2022 AIME II Solutions

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1. Adults made up $\frac{5}{12}$ of the crowd of people at a concert. After a bus carrying 50 more people arrived, adults made up $\frac{11}{25}$ of the people at the concert. Find the minimum number of adults who could have been at the concert after the bus arrived.

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Solution:

Let the original crowd have $12k$ people, of whom $5k$ are adults. After the bus arrives there are $12k + 50$ people, and the number of adults is $\frac{11}{25}(12k + 50)$. For this to be an integer, 25 must divide $12k + 50$, so $25 \mid 12k$, and since $\gcd(12, 25) = 1$ this means k is a multiple of 25.

The adult count $\frac{11}{25}(12k + 50)$ increases with k, so the minimum occurs at $k = 25$: the new total is 350 and the number of adults is $\frac{11}{25} \cdot 350 = 154$. This is achievable, for example if the bus carries 29 adults and 21 non-adults, so the answer is 154.

2. Azar, Carl, Jon, and Sergey are the four players left in a singles tennis tournament. They are randomly assigned opponents in the semifinal matches, and the winners of those matches play each other in the final match to determine the winner of the tournament. When Azar plays Carl, Azar will win the match with probability $\frac{2}{3}$. When either Azar or Carl plays either Jon or Sergey, Azar or Carl will win the match with probability $\frac{3}{4}$. Assume that outcomes of different matches are independent. The probability that Carl will win the tournament is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

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Solution:

The three ways to pair the four players are equally likely, so Carl plays Azar in the semifinal with probability $\frac{1}{3}$. In that case Carl beats Azar with probability $\frac{1}{3}$ and then beats the Jon–Sergey winner with probability $\frac{3}{4}$, so Carl wins the tournament with probability $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$.

Otherwise (probability $\frac{2}{3}$) Carl plays Jon or Sergey and wins with probability $\frac{3}{4}$. His opponent in the final is Azar with probability $\frac{3}{4}$ (Carl then wins with probability $\frac{1}{3}$) and is Jon or Sergey with probability $\frac{1}{4}$ (Carl then wins with probability $\frac{3}{4}$). So in this case Carl wins the tournament with probability

$$\frac{3}{4}\left(\frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{3}{4}\right) = \frac{3}{4} \cdot \frac{7}{16} = \frac{21}{64}.$$

The total probability is $\frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{21}{64} = \frac{1}{12} + \frac{7}{32} = \frac{29}{96}$, so $p + q = 29 + 96 = 125$.

3. A right square pyramid with volume 54 has a base with side length 6. The five vertices of the pyramid all lie on a sphere with radius $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

The base has area 36, so $\frac{1}{3} \cdot 36 \cdot h = 54$ gives height $h = \frac{9}{2}$. By symmetry the sphere's center lies on the pyramid's axis, say at height z above the base. Each base vertex is at distance $3\sqrt{2}$ from the axis, so the center's distance to a base vertex is $\sqrt{z^2 + 18}$, while its distance to the apex is $\frac{9}{2} - z$.

Setting $\left(\frac{9}{2} - z\right)^2 = z^2 + 18$ gives $\frac{81}{4} - 9z = 18$, so $z = \frac{1}{4}$. The radius is $\frac{9}{2} - \frac{1}{4} = \frac{17}{4}$, and $m + n = 17 + 4 = 21$.

4. There is a positive real number x not equal to either $\frac{1}{20}$ or $\frac{1}{2}$ such that

$$\log_{20x}(22x) = \log_{2x}(202x).$$

The value $\log_{20x}(22x)$ can be written as $\log_{10}\left(\frac{m}{n}\right)$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Let y be the common value. In natural logarithms,

$$y = \frac{\ln 22x}{\ln 20x} = \frac{\ln 202x}{\ln 2x}.$$

When two fractions are equal, each also equals the quotient of the differences of numerators and denominators:

$$y = \frac{\ln 202x - \ln 22x}{\ln 2x - \ln 20x} = \frac{\ln \frac{101}{11}}{\ln \frac{1}{10}} = -\log_{10}\frac{101}{11} = \log_{10}\frac{11}{101}.$$

(Such an x exists: the equation rearranges to a solvable condition, and the excluded values $\frac{1}{20}, \frac{1}{2}$ only rule out degenerate bases.) Since $\gcd(11, 101) = 1$, we get $m + n = 11 + 101 = 112.$

5. Twenty distinct points are marked on a circle and labeled 1 through 20 in clockwise order. A line segment is drawn between every pair of points whose labels differ by a prime number. Find the number of triangles formed whose vertices are among the original 20 points.

