$ and extending $A \to M$ by that length gives $P = \left(\frac{567}{74}, -\frac{147}{37}\right)$. The direction of $\overrightarrow{BP}$ is proportional to $(27, -14)$, and the direction of $\overrightarrow{CP}$ is proportional to $-(67, 42)$.

Write $Q = (5 + 2t,\ 12 - 12t)$ for $t \in (0,1)$, so that $AQ = t \cdot AM$. Using $\tan\theta = \frac{|u \times v|}{u \cdot v}$ for the angle between rays,

$$\tan\angle PBQ = \frac{394 - 296t}{222t - 33}, \qquad \tan\angle PCQ = \frac{1182 - 888t}{370t + 99},$$

and the second numerator is exactly $3(394 - 296t)$. Setting the two tangents equal cancels this common factor and leaves $370t + 99 = 3(222t - 33)$, so $296t = 198$ and $t = \frac{99}{148}$.

Then $AQ = \frac{99}{148} \cdot 2\sqrt{37} = \frac{99}{148}\sqrt{148} = \frac{99}{\sqrt{148}}$, and since $\gcd(99, 148) = 1$, the answer is $99 + 148 = 247$.

13. Let A be an acute angle such that $\tan A = 2\cos A$. Find the number of positive integers n less than or equal to 1000 such that $\sec^n A + \tan^n A$ is a positive integer whose units digit is 9.

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Solution:

Let $s = \sec A$ and $t = \tan A$. The hypothesis $\tan A = 2\cos A$ says $\tan A \sec A = 2$, i.e. $st = 2$, and always $s^2 - t^2 = 1$. Then $(s^2+t^2)^2 = (s^2-t^2)^2 + 4s^2t^2 = 17$, so $u = s^2$ and $v = t^2$ satisfy $u + v = \sqrt{17}, uv = 4$. The sums $w_m = u^m + v^m$ obey $w_{m+1} = \sqrt{17}\, w_m - 4w_{m-1}$ with $w_0 = 2, w_1 = \sqrt{17}$; by induction w_m is a positive integer for even m and an integer times $\sqrt{17}$ for odd m.

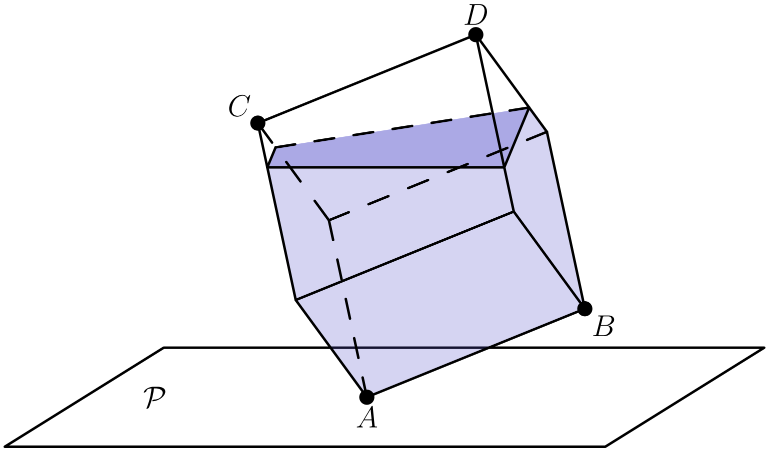
For even $n = 2m, s^n + t^n = w_m$, an integer exactly when m is even, i.e. $4 \mid n$. For odd n, $(s^n + t^n)^2 = w_n + 2(st)^n = w_n + 2^{n+1}$ is irrational, so $s^n + t^n$ is not an integer. Thus write $n = 4j$ and $x_j = w_{2j}$. Since $u^2 + v^2 = 9$ and $u^2v^2 = 16$, the integers x_j satisfy

$$x_{j+1} = 9x_j - 16x_{j-1}, \qquad x_0 = 2, \quad x_1 = 9,$$

giving $9, 49, 297, 1889, \ldots$ whose units digits repeat with period three: $9, 9, 7$. The units digit is 7 when $3 \mid j$ and 9 otherwise.

The valid $n \leq 1000$ are $n = 4j$ with $1 \leq j \leq 250$ and $3 \nmid j$: there are $250 - 83 = 167$ of them.

