2023 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Five men and nine women stand equally spaced around a circle in random order. The probability that every man stands diametrically opposite a woman is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

The 14 positions split into 7 diametrically opposite pairs. Only the set of positions occupied by the men matters, and all $\binom{14}{5} = 2002$ five-element sets are equally likely. Every man stands opposite a woman exactly when no pair contains two men, so choose which 5 of the 7 pairs contain a man ($\binom{7}{5} = 21$ ways) and which position of each chosen pair the man occupies ($2^5 = 32$ ways), for $21 \cdot 32 = 672$ favorable sets.

The probability is $\frac{672}{2002} = \frac{48}{143}$, so $m + n = 48 + 143 = 191.$

2. Positive real numbers $b \neq 1$ and n satisfy the equations

$$\sqrt{\log_b n} = \log_b \sqrt{n} \qquad \text{and} \qquad b \cdot \log_b n = \log_b (bn).$$

The value of n is $\frac{j}{k}$, where j and k are relatively prime positive integers. Find $j + k$.

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Solution:

Let $x = \log_b n$. The first equation says $\sqrt{x} = \log_b n^{1/2} = \frac{x}{2}$, so $x = \frac{x^2}{4}$, giving $x = 0$ or $x = 4$. If $x = 0$ then $n = 1$, and the second equation would read $0 = \log_b b = 1$, impossible; so $x = 4$.

The second equation says $bx = \log_b b + \log_b n = 1 + x$, so $4b = 5$ and $b = \frac{5}{4}$. Then

$$n = b^4 = \left(\frac{5}{4}\right)^4 = \frac{625}{256},$$

which is in lowest terms, so $j + k = 625 + 256 = 881.$

3. A plane contains 40 lines, no 2 of which are parallel. Suppose that there are 3 points where exactly 3 lines intersect, 4 points where exactly 4 lines intersect, 5 points where exactly 5 lines intersect, 6 points where exactly 6 lines intersect, and no points where more than 6 lines intersect. Find the number of points where exactly 2 lines intersect.

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Solution:

Since no two of the 40 lines are parallel, every two lines cross, giving $\binom{40}{2} = 780$ pairs of lines, and each pair meets at exactly one point. A point where exactly k lines meet accounts for exactly $\binom{k}{2}$ of these pairs.

The given points account for $3\binom{3}{2} + 4\binom{4}{2} + 5\binom{5}{2} + 6\binom{6}{2} = 9 + 24 + 50 + 90 = 173$ pairs of lines. Each remaining pair meets at a point where exactly 2 lines intersect, one point per pair, so there are $780 - 173 = 607$ such points.

4. The sum of all positive integers m such that $\frac{13!}{m}$ is a perfect square can be written as $2^a 3^b 5^c 7^d 11^e 13^f$, where $a, b, c, d, e,$ and f are positive integers. Find $a+b+c+d+e+f$.

Solution:

Since $13! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, a valid $m = 2^x 3^y 5^z 7^w 11^u 13^v$ must leave every exponent of $\frac{13!}{m}$ even: $x \in \{0, 2, 4, 6, 8, 10\}$, $y \in \{1, 3, 5\}$, $z \in \{0, 2\}$, and $w = u = v = 1$.

The choices are independent, so the sum of all such m factors as

$$(1 + 4 + \cdots + 1024)(3 + 27 + 243)(1 + 25) \cdot 7 \cdot 11 \cdot 13 = 1365 \cdot 273 \cdot 26 \cdot 1001.$$

Since $1365 = 3 \cdot 5 \cdot 7 \cdot 13$, $273 = 3 \cdot 7 \cdot 13$, $26 = 2 \cdot 13$, and $1001 = 7 \cdot 11 \cdot 13$, the sum equals $2^1 3^2 5^1 7^3 11^1 13^4$, and $a+b+c+d+e+f = 1+2+1+3+1+4 = 12$.