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Solution:

A triangle has vertices $i < j < k$ where $j - i, k - j$, and $k - i$ are all prime. Since $k - i = (j - i) + (k - j)$ is a prime that is a sum of two primes, and the sum of two odd primes is even, one of the two smaller differences must equal 2. So the differences are $\{2, p\}$ in some order with p and $p + 2$ both prime: the twin prime pairs with $p + 2 \leq 19$ are $(3, 5), (5, 7), (11, 13)$, and $(17, 19)$.

For each pair, the middle vertex can be at distance 2 or at distance p from the smallest, and the total span is $p + 2$, so there are $2\big(20 - (p + 2)\big)$ triangles. This gives $2 \cdot 15 = 30, 2 \cdot 13 = 26, 2 \cdot 7 = 14$, and $2 \cdot 1 = 2$ for the four pairs.

The total is $30 + 26 + 14 + 2 = 72.$

6. Let $x_1 \le x_2 \le \cdots \le x_{100}$ be real numbers such that $|x_1| + |x_2| + \cdots + |x_{100}| = 1$ and $x_1 + x_2 + \cdots + x_{100} = 0$. Among all such 100-tuples of numbers, the greatest value that $x_{76} - x_{16}$ can achieve is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Since the terms sum to 0 while their absolute values sum to 1, the positive terms sum to $\frac{1}{2}$ and the negative terms sum to $-\frac{1}{2}$. If $x_{16} < -\frac{1}{32}$, then $x_1, \ldots, x_{16}$ are all less than $-\frac{1}{32}$ and would sum below $-\frac{1}{2}$, a contradiction; hence $x_{16} \ge -\frac{1}{32}$. Similarly, if $x_{76} > \frac{1}{50}$ then $x_{76}, \ldots, x_{100}$ are 25 terms each exceeding $\frac{1}{50}$, summing above $\frac{1}{2}$; hence $x_{76} \le \frac{1}{50}$.

Therefore $x_{76} - x_{16} \le \frac{1}{50} + \frac{1}{32} = \frac{16+25}{800} = \frac{41}{800}$, and this is achieved by taking $x_1 = \cdots = x_{16} = -\frac{1}{32}$, $x_{17} = \cdots = x_{75} = 0$, and $x_{76} = \cdots = x_{100} = \frac{1}{50}$.

Since $\gcd(41, 800) = 1$, the answer is $41 + 800 = 841$.

7. A circle with radius 6 is externally tangent to a circle with radius 24. Find the area of the triangular region bounded by the three common tangent lines of these two circles.

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Solution:

The centers O_1 (radius 24) and O_2 (radius 6) are 30 apart. The two external tangents meet at a point P on line O_1O_2 beyond the small circle, with $\frac{PO_1}{PO_2} = \frac{24}{6} = 4$. Combined with $PO_1 - PO_2 = 30$, this gives $PO_1 = 40$ and $PO_2 = 10$. Each external tangent makes angle θ with the center line, where $\sin\theta = \frac{24}{40} = \frac{3}{5}$, so $\tan\theta = \frac{3}{4}$.

The third common tangent is the tangent at the point of tangency T, which is perpendicular to O_1O_2 at distance 24 from O_1. The triangle bounded by the three tangents has apex P and base on this line, with height $PT = 40 - 24 = 16$ and half-base $16\tan\theta = 12$.

Its area is $\frac{1}{2} \cdot 24 \cdot 16 = 192$.

8. Find the number of positive integers $n \leq 600$ whose value can be uniquely determined among all positive integers when the values of $\left\lfloor \frac{n}{4} \right\rfloor$, $\left\lfloor \frac{n}{5} \right\rfloor$, and $\left\lfloor \frac{n}{6} \right\rfloor$ are given, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real number x.

