2022 AIME I Solutions

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1. Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients of 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16, 54)$ and $(20, 53)$. Find $P(0) + Q(0)$.

Solution:

Let $R(x) = P(x) + Q(x)$. The leading coefficients 2 and -2 cancel, so R is a linear function. Since both graphs pass through $(16, 54)$ and $(20, 53)$, we get $R(16) = 108$ and $R(20) = 106$.

The slope of R is $\frac{106-108}{20-16} = -\frac{1}{2}$, so

$$P(0) + Q(0) = R(0) = R(16) + 16 \cdot \frac{1}{2} = 108 + 8 = 116.$$

2. Find the three-digit positive integer $\underline{a}\,\underline{b}\,\underline{c}$ whose representation in base nine is $\underline{b}\,\underline{c}\,\underline{a}_{\,\text{nine}}$, where $a, b,$ and c are (not necessarily distinct) digits.

Solution:

The condition says $100a + 10b + c = 81b + 9c + a$, which simplifies to $99a = 71b + 8c$. Since the digits also appear in a base-nine numeral, each is at most 8. Reducing modulo 8 gives $3a \equiv -b \pmod 8$, so $b \equiv 5a \pmod 8$.

For $a = 1, b = 5$ makes $71b$ exceed 99; for $a = 2, b = 2$ gives $8c = 198 - 142 = 56$, so $c = 7$. For each $a \geq 3$, the required b forces $99a - 71b$ outside the range $[0, 64]$, so there is no other solution.

The number is 227, and indeed $227 = 2 \cdot 81 + 7 \cdot 9 + 2 = 272_{\text{nine}}$.

3. In isosceles trapezoid $ABCD$, parallel bases $\overline{AB}$ and $\overline{CD}$ have lengths 500 and 650, respectively, and $AD = BC = 333.$ The angle bisectors of $\angle A$ and $\angle D$ meet at P, and the angle bisectors of $\angle B$ and $\angle C$ meet at Q. Find PQ.

Solution:

Let the bisector of $\angle A$ meet $\overline{CD}$ at A'. Since $\overline{AB} \parallel \overline{CD}$, we have $\angle DA'A = \angle A'AB = \angle A'AD$, so triangle ADA' is isosceles with $DA' = DA = 333.$ The bisector of $\angle D$ is then the median from D in this triangle, so P, which lies on both bisectors, is the midpoint of $\overline{AA'}$. Symmetrically, Q is the midpoint of $\overline{BB'}$, where B' is on $\overline{CD}$ with $CB' = 333.$

Place $D = (0, 0)$ and $C = (650, 0)$, so $A = (75, h)$ and $B = (575, h)$ for the appropriate height h. Then $A' = (333, 0)$ and $B' = (650 - 333, 0) = (317, 0)$, so

$$P = \left(\frac{75 + 333}{2}, \frac{h}{2}\right) = \left(204, \frac{h}{2}\right), \qquad Q = \left(\frac{575 + 317}{2}, \frac{h}{2}\right) = \left(446, \frac{h}{2}\right).$$

Therefore $PQ = 446 - 204 = 242.$

4. Let $w = \frac{\sqrt{3}+i}{2}$ and $z = \frac{-1+i\sqrt{3}}{2}$, where $i = \sqrt{-1}$. Find the number of ordered pairs (r, s) of positive integers not exceeding 100 that satisfy the equation $i \cdot w^r = z^s$.

Solution:

Both w and z have modulus 1 : in polar form $w = \text{cis}\,30^\circ$ and $z = \text{cis}\,120^\circ$, while $i = \text{cis}\,90^\circ$. The equation $i \cdot w^r = z^s$ is therefore a statement about arguments:

$$90 + 30r \equiv 120s \pmod{360}, \qquad \text{i.e.} \qquad r + 3 \equiv 4s \pmod{12}.$$

For each s, this determines $r \pmod{12}$: the residue $4s - 3$ is $1, 5,$ or 9 modulo 12 according as $s \equiv 1, 2,$ or $0 \pmod 3$. Among $1 \le r \le 100$ there are 9 values with $r \equiv 1 \pmod{12}$ and 8 values each with $r \equiv 5$ or $r \equiv 9 \pmod{12}$. Among $1 \le s \le 100$ there are 34 values with $s \equiv 1 \pmod 3$ and 33 values in each of the other two classes.