5. Let P be a point on the circle circumscribing square $ABCD$ that satisfies $PA \cdot PC = 56$ and $PB \cdot PD = 90$. Find the area of $ABCD$.

Solution:

Let the circle have center O and radius R, with $A = (R, 0)$, $B = (0, R)$, $C = (-R, 0)$, $D = (0, -R)$, and $P = (R\cos\theta, R\sin\theta)$. Then $PA^2 = 2R^2(1 - \cos\theta)$ and $PC^2 = 2R^2(1 + \cos\theta)$, so $PA \cdot PC = 2R^2|\sin\theta| = 56$. In the same way $PB \cdot PD = 2R^2|\cos\theta| = 90$.

Squaring and adding, $4R^4 = 56^2 + 90^2 = 11236$, so $2R^2 = \sqrt{11236} = 106$. The square has diagonal $2R$, hence area $\frac{(2R)^2}{2} = 2R^2 = 106$.

6. Alice knows that 3 red cards and 3 black cards will be revealed to her one at a time in random order. Before each card is revealed, Alice must guess its color. If Alice plays optimally, the expected number of cards she will guess correctly is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Whatever Alice guesses, the deck evolves the same way; only the immediate success probability depends on her guess, so it is optimal to guess a color with the most cards remaining. Let $E(r, b)$ be the expected number of correct guesses from a state with r red and b black cards left. Then $E(r, 0) = r$, $E(0, b) = b$, and

$$E(r, b) = \frac{\max(r, b)}{r + b} + \frac{r\,E(r-1, b) + b\,E(r, b-1)}{r + b}.$$

By symmetry $E(r, b) = E(b, r)$. Computing upward: $E(1, 1) = \frac{3}{2}$, $E(2, 1) = \frac{2}{3} + \frac{2\cdot\frac{3}{2}+2}{3} = \frac{7}{3}$, $E(2, 2) = \frac{1}{2} + \frac{7}{3} = \frac{17}{6}$, $E(3, 1) = \frac{3}{4} + \frac{3\cdot\frac{7}{3}+3}{4} = \frac{13}{4}$, $E(3, 2) = \frac{3}{5} + \frac{3\cdot\frac{17}{6}+2\cdot\frac{13}{4}}{5} = \frac{18}{5}$, and finally $E(3, 3) = \frac{1}{2} + \frac{18}{5} = \frac{41}{10}$.

So the expected number of correct guesses is $\frac{41}{10}$, and $m + n = 41 + 10 = 51$.

7. Call a positive integer n *extra-distinct* if the remainders when n is divided by $2, 3, 4, 5,$ and 6 are distinct. Find the number of extra-distinct positive integers less than $1000.$

Solution:

Write r_k for the remainder of n modulo $k,$ and note $r_2 \equiv r_4 \pmod 2), r_2 \equiv r_6 \pmod 2),$ and $r_3 \equiv r_6 \pmod 3).$ If $r_2 = 0:$ then r_4 is even and different from $r_2,$ so $r_4 = 2;$ then r_6 is even and avoids $\{0, 2\},$ so $r_6 = 4,$ which gives $r_3 = 1;$ finally r_5 avoids $\{0, 1, 2, 4\},$ so $r_5 = 3.$ These say $n \equiv -2$ modulo each of $2, 3, 4, 5, 6,$ i.e. $n \equiv 58 \pmod{60}.$

If $r_2 = 1:$ similarly $r_4 = 3,$ then $r_6 = 5,$ giving $r_3 = 2,$ and r_5 avoids $\{1, 2, 3, 5\},$ so $r_5 \in \{0, 4\}.$ The choice $r_5 = 4$ gives $n \equiv -1 \pmod{60},$ i.e. $n \equiv 59;$ the choice $r_5 = 0$ gives $n \equiv 11 \pmod{12}$ with $5 \mid n,$ i.e. $n \equiv 35 \pmod{60}.$

Below 1000 there are 17 integers congruent to $35, 16$ congruent to $58,$ and 16 congruent to 59 modulo $60,$ for a total of $17 + 16 + 16 = 49.$

8. Rhombus $ABCD$ has $\angle BAD < 90^\circ$. There is a point P on the incircle of the rhombus such that the distances from P to the lines DA, AB, and BC are $9, 5$, and 16, respectively. Find the perimeter of $ABCD$.