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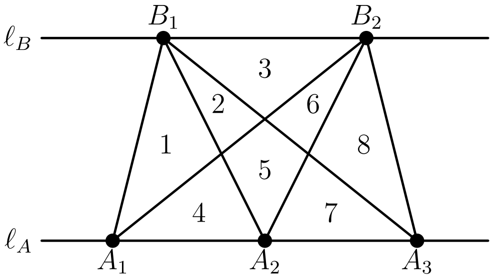
Solution:

The set of positive integers sharing a given triple $\left(\left\lfloor \frac{n}{4} \right\rfloor, \left\lfloor \frac{n}{5} \right\rfloor, \left\lfloor \frac{n}{6} \right\rfloor\right)$ is an intersection of three intervals, hence a block of consecutive integers. So n is uniquely determined exactly when neither $n-1$ nor $n+1$ gives the same triple: some floor must drop at $n-1$, meaning $4, 5,$ or 6 divides n, and some floor must jump at $n+1$, meaning $4, 5,$ or 6 divides $n+1$.

Since n and $n+1$ cannot both be even, the divisor pairs for $(n, n+1)$ are $(4,5), (5,4), (5,6)$, and $(6,5)$. Working modulo $60 : 4 \mid n,\ \ 5 \mid n+1$ gives $n \equiv 4, 24, 44; 5 \mid n,\ \ 4 \mid n+1$ gives $n \equiv 15, 35, 55; 5 \mid n,\ \ 6 \mid n+1$ gives $n \equiv 5, 35$; and $6 \mid n,\ \ 5 \mid n+1$ gives $n \equiv 24, 54$. The union is the 8 residues $\{4, 5, 15, 24, 35, 44, 54, 55\}$ modulo 60.

Each residue occurs 10 times among $1 \leq n \leq 600$, so the count is $8 \cdot 10 = 80$. (Note $n = 600$ fails: 601 is divisible by none of $4, 5, 6$, so $601, 602, 603$ share 600's triple.)

9. Let ℓ_A and ℓ_B be two distinct parallel lines. For positive integers m and n, distinct points $A_1, A_2, A_3, \ldots, A_m$ lie on ℓ_A, and distinct points $B_1, B_2, B_3, \ldots, B_n$ lie on ℓ_B. Additionally, when segments $\overline{A_iB_j}$ are drawn for all $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$, no point strictly between ℓ_A and ℓ_B lies on more than two of the segments. Find the number of bounded regions into which this figure divides the plane when $m = 7$ and $n = 5$. The figure shows that there are 8 regions when $m = 3$ and $n = 2$.



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Solution:

Two segments $\overline{A_iB_j}$ and $\overline{A_kB_l}$ cross strictly between the lines exactly when one of the A's comes first and the other's B comes first, which happens for exactly one pairing of any two A's with any two B's. By the general-position hypothesis these crossings are distinct, so there are $X = \binom{m}{2}\binom{n}{2}$ of them.

Clip the two lines to long segments and apply Euler's formula. The vertices are the $m+n$ marked points, the X crossings, and the 4 clipped line ends, so $V = m + n + X + 4$. Line ℓ_A is divided into $m+1$ edges and ℓ_B into $n+1$; each crossing splits two segments, so the drawn segments contribute $mn + 2X$ edges, giving $E = mn + m + n + 2X + 2$. Then

$$F = E - V + 2 = mn + X,$$

of which one face is unbounded, so there are $mn + X - 1$ bounded regions. For $m = 3, n = 2$ this gives $6 + 3 - 1 = 8$, matching the figure.

For $m = 7$ and $n = 5 : 35 + \binom{7}{2}\binom{5}{2} - 1 = 35 + 21 \cdot 10 - 1 = 244.$

10. Find the remainder when

$$\binom{\binom{3}{2}}{2} + \binom{\binom{4}{2}}{2} + \cdots + \binom{\binom{40}{2}}{2}$$

is divided by 1000.