The count is $34 \cdot 9 + 33 \cdot 8 + 33 \cdot 8 = 306 + 264 + 264 = 834.$

5. A straight river that is 264 meters wide flows from west to east at a rate of 14 meters per minute. Melanie and Sherry sit on the south bank of the river with Melanie a distance of D meters downstream from Sherry. Relative to the water, Melanie swims at 80 meters per minute, and Sherry swims at 60 meters per minute. At the same time, Melanie and Sherry begin swimming in straight lines to a point on the north bank of the river that is equidistant from their starting positions. The two women arrive at this point simultaneously. Find D.

Solution:

Put Sherry at the origin and Melanie at $(D, 0)$ on the south bank. A point on the north bank equidistant from both is $\left(\frac{D}{2}, 264\right)$. If both arrive at time t, then each swimmer's velocity relative to the water is her ground velocity minus the current $(14, 0)$, so

$$\left(\frac{D}{2t} - 14\right)^2 + \left(\frac{264}{t}\right)^2 = 60^2, \qquad \left(-\frac{D}{2t} - 14\right)^2 + \left(\frac{264}{t}\right)^2 = 80^2.$$

Subtracting, with $u = \frac{D}{2t}$: $(u + 14)^2 - (u - 14)^2 = 56u = 6400 - 3600 = 2800$, so $u = 50$. Substituting back, $(50 - 14)^2 + \left(\frac{264}{t}\right)^2 = 3600$ gives $\frac{264}{t} = 48$, so $t = \frac{11}{2}$.

Therefore $D = 2ut = 100t = 550$.

6. Find the number of ordered pairs of integers (a, b) such that the sequence

$$3, 4, 5, a, b, 30, 40, 50$$

is strictly increasing and no set of four (not necessarily consecutive) terms forms an arithmetic progression.

Solution:

The sequence is increasing exactly when $5 < a < b < 30$, giving $\binom{24}{2} = 276$ pairs. The six fixed terms contain no four-term arithmetic progression, so every progression must involve a or b. If only one of them is involved, three fixed terms must already be in progression: $3, 4, 5$ extends only by 6, and $30, 40, 50$ extends only by 20. So the single-variable violations are $a = 6$ (23 pairs) and $20 \in \{a, b\}$ (23 pairs), which overlap in the pair $(6, 20)$.

If both a and b are involved, two fixed terms complete the progression. Checking the possible positions: $(4, 5, a, b)$ gives $(6, 7)$; $(3, 5, a, b)$ gives $(7, 9)$; $(3, a, b, 30)$ gives $(12, 21)$; $(4, a, b, 40)$ gives $(16, 28)$; $(5, a, b, 50)$ gives $(20, 35)$, out of range; and $(a, b, 30, 40)$ gives $(10, 20)$. Of these, $(6, 7)$ and $(10, 20)$ are already counted, so $(7, 9)$, $(12, 21)$, and $(16, 28)$ are the only new bad pairs.

The number of valid pairs is $276 - (23 + 23 - 1) - 3 = 276 - 48 = 228$.

7. Let $a, b, c, d, e, f, g, h, i$ be distinct integers from 1 to 9. The minimum possible positive value of

$$\frac{a \cdot b \cdot c - d \cdot e \cdot f}{g \cdot h \cdot i}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Try to make the numerator equal to 1 while keeping large digits in the denominator. The products $2 \cdot 3 \cdot 6 = 36$ and $1 \cdot 5 \cdot 7 = 35$ differ by 1 and leave $4, 8, 9$ for the denominator, giving the value

$$\frac{36 - 35}{4 \cdot 8 \cdot 9} = \frac{1}{288}.$$

To beat this, a fraction would need numerator 1 with denominator greater than 288. The denominators exceeding 288 are $\{7, 8, 9\}$, $\{6, 8, 9\}$, $\{5, 8, 9\}$, $\{6, 7, 9\}$, $\{5, 7, 9\}$, and $\{6, 7, 8\}$. Splitting the remaining six digits into two triples in each case, the closest product pairs are 30 and 24, 30 and 28, 36 and 28, 30 and 28, 36 and 32, and 36 and 30, respectively — differences of at least 2, and even $\frac{2}{432} = \frac{1}{216}$ exceeds $\frac{1}{288}$.