Solution:

The distances from an interior point to the parallel lines DA and BC add up to the distance between them, the height of the rhombus. So the height is $9 + 16 = 25$, and the incircle, tangent to both lines, has radius $\frac{25}{2}$. Center the incircle at the origin with $DA: \; y = \frac{25}{2}$ and $BC: \; y = -\frac{25}{2}$. Then P has y-coordinate $\frac{25}{2} - 9 = \frac{7}{2}$, and $x^2 + \left(\frac{7}{2}\right)^2 = \left(\frac{25}{2}\right)^2$ gives $x = \pm 12$.

Let $\alpha = \angle BAD$. Line AB is tangent to the incircle and makes angle α with the horizontal, so (orienting the figure suitably) it is $x \sin\alpha + y\cos\alpha = -\frac{25}{2}$, and interior points satisfy $x\sin\alpha + y\cos\alpha + \frac{25}{2} > 0$. The condition $\operatorname{dist}(P, AB) = 5$ reads $x\sin\alpha + \frac{7}{2}\cos\alpha + \frac{25}{2} = 5$. For $x = 12$ the left side exceeds $\frac{25}{2}$, so $x = -12$, and the equation becomes $24\sin\alpha - 7\cos\alpha = 15$.

Substituting $7\cos\alpha = 24\sin\alpha - 15$ into $\sin^2\alpha + \cos^2\alpha = 1$ yields $625\sin^2\alpha - 720\sin\alpha + 176 = 0$, so $\sin\alpha = \frac{4}{5}$ or $\frac{44}{125}$. The root $\frac{44}{125}$ makes $\cos\alpha = \frac{24\sin\alpha - 15}{7}$ negative, contradicting $\angle BAD < 90^\circ$. So $\sin\alpha = \frac{4}{5}$, the side length is $\frac{25}{\sin\alpha} = \frac{125}{4}$, and the perimeter is $4 \cdot \frac{125}{4} = 125$.

9. Find the number of cubic polynomials $p(x) = x^3 + ax^2 + bx + c$, where a, b, and c are integers in $\{-20, -19, -18, \ldots, 18, 19, 20\}$, such that there is a unique integer $m \neq 2$ with $p(m) = p(2)$.

Solution:

Since $p(m) - p(2)$ does not involve c, each valid pair (a, b) contributes 41 polynomials. Factoring,

$$p(x) - p(2) = (x - 2)\big(x^2 + (a + 2)x + (2a + b + 4)\big),$$

so we need the quadratic factor $q(x)$ to have exactly one integer root different from 2. If q has any integer root, its other root is also an integer (their sum $-(a + 2)$ is an integer); if q has no integer root, then no m exists at all. So either q has roots 2 and k with $k \neq 2$, or a double root $k \neq 2$.

Roots 2 and k : Vieta's formulas give $a = -4 - k$ and $b = 4k + 4$. The constraint $-20 \leq b \leq 20$ forces $-6 \leq k \leq 4$ (and then a is automatically in range), so excluding $k = 2$ leaves 10 pairs. Double root k : here $a = -2k - 2$ and $b = k^2 + 4k$, and $b \leq 20$ forces $-6 \leq k \leq 2$, all valid for a, so excluding $k = 2$ leaves 8 pairs.

That is 18 pairs (a, b), hence $18 \cdot 41 = 738$ polynomials.

10. There exists a unique positive integer a for which the sum

$$U = \sum_{n=1}^{2023} \left\lfloor \frac{n^2 - na}{5} \right\rfloor$$

is an integer strictly between -1000 and 1000. For that unique a, find $a + U$.

(Note that $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.)