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Solution:

Since $\binom{n}{2} = \frac{n(n-1)}{2}$ and $\binom{n}{2} - 1 = \frac{(n+1)(n-2)}{2}$,

$$\binom{\binom{n}{2}}{2} = \frac{1}{2} \cdot \frac{n(n-1)}{2} \cdot \frac{(n+1)(n-2)}{2} = \frac{(n+1)n(n-1)(n-2)}{8} = 3\binom{n+1}{4}.$$

By the hockey stick identity,

$$\sum_{n=3}^{40} 3\binom{n+1}{4} = 3\sum_{k=4}^{41}\binom{k}{4} = 3\binom{42}{5} = 3 \cdot 850668 = 2552004.$$

The remainder upon division by 1000 is 4.

11. Let $ABCD$ be a convex quadrilateral with $AB = 2$, $AD = 7$, and $CD = 3$ such that the bisectors of acute angles $\angle DAB$ and $\angle ADC$ intersect at the midpoint of $\overline{BC}$. Find the square of the area of $ABCD$.

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Solution:

Place $A = (0,0)$ and $D = (7,0)$ with B, C above the axis, and let M be the midpoint of $\overline{BC}$. Reflecting B over the bisector line AM carries ray AB to ray AD, so B maps to $B' = (2,0)$, and reflecting C over the bisector DM gives $C' = (4,0)$. Since M lies on both mirror lines, $MB' = MB = MC = MC'$, so M is equidistant from B' and C' and hence $M = (3,h)$ for some $h > 0$.

Write $\angle DAB = 2\alpha$ and $\angle ADC = 2\delta$, so $\tan\alpha = \frac{h}{3}$ and $\tan\delta = \frac{h}{4}$. Then $B = (2\cos 2\alpha,\ 2\sin 2\alpha)$ and $C = (7 - 3\cos 2\delta,\ 3\sin 2\delta)$, and the midpoint condition on the x-coordinates reads $2\cos 2\alpha - 3\cos 2\delta = -1$. Substituting $\cos 2\alpha = \frac{9-h^2}{9+h^2}$ and $\cos 2\delta = \frac{16-h^2}{16+h^2}$ and clearing denominators gives $2h^4 = 10h^2$, so $h^2 = 5$. (The y-coordinate condition is then satisfied automatically: $2\sin 2\alpha + 3\sin 2\delta = \frac{6\sqrt{5}}{7} + \frac{8\sqrt{5}}{7} = 2h$.)

Now $\cos 2\alpha = \frac{2}{7}$, $\sin 2\alpha = \frac{3\sqrt{5}}{7}$, $\cos 2\delta = \frac{11}{21}$, $\sin 2\delta = \frac{8\sqrt{5}}{21}$, so $B = \left(\frac{4}{7}, \frac{6\sqrt{5}}{7}\right)$ and $C = \left(\frac{38}{7}, \frac{8\sqrt{5}}{7}\right)$. The shoelace formula on A, B, C, D gives area $6\sqrt{5}$, whose square is 180.

12. Let $a, b, x,$ and y be real numbers with $a > 4$ and $b > 1$ such that

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 16} = \frac{(x-20)^2}{b^2-1} + \frac{(y-11)^2}{b^2} = 1.$$

Find the least possible value of $a + b$.

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Solution:

The first ellipse has $c^2 = a^2 - (a^2 - 16) = 16$, hence foci $F_1 = (-4, 0)$ and $F_2 = (4, 0)$, with distance sum $2a$. The second is centered at $(20, 11)$ with vertical major axis and $c^2 = b^2 - (b^2 - 1) = 1$, hence foci $G_1 = (20, 10)$ and $G_2 = (20, 12)$, with distance sum $2b$. If $P = (x, y)$ lies on both, then

$$2a + 2b = (PF_1 + PG_1) + (PF_2 + PG_2) \geq F_1G_1 + F_2G_2 = \sqrt{24^2 + 10^2} + \sqrt{16^2 + 12^2} = 26 + 20 = 46,$$

so $a + b \geq 23$.

Equality requires P to lie on both segments $\overline{F_1G_1}$ and $\overline{F_2G_2}$. These segments do intersect, at $P = \left(14, \frac{15}{2}\right)$: then $PF_1 = \frac{39}{2}, PF_2 = \frac{25}{2}$, so $2a = 32, a = 16 > 4$; and $PG_1 = \frac{13}{2}, PG_2 = \frac{15}{2}$, so $2b = 14, b = 7 > 1$.