So the minimum positive value is $\frac{1}{288}$, and $m + n = 1 + 288 = 289$.

8. Equilateral triangle $\triangle ABC$ is inscribed in circle ω with radius 18. Circle ω_A is tangent to sides $\overline{AB}$ and $\overline{AC}$ and is internally tangent to ω. Circles ω_B and ω_C are defined analogously. Circles $\omega_A, \omega_B,$ and ω_C meet in six points — two points for each pair of circles. The three intersection points closest to the vertices of $\triangle ABC$ are the vertices of a large equilateral triangle in the interior of $\triangle ABC,$ and the other three intersection points are the vertices of a smaller equilateral triangle in the interior of $\triangle ABC.$ The side length of the smaller equilateral triangle can be written as $\sqrt{a} - \sqrt{b},$ where a and b are positive integers. Find $a + b.$

Solution:

Let O be the center of ω. The center of ω_A lies on line AO (the bisector of $\angle A$) at some distance d from A; since $\overline{AB}$ makes a 30° angle with AO, the radius is $r = d \sin 30^\circ = \frac{d}{2}$. Internal tangency to ω requires the center to be $18 - r$ from O, which forces the center past $O : d - 18 = 18 - \frac{d}{2}$, so $d = 24, r = 12,$ and the center is 6 beyond O.

Place O at the origin with $A = (0, 18)$. Then the three centers are $O_A = (0, -6)$ and $O_B, O_C = (\pm 3\sqrt{3}, 3)$, all with radius 12. The intersections of ω_B and ω_C lie on the y-axis: $27 + (y-3)^2 = 144$ gives $y = 3 \pm \sqrt{117}$. The point $(0, 3 + \sqrt{117})$ is closer to A and belongs to the larger triangle, so the smaller triangle has vertex $(0, 3 - \sqrt{117})$, at distance $\sqrt{117} - 3$ from O.

By symmetry the smaller triangle is equilateral with circumradius $\sqrt{117} - 3$, so its side is $\sqrt{3}\left(\sqrt{117} - 3\right) = \sqrt{351} - \sqrt{27}$. Thus $a + b = 351 + 27 = 378.$

9. Ellina has twelve blocks, two each of red ($\mathbf{R}$), blue ($\mathbf{B}$), yellow ($\mathbf{Y}$), green ($\mathbf{G}$), orange ($\mathbf{O}$), and purple ($\mathbf{P}$). Call an arrangement of blocks *even* if there is an even number of blocks between each pair of blocks of the same color. For example, the arrangement

$$\mathbf{R\,B\,B\,Y\,G\,G\,Y\,R\,O\,P\,P\,O}$$

is even. Ellina arranges her blocks in a row in random order. The probability that her arrangement is even is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution:

If a color occupies positions $i<j$, the number of blocks between them is $j-i-1$, which is even exactly when i and j have opposite parity. So an arrangement is even precisely when every color occupies one odd position and one even position — that is, the six odd slots contain each color exactly once, and so do the six even slots.

Counting arrangements of the twelve blocks (blocks of the same color identical), there are $\frac{12!}{2^6}$ in total, and $6!\cdot 6!$ even ones (a permutation of the six colors in the odd slots and another in the even slots). The probability is

$$\frac{6!\cdot 6!\cdot 2^6}{12!}=\frac{16}{231}.$$

Since $\gcd(16,231)=1$, the answer is $m+n=16+231=247.$

10. Three spheres with radii $11, 13,$ and 19 are mutually externally tangent. A plane intersects the spheres in three congruent circles centered at $A, B,$ and $C,$ respectively, and the centers of the spheres all lie on the same side of this plane. Suppose that $AB^2 = 560$. Find AC^2.