Solution:

Ignoring the floors, $\sum_{n=1}^{2023} \frac{n^2-na}{5} = \frac{1}{5}\left(\sum n^2 - a\sum n\right)$ vanishes exactly when $a = \frac{\sum n^2}{\sum n} = \frac{2 \cdot 2023+1}{3} = 1349$, an integer. For any other integer a the raw sum has absolute value at least $\frac{1}{5}\sum n = \frac{2023 \cdot 1012}{5} \approx 409455$, while taking floors changes the total by less than 2023, so only $a = 1349$ can put U strictly between -1000 and 1000.

With $a = 1349$, each term is $\frac{n^2-1349n-r_n}{5}$ with $r_n = (n^2 - 1349n) \bmod 5$, so $U = -\frac{1}{5}\sum r_n$. Since $1349 \equiv 4 \pmod 5$, we have $n^2 - 1349n \equiv n(n+1) \pmod 5$, whose residues for $n \equiv 0, 1, 2, 3, 4$ are $0, 2, 1, 2, 0$, summing to 5 per block of five. With $2023 = 5 \cdot 404 + 3$, the leftover terms $n \equiv 1, 2, 3$ contribute $2 + 1 + 2 = 5$, so $\sum r_n = 405 \cdot 5 = 2025$.

So $U = -\frac{2025}{5} = -405$, which indeed lies strictly between -1000 and 1000, and $a + U = 1349 - 405 = 944$.

11. Find the number of subsets of $\{1, 2, 3, \ldots, 10\}$ that contain exactly one pair of consecutive integers. Examples of such subsets are $\{\mathbf{1}, \mathbf{2}, 5\}$ and $\{1, 3, \mathbf{6}, \mathbf{7}, 10\}$.

Solution:

First, the number of subsets of a block of m consecutive integers containing no two consecutive elements is the Fibonacci number F_{m+2} (with $F_1 = F_2 = 1$): conditioning on whether the last element is used gives the Fibonacci recursion, and the counts start $1, 2, 3, 5, \ldots$

Suppose the unique consecutive pair is $\{k, k+1\}$ for some $1 \leq k \leq 9$. The remaining elements must exclude $k-1$ and $k+2$ (either would create a second consecutive pair) and must contain no consecutive pair within $\{1, \ldots, k-2\}$ or within $\{k+3, \ldots, 10\}$, blocks of sizes $k-2$ and $8-k$. So the count for this k is $F_k \cdot F_{10-k}$.

Summing over $k = 1, \ldots, 9:$

$$\sum_{k=1}^{9} F_k F_{10-k} = 34 + 21 + 26 + 24 + 25 + 24 + 26 + 21 + 34 = 235.$$

12. Let $\triangle ABC$ be an equilateral triangle with side length 55. Points $D, E,$ and F lie on $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, with $BD = 7, CE = 30,$ and $AF = 40$. Point P inside $\triangle ABC$ has the property that

$$\angle AEP = \angle BFP = \angle CDP.$$

Find $\tan^2(\angle AEP)$.

Solution:

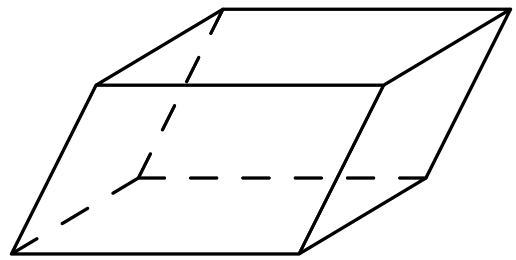
Place $B = (0,0), C = (55,0), A = \left(\frac{55}{2}, \frac{55\sqrt{3}}{2}\right)$, so that $D = (7,0), E = (40, 15\sqrt{3})$, and $F = \left(\frac{15}{2}, \frac{15\sqrt{3}}{2}\right)$. Let θ be the common angle and let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be the unit vectors in the directions $C \to A, A \to B, B \to C$ – the directions from E toward A, from F toward B, and from D toward C. Splitting $P - E$ into its component along $\mathbf{u}_1$ and its component perpendicular to side CA, whose length is the distance d_1 from P to line CA, the angle condition gives $(P - E) \cdot \mathbf{u}_1 = d_1 \cot\theta$; similarly $(P - F) \cdot \mathbf{u}_2 = d_2 \cot\theta$ and $(P - D) \cdot \mathbf{u}_3 = d_3 \cot\theta$.