Hence the least possible value of $a + b$ is $16 + 7 = 23$.

13. There is a polynomial $P(x)$ with integer coefficients such that

$$P(x) = \frac{(x^{2310}-1)^6}{(x^{105}-1)(x^{70}-1)(x^{42}-1)(x^{30}-1)}$$

holds for every $0 < x < 1$. Find the coefficient of x^{2022} in $P(x)$.

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Solution:

For $0 < x < 1$,

$$P(x) = (1-x^{2310})^6 \cdot \frac{1}{(1-x^{105})(1-x^{70})(1-x^{42})(1-x^{30})},$$

and each factor $\frac{1}{1-x^k}$ expands as a geometric series. Since $2022 < 2310$, the factor $(1-x^{2310})^6$ contributes only its constant term 1, so the coefficient of x^{2022} is the number of nonnegative integer solutions of $105a + 70b + 42c + 30d = 2022$.

Reducing modulo 2 gives $105a \equiv 2022$, so a is even; modulo 3 gives $70b \equiv 2022 \equiv 0$, so $3 \mid b$; modulo 5 gives $2c \equiv 2022 \equiv 2$, so $c \equiv 1 \pmod 5$; modulo 7 gives $2d \equiv 2022 \equiv 6$, so $d \equiv 3 \pmod 7$. Writing $a = 2a', b = 3b', c = 5c' + 1, d = 7d' + 3$ turns the equation into

$$210(a'+b'+c'+d') + 42 + 90 = 2022, \qquad \text{so} \qquad a'+b'+c'+d' = 9.$$

By stars and bars there are $\binom{12}{3} = 220$ solutions, so the coefficient is 220.

14. For positive integers a, b, and c with $a < b < c$, consider collections of postage stamps in denominations a, b, and c cents that contain at least one stamp of each denomination. If there exists such a collection that contains sub-collections worth every whole number of cents up to 1000 cents, let $f(a, b, c)$ be the minimum number of stamps in such a collection. Find the sum of the three least values of c such that $f(a, b, c) = 97$ for some choice of a and b.

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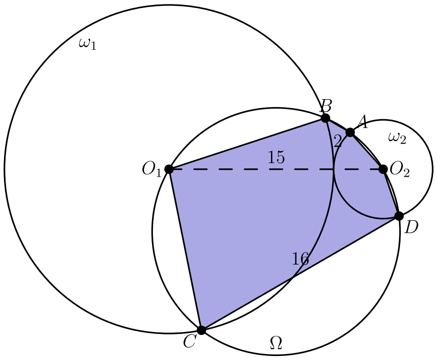
Solution:

To form 1 cent we need $a = 1$. Suppose the collection has x ones, y stamps of b, and z of c. The value $b - 1$ must be made from ones alone, so $x \geq b - 1$; the value $c - 1$ must be made from ones and b's, so $x + yb \geq c - 1$; and the total $x + yb + zc$ must be at least 1000. Conversely these three conditions suffice: with $x \geq b - 1$ the ones and b's make every value up to $x + yb$, and then c's extend this to every value up to the total. So the optimum takes $x = b - 1$, then the least y with $x + yb \geq c - 1$, then the least z reaching 1000.

For fixed c, the count $f(1, b, c)$ is maximized at $b = c - 1$ (many ones, which cover value least efficiently), where the optimal collection is $c - 2$ ones, one stamp of $c - 1$, and $\left\lceil \frac{1003-2c}{c} \right\rceil$ stamps of c, totaling $c - 3 + \left\lceil \frac{1003}{c} \right\rceil$. For $12 \leq c \leq 87$ this maximum is at most 96 (it is 93 at $c = 12$, decreases in the middle, and returns to 96 at $c = 87$), so no b gives 97; a quick check of $c \leq 10$ shows the possible counts skip 97 there as well.