14. A cube-shaped container has vertices $A, B, C,$ and $D,$ where $\overline{AB}$ and $\overline{CD}$ are parallel edges of the cube, and $\overline{AC}$ and $\overline{BD}$ are diagonals of faces of the cube, as shown. Vertex A of the cube is set on a horizontal plane $\mathcal{P}$ so that the plane of the rectangle $ABDC$ is perpendicular to $\mathcal{P}$, vertex B is 2 meters above $\mathcal{P}$, vertex C is 8 meters above $\mathcal{P}$, and vertex D is 10 meters above $\mathcal{P}$. The cube contains water whose surface is parallel to $\mathcal{P}$ at a height of 7 meters above $\mathcal{P}$. The volume of water is $\frac{m}{n}$ cubic meters, where m and n are relatively prime positive integers. Find $m+n.$



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Solution:

Give the cube coordinates so that A is the origin, the edges lie along the axes, and the edge length is s : then $B=(s,0,0), C=(0,s,s), D=(s,s,s)$ satisfy the description ($\overline{AB} \parallel \overline{CD}$ are edges and $\overline{AC}, \overline{BD}$ are face diagonals). Height above $\mathcal{P}$ is a linear function $h(x,y,z)=u_1x+u_2y+u_3z$ for some unit vector u. The plane of rectangle $ABDC$ has normal direction $(0,1,-1)$, and perpendicularity to $\mathcal{P}$ means the vertical direction u lies in that plane, so $u_2=u_3$. The heights of B and C give $su_1=2$ and $s(u_2+u_3)=8$, so $su_2=su_3=4$, and $|u|=1$ forces $s^2=2^2+4^2+4^2=36$. Thus $s=6$ and $u=\frac{1}{3}(1,2,2)$ (and indeed $h(D)=10$).

The water is the region of $[0,6]^3$ where $h \leq 7$, i.e. $x+2y+2z \leq 21$. For fixed $x=a$, the slice is $\{(y,z) \in [0,6]^2 : y+z \leq \frac{21-a}{2}\}$, and since $\frac{21-a}{2}$ lies between 6 and 12, its area is $36-\frac{1}{2}\left(12-\frac{21-a}{2}\right)^2=36-\frac{(3+a)^2}{8}$.

Integrating,

$$V = \int_0^6 \left(36 - \frac{(3+a)^2}{8}\right) da = 216 - \frac{9^3 - 3^3}{24} = 216 - \frac{702}{24} = \frac{747}{4},$$

so $m + n = 747 + 4 = 751.$

15. For each positive integer n let a_n be the least positive integer multiple of 23 such that $a_n \equiv 1 \pmod{2^n}$. Find the number of positive integers n less than or equal to 1000 that satisfy $a_n = a_{n+1}$.

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Solution:

Write $a_n = 23b_n$, where b_n is the unique integer in $[1, 2^n]$ with $23b_n \equiv 1 \pmod{2^n}$, i.e. the inverse of 23 mod 2^n. Reducing mod 2^n shows $b_{n+1} \in \{b_n,\ b_n + 2^n\}$, so $a_n = a_{n+1}$ exactly when $b_{n+1} = b_n$, which happens exactly when the binary digit of weight 2^n in b_{n+1} is 0.

Since $23 \cdot 89 = 2047 = 2^{11} - 1$, setting $T_k = 1 + 2^{11} + 2^{22} + \cdots + 2^{11(k-1)}$ gives $23 \cdot 89\, T_k = 2^{11k} - 1$, so

$$23\left(2^{11k} - 89\, T_k\right) \equiv 1 \pmod{2^{11k}},$$

and every b_n with $n \le 11k$ is the reduction of $2^{11k} - 89\, T_k$ mod 2^n. In binary, $89 = 1011001_{\text{two}}$ occupies positions $0, 3, 4, 6$ of each 11-bit block of $89\, T_k$. Since $89\, T_k$ is odd, $2^{11k} - 89\, T_k = \left(2^{11k} - 1 - 89\, T_k\right) + 1$ keeps digit 0 equal to 1 and complements every digit in positions 1 through $11k - 1$.

So for $n \ge 1$, the digit of weight 2^n is 0 exactly when $n \equiv 0, 3, 4, 6 \pmod{11}$. Among $1 \le n \le 1000 = 90 \cdot 11 + 10$, the residue 0 occurs 90 times and the residues $3, 4, 6$ occur 91 times each, for a total of $90 + 3 \cdot 91 = 363$.

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