Solution:

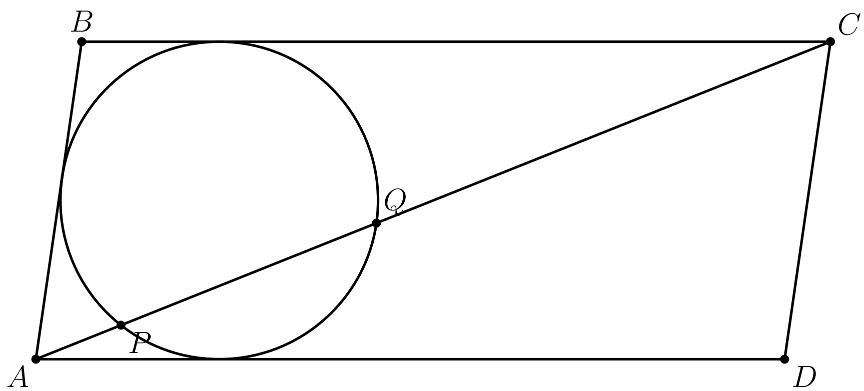
Let the sphere centers be at heights h_1, h_2, h_3 above the plane. Each circle's center is the foot of the perpendicular from the sphere's center, and the common circle radius ρ satisfies $\rho^2 = 11^2 - h_1^2 = 13^2 - h_2^2 = 19^2 - h_3^2$.

The first two spheres are tangent, so their centers are $11 + 13 = 24$ apart, and projecting onto the plane, $AB^2 = 24^2 - (h_2 - h_1)^2$. Thus $(h_2 - h_1)^2 = 576 - 560 = 16$. Congruence gives $h_2^2 - h_1^2 = 169 - 121 = 48$, so $h_2 - h_1 = 4$ and $h_2 + h_1 = 12$ (the other sign gives a negative sum), yielding $h_1 = 4, h_2 = 8,$ and $\rho^2 = 121 - 16 = 105$. Then $h_3^2 = 361 - 105 = 256,$ so $h_3 = 16$.

The first and third centers are $11 + 19 = 30$ apart, so

$$AC^2 = 30^2 - (h_3 - h_1)^2 = 900 - 144 = 756.$$

11. Let $ABCD$ be a parallelogram with $\angle BAD < 90^\circ$. A circle tangent to sides $\overline{DA}$, $\overline{AB}$, and $\overline{BC}$ intersects diagonal $\overline{AC}$ at points P and Q with $AP < AQ$, as shown. Suppose that $AP = 3$, $PQ = 9$, and $QC = 16$. Then the area of $ABCD$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.



Solution:

By power of a point, $AP \cdot AQ = 3 \cdot 12 = 36$ and $CQ \cdot CP = 16 \cdot 25 = 400$, so the tangent lengths from A and C are 6 and 20. The tangent point on $\overline{AB}$ is 6 from A, hence $AB - 6$ from B; equal tangents from B put the tangent point on $\overline{BC}$ at that same distance from B, so its distance from C is $BC - (AB - 6) = 20$, giving $BC = AB + 14$.

Let $\angle BAD = 2\theta$. The center lies on the bisector of $\angle A$ with the tangent length from A equal to 6, so the radius is $\rho = 6\tan\theta$. The circle is tangent to both parallel lines AD and BC, whose distance apart is $AB \sin 2\theta$, so $AB \sin 2\theta = 2\rho = 12\tan\theta$, which simplifies to $AB\cos^2\theta = 6$. In triangle ABC, $\angle ABC = 180^\circ - 2\theta$ and $AC = 3 + 9 + 16 = 28$, so the law of cosines gives

$$784 = AB^2 + BC^2 + 2 \cdot AB \cdot BC \cos 2\theta.$$

Substituting $BC = AB + 14$ and $\cos 2\theta = 2\cos^2\theta - 1$, the AB^2 terms cancel and, using $AB\cos^2\theta = 6$, the equation collapses to $24\,AB + 336 + 196 = 784$, so $AB = \frac{21}{2}$ and $\cos^2\theta = \frac{4}{7}$.