Now add all three relations. Since $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$ (the directed sides of a triangle close up), P drops out, and by Viviani's theorem $d_1 + d_2 + d_3$ equals the height $\frac{55\sqrt{3}}{2}$. With $\mathbf{u}_1 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\mathbf{u}_2 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, and $\mathbf{u}_3 = (1,0)$, we get $E \cdot \mathbf{u}_1 = \frac{5}{2}, F \cdot \mathbf{u}_2 = -15$, and $D \cdot \mathbf{u}_3 = 7$, so

$$\frac{55\sqrt{3}}{2}\cot\theta = -\left(\frac{5}{2} - 15 + 7\right) = \frac{11}{2}.$$

Hence $\cot\theta = \frac{11}{55\sqrt{3}} = \frac{1}{5\sqrt{3}}$, so $\tan^2(\angle AEP) = \left(5\sqrt{3}\right)^2 = 75$.

13. Each face of two noncongruent parallelepipeds is a rhombus whose diagonals have lengths $\sqrt{21}$ and $\sqrt{31}$. The ratio of the volume of the larger of the two polyhedra to the volume of the smaller is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$. A parallelepiped is a solid with six parallelogram faces such as the one shown below.



Solution:

A rhombus with diagonals $\sqrt{21}$ and $\sqrt{31}$ has side $\sqrt{\frac{21}{4}+\frac{31}{4}}=\sqrt{13}$, so the three edge vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ all have squared length 13. In the face spanned by $\mathbf{u}$ and $\mathbf{v}$ the diagonals are $\mathbf{u}\pm\mathbf{v}$, with $|\mathbf{u}\pm\mathbf{v}|^2=26\pm 2\,\mathbf{u}\cdot\mathbf{v}$; matching $\{21,31\}$ gives $\mathbf{u}\cdot\mathbf{v}=\pm\frac{5}{2}$, and likewise for the other two pairs.

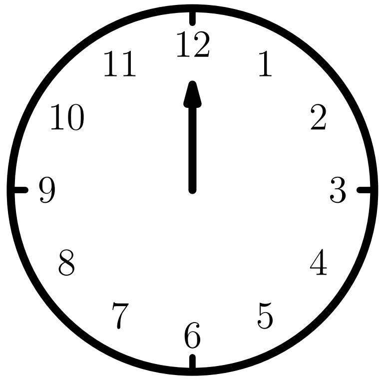
The squared volume is the Gram determinant

$$V^2=\det\begin{pmatrix}13 & x & y\\ x & 13 & z\\ y & z & 13\end{pmatrix}=2197-13(x^2+y^2+z^2)+2xyz=2197-\frac{975}{4}+2xyz$$

with $x,y,z\in\left\{\pm\frac{5}{2}\right\}$. Negating an edge vector flips the signs of two of x,y,z, so only the sign of xyz matters: $2xyz=\pm\frac{125}{4}$, giving $V^2=\frac{7938}{4}$ or $\frac{7688}{4}$.

The ratio of the volumes is $\sqrt{\frac{7938}{7688}}=\sqrt{\frac{3969}{3844}}=\frac{63}{62}$, already in lowest terms, so $m+n=63+62=125$.

14. The following analog clock has two hands that can move independently of each other.



Initially, both hands point to the number 12. The clock performs a sequence of hand movements so that on each movement, one of the two hands moves clockwise to the next number on the clock face while the other hand does not move.

Let N be the number of sequences of 144 hand movements such that during the sequence, every possible positioning of the hands appears exactly once, and at the end of the 144 movements, the hands have returned to their initial position. Find the remainder when N is divided by 1000.