For $c = 11$, taking $b = 7$ gives 6 ones, one 7 (reaching $13 \geq 10$), and $\left\lceil \frac{987}{11} \right\rceil = 90$ elevens: $f(1, 7, 11) = 6 + 1 + 90 = 97$. For $c = 88$ and $c = 89$, taking $b = 87$ gives 86 ones, one 87 (reaching 173), and $\left\lceil \frac{827}{88} \right\rceil = \left\lceil \frac{827}{89} \right\rceil = 10$ stamps of c, for $86 + 1 + 10 = 97$ in both cases. So the three least values of c are $11, 88, 89$, with sum 188.

15. Two externally tangent circles ω_1 and ω_2 have centers O_1 and O_2, respectively. A third circle Ω passing through O_1 and O_2 intersects ω_1 at B and C and ω_2 at A and D, as shown. Suppose that $AB = 2, O_1O_2 = 15, CD = 16$, and ABO_1CDO_2 is a convex hexagon. Find the area of this hexagon.



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Solution:

All six hexagon vertices lie on $\Omega: O_1$ and O_2 by hypothesis, and A, B, C, D as intersection points with Ω. Let R be the radius of Ω, and let the arcs cut off by the sides $AB, BO_1, O_1C, CD, DO_2, O_2A$ be $2\alpha, 2\beta, 2\beta, 2\gamma, 2\delta, 2\delta$ (the two β's because chords $BO_1 = O_1C = r_1$, the radius of ω_1, and likewise $O_2A = O_2D = r_2$), so $\alpha + 2\beta + \gamma + 2\delta = \pi$. Each chord equals $2R\sin(\text{half its arc}): 2R\sin\alpha = 2, 2R\sin\gamma = 16$, and the chord $\overline{O_1O_2}$ subtends $2\beta + 2\gamma + 2\delta$, giving $2R\sin(\alpha+\sigma) = 15$ where $\sigma = \beta + \delta$. External tangency gives a second equation worth $15: r_1 + r_2 = 2R(\sin\beta + \sin\delta) = 15$.

Since $\gamma = \pi - \alpha - 2\sigma$, we have $\sin\gamma = \sin(\alpha + 2\sigma)$. Sum-to-product then gives

$$18 = 2R\left[\sin(\alpha+2\sigma) + \sin\alpha\right] = 4R\sin(\alpha+\sigma)\cos\sigma = 30\cos\sigma,$$

so $\cos\sigma = \frac{3}{5}$, and similarly $14 = 4R\cos(\alpha+\sigma)\sin\sigma$ gives $R\cos(\alpha+\sigma) = \frac{35}{8}$. Combining with $2R\sin(\alpha+\sigma) = 15$ yields $4R^2 = 225 + \frac{1225}{16}$, so $R^2 = \frac{4825}{64}$. Also $15 = 2R(\sin\beta + \sin\delta) = 4R\sin\frac{\sigma}{2}\cos\frac{\beta-\delta}{2}$ with $\sin\frac{\sigma}{2} = \frac{1}{\sqrt{5}}$, so $\cos\frac{\beta-\delta}{2} = \frac{15\sqrt{5}}{4R}$ and $\cos(\beta-\delta) = \frac{1125}{8R^2} - 1$.

Joining the center of Ω to the six vertices splits the hexagon into six triangles, so its area is $\frac{1}{2}R^2\left[\sin 2\alpha + 2\sin 2\beta + 2\sin 2\delta + \sin 2\gamma\right]$. Now $R^2(\sin 2\beta + \sin 2\delta) = 2R^2\sin\sigma\cos(\beta-\delta) = \frac{8}{5}\left(\frac{1125}{8} - R^2\right) = \frac{835}{8}$, while $\sin 2\alpha + \sin 2\gamma = -2\sin 2\sigma\cos\big(2(\alpha+\sigma)\big) = -2\cdot\frac{24}{25}\cdot\left(-\frac{95}{193}\right)$ since $\cos\big(2(\alpha+\sigma)\big) = 1 - \frac{2\cdot 225}{4R^2} = -\frac{95}{193}$, so $\frac{1}{2}R^2(\sin 2\alpha + \sin 2\gamma) = \frac{1}{2}\cdot\frac{4825}{64}\cdot\frac{912}{965} = \frac{285}{8}$. The area is $\frac{835}{8} + \frac{285}{8} = 140$.

Problems: https://live.poshenloh.com/past-contests/aime/2022II