Then $\sin 2\theta = 2\sqrt{\frac{3}{7}}\sqrt{\frac{4}{7}} = \frac{4\sqrt{3}}{7}$, and the area is

$$AB \cdot BC \sin 2\theta = \frac{21}{2} \cdot \frac{49}{2} \cdot \frac{4\sqrt{3}}{7} = 147\sqrt{3},$$

so $m + n = 147 + 3 = 150.$

12. For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \ldots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\},$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

Solution:

Count element by element: S_n equals the number of triples (x, A, B) with $|A| = |B|$ and $x \in A \cap B$. For a fixed x and size k, there are $\binom{n-1}{k-1}$ choices for each of A and B containing x, so by the Vandermonde identity

$$S_n = n \sum_{k=1}^{n} \binom{n-1}{k-1}^2 = n\binom{2n-2}{n-1}.$$

Therefore

$$\frac{S_{2022}}{S_{2021}} = \frac{2022\binom{4042}{2021}}{2021\binom{4040}{2020}} = \frac{2022}{2021} \cdot \frac{4042 \cdot 4041}{2021^2} = \frac{2 \cdot 2022 \cdot 4041}{2021^2}.$$

Since $2021 = 43 \cdot 47$ divides neither $2022, 4041 = 3^2 \cdot 449$, nor 2, this fraction is in lowest terms: $p = 2 \cdot 2022 \cdot 4041 = 16341804$ and $q = 2021^2 = 4084441$.

Then $p + q = 20426245$, whose remainder modulo 1000 is 245.

13. Let S be the set of all rational numbers that can be expressed as a repeating decimal in the form $0.\overline{abcd}$, where at least one of the digits $a, b, c,$ or d is nonzero. Let N be the number of distinct numerators obtained when numbers in S are written as fractions in lowest terms. For example, both 4 and 410 are counted among the distinct numerators for numbers in S because $0.\overline{3636} = \frac{4}{11}$ and $0.\overline{1230} = \frac{410}{3333}$. Find the remainder when N is divided by 1000.

Solution:

Every element of S equals $\frac{k}{9999}$ for some $1 \leq k \leq 9999$, where $9999 = 3^2 \cdot 11 \cdot 101.$ In lowest terms this is $\frac{m}{D}$ where $D \mid 9999, m \leq D,$ and $\gcd(m, D) = 1;$ conversely any such $\frac{m}{D}$ arises from $k = m \cdot \frac{9999}{D}$. So N counts the integers m that are at most, and coprime to, some divisor D of $9999.$

Classify m by which of the primes $3, 11, 101$ divide it, always using the largest divisor D coprime to $m.$ If $\gcd(m, 9999) = 1,$ take $D = 9999:$ there are $\varphi(9999) = 6000$ such $m.$ If $3 \mid m$ only, take $D = 11 \cdot 101 = 1111:$ multiples of 3 up to 1111 avoiding 11 and 101 number $370 - 33 - 3 = 334.$ If $11 \mid m$ only, take $D = 9 \cdot 101 = 909:$ that gives $82 - 27 = 55.$ If $101 \mid m$ only, then $D = 99 < 101$ admits none. If $33 \mid m$ but $101 \nmid m,$ take $D = 101:$ the values $33, 66, 99$ give 3 more, and any m divisible by $3 \cdot 101$ or $11 \cdot 101$ would need $D \leq 11,$ which is impossible.

Therefore $N = 6000 + 334 + 55 + 3 = 6392,$ and the remainder modulo 1000 is $392.$

14. Given $\triangle ABC$ and a point P on one of its sides, call line ℓ the *splitting line* of $\triangle ABC$ through P if ℓ passes through P and divides $\triangle ABC$ into two polygons of equal perimeter. Let $\triangle ABC$ be a triangle where $BC = 219$ and AB and AC are positive integers. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively, and suppose that the splitting lines of $\triangle ABC$ through M and N intersect at 30°. Find the perimeter of $\triangle ABC$.