Solution:

Record the hands as an ordered pair $(a, b) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12}$; each movement replaces (a, b) by $(a+1, b)$ or $(a, b+1)$, so a valid sequence is a closed tour through all 144 positions — equivalently, a choice, at each position, of which hand moves next. Sort the positions into 12 rows according to b, and let $S_b \subseteq \mathbb{Z}_{12}$ be the set of a-values at which the tour leaves row b (a b-move). A b-move from row $b-1$ enters row b at the same a-value, and the tour then runs through consecutive a-values until its next exit. For these runs to cover row b exactly once they must partition $\mathbb{Z}_{12}$, which forces each entry point to sit one step past an exit: $S_{b-1} = S_b + 1$. In particular every S_b has the same size c, and $S_b = S_0 - b$.

Leaving row b at its i-th exit (in cyclic order) leads to a run ending at the $(i+1)$-st exit of row $b+1$: each b-move advances the row index by 1 modulo 12 and the exit index by 1 modulo c. The tour therefore closes after $\operatorname{lcm}(12, c)$ b-moves, while a full tour must use all $12c$ exits, so the tour is a single cycle through all 144 positions precisely when $\gcd(c, 12) = 1$. Conversely, every choice of S_0 with $\gcd(|S_0|, 12) = 1$ yields exactly one valid movement sequence from the starting position.

Hence

$$N = \binom{12}{1} + \binom{12}{5} + \binom{12}{7} + \binom{12}{11} = 12 + 792 + 792 + 12 = 1608,$$

and the remainder when N is divided by 1000 is 608.

15. Find the largest prime number $p < 1000$ for which there exists a complex number z satisfying

- the real and imaginary part of z are both integers;

- $|z| = \sqrt{p}$, and

- there exists a triangle whose three side lengths are p, the real part of z^3, and the imaginary part of z^3.

Solution:

Write $z = a + bi$ with $a^2 + b^2 = p$, so $p = 2$ or $p \equiv 1 \pmod 4$, and the pair $\{|a|, |b|\}$ is then unique. Replacing z by $\pm z, \pm \bar{z}, \pm iz, \pm i\bar{z}$ only changes the real and imaginary parts of z^3 by signs and swaps, so we may take $a > b > 0$, and the two candidate side lengths are $|\operatorname{Re} z^3|$ and $|\operatorname{Im} z^3|$. Expanding $z^3 = (a^3 - 3ab^2) + (3a^2b - b^3)i$ and factoring,

$$\operatorname{Re} z^3 + \operatorname{Im} z^3 = (a-b)(p+4ab), \qquad \operatorname{Re} z^3 - \operatorname{Im} z^3 = (a+b)(p-4ab).$$

The triangle exists exactly when $\big||\operatorname{Re}| - |\operatorname{Im}|\big| < p < |\operatorname{Re}| + |\operatorname{Im}|$, and those two quantities are, in some order, the absolute values above. Since $a > b$ forces $(a-b)(p+4ab) > p$, the whole condition reduces to $(a+b)\,|p-4ab| < p$.

Because $a + b > \sqrt{a^2+b^2} = \sqrt{p}$, this requires $|p - 4ab| < \sqrt{p} < 32$: the products $4ab$ and $a^2 + b^2$ must nearly coincide. Checking the primes $p \equiv 1 \pmod 4$ below 1000 from the top down, each with its unique representation (for instance $997 = 31^2 + 6^2$ gives $(a+b)|p-4ab| = 37 \cdot 253$, far too big), the condition fails for every prime greater than 349 and holds for $349 = 18^2 + 5^2$, where $(a+b)\,|p-4ab| = 23 \cdot |349 - 360| = 253 < 349$.

Indeed for $z = 18 + 5i$ we get $z^3 = 4482 + 4735i$, and the lengths $349, 4482, 4735$ form a valid triangle. The answer is 349.

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