Solution:

Write $a = BC = 219, b = CA, c = AB$, and s for the semiperimeter. The splitting line through M meets $\overline{BC}$ at the point X with $BX = s - \frac{c}{2}$ (then each piece has perimeter s). In triangle BMX, the law of sines shows $\angle BXM = \frac{C}{2}$: this needs $c \sin\left(B + \frac{C}{2}\right) = (a+b)\sin\frac{C}{2}$, which reduces via $a + b = 2R(\sin A + \sin B) = 4R\cos\frac{C}{2}\cos\frac{A-B}{2}$ and $c = 4R\sin\frac{C}{2}\cos\frac{C}{2}$ to $\sin\left(B + \frac{C}{2}\right) = \cos\frac{A-B}{2}$, true because those angles are complementary. Hence the splitting line through M is parallel to the angle bisector from C, and likewise the one through N is parallel to the bisector from B.

The internal bisectors from B and C meet at $90^\circ + \frac{A}{2} > 90^\circ$, so the acute angle between the two splitting lines is $90^\circ - \frac{A}{2} = 30^\circ$, forcing $\angle A = 120^\circ$. The law of cosines gives

$$219^2 = b^2 + c^2 + bc = (b+c)^2 - bc.$$

Set $p = b + c$, so $bc = p^2 - 219^2$ and b, c are roots of $t^2 - pt + (p^2 - 219^2)$, requiring $4 \cdot 219^2 - 3p^2$ to be a perfect square k^2. Then $3 \mid k$ and $3 \mid p$; writing $p = 3r$ and $k = 3m$ turns the condition into $m^2 + 3r^2 = 146^2$. The triangle inequality $p > 219$ and $4 \cdot 219^2 \geq 3p^2$ restrict $74 \leq r \leq 84$, and checking these, only $r = 80$ works, with $m = 46$.

So $b + c = 240$ and $bc = 240^2 - 47961 = 9639$, giving $\{b, c\} = \{51, 189\}$ — a valid triangle. The perimeter is $219 + 240 = 459$.

15. Let x, y, and z be positive real numbers satisfying the system of equations

$$\sqrt{2x - xy} + \sqrt{2y - xy} = 1$$
$$\sqrt{2y - yz} + \sqrt{2z - yz} = \sqrt{2}$$
$$\sqrt{2z - zx} + \sqrt{2x - zx} = \sqrt{3}.$$

Then $\left[(1-x)(1-y)(1-z)\right]^2$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Each radicand factors: $2x - xy = x(2 - y)$, and so on, so $0 < x, y, z \leq 2$. Substitute $x = 2\sin^2\alpha, y = 2\sin^2\beta, z = 2\sin^2\gamma$ with $\alpha, \beta, \gamma \in (0°, 90°]$. Then $\sqrt{x(2-y)} = \sqrt{4\sin^2\alpha\cos^2\beta} = 2\sin\alpha\cos\beta$, and each equation collapses by the sine addition formula:

$$2\sin(\alpha+\beta) = 1, \qquad 2\sin(\beta+\gamma) = \sqrt{2}, \qquad 2\sin(\gamma+\alpha) = \sqrt{3}.$$

Taking $\alpha + \beta = 30°, \beta + \gamma = 45°, \gamma + \alpha = 60°$ and solving, $\alpha = 22.5°, \beta = 7.5°$, $\gamma = 37.5°$. (The supplementary branch choices consistent with the angle ranges lead to the same value of the final square.) By the double-angle identity, $1 - x = \cos 2\alpha = \cos 45°$, $1 - y = \cos 15°$, and $1 - z = \cos 75°$.

Therefore

$$(1-x)(1-y)(1-z) = \frac{\sqrt{2}}{2}\cos 15°\sin 15° = \frac{\sqrt{2}}{2} \cdot \frac{\sin 30°}{2} = \frac{\sqrt{2}}{8},$$

whose square is $\frac{2}{64} = \frac{1}{32}$. Thus $m + n = 1 + 32 = 33$.

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