2020 AIME I Solutions

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1. In $\triangle ABC$ with $AB = AC$, point D lies strictly between A and C on side $\overline{AC}$, and point E lies strictly between A and B on side $\overline{AB}$ such that $AE = ED = DB = BC$. The degree measure of $\angle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Let $\angle BAC = \alpha$. Since $AE = ED$, triangle AED is isosceles with $\angle ADE = \angle DAE = \alpha$, so the exterior angle at E gives $\angle DEB = 2\alpha$. Since $ED = DB$, triangle EDB has $\angle DBE = \angle DEB = 2\alpha$, hence $\angle EDB = 180^\circ - 4\alpha$.

The three angles at D on segment $\overline{AC}$ sum to a straight angle: $\alpha + (180^\circ - 4\alpha) + \angle BDC = 180^\circ$, so $\angle BDC = 3\alpha$. Since $DB = BC$, also $\angle BCD = \angle BDC = 3\alpha$. But $AB = AC$ makes $\angle ABC = \angle ACB = 3\alpha$, so the angle sum of $\triangle ABC$ gives $\alpha + 3\alpha + 3\alpha = 180^\circ$, hence $\alpha = \frac{180}{7}$ degrees.

Then $\angle ABC = 3\alpha = \frac{540}{7}$ degrees, and $m + n = 540 + 7 = 547$.

2. There is a unique positive real number x such that the three numbers $\log_8(2x), \log_4 x,$ and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

Let $t = \log_2 x$. Then $\log_4 x = \frac{t}{2}$ and $\log_8(2x) = \frac{1+t}{3}$. In a geometric progression the middle term squared equals the product of the outer terms:

$$\left(\frac{t}{2}\right)^2 = \frac{1+t}{3} \cdot t.$$

Since $t = 0$ gives no valid ratio, divide by $t : \frac{t}{4} = \frac{1+t}{3}$, so $3t = 4 + 4t$ and $t = -4$. Thus $x = 2^{-4} = \frac{1}{16}$, and the progression is $-1, -2, -4$ with common ratio 2, which is positive as required.

Therefore $m + n = 1 + 16 = 17.$

3. A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

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Solution:

Equating the two representations in base ten gives $121a + 11b + c = 512 + 64b + 8c + a$, which simplifies to

$$120a = 512 + 53b + 7c.$$

All of a, b, c are base-eight digits, so $0 \leq a, b, c \leq 7$ (and $a \geq 1$ since it leads the base-eleven representation).

The right side is at least 512, so $a \geq 5$. Since $N = 121a + 11b + c$ increases with a, try $a = 5$: then $53b + 7c = 88$. Here $b = 0$ gives $7c = 88$, impossible, and $b \geq 2$ overshoots, so $b = 1$ and $7c = 35$, giving $c = 5$.

Thus $N = 121 \cdot 5 + 11 + 5 = 621$, whose base-eight representation is 1155 and base-eleven representation is 515, as required. The least such N is 621.

4. Let S be the set of positive integers N with the property that the last four digits of N are $2020,$ and when the last four digits are removed, the result is a divisor of $N.$ For example, $42{,}020$ is in S because 4 is a divisor of $42{,}020.$ Find the sum of all the digits of all the numbers in $S.$ For example, the number $42{,}020$ contributes $4+2+0+2+0=8$ to this total.

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Solution:

If removing the last four digits leaves $k \geq 1,$ then $N = 10000k + 2020,$ and the condition $k \mid N$ is equivalent to $k \mid 2020.$ Since $2020 = 2^2 \cdot 5 \cdot 101,$ there are 12 choices of $k : 1, 2, 4, 5, 10, 20, 101, 202, 404, 505, 1010, 2020.$

Each member of S has digit sum equal to the digit sum of k plus $2+0+2+0=4.$ The digit sums of the twelve divisors are $1, 2, 4, 5, 1, 2, 2, 4, 8, 10, 2, 4,$ totaling $45.$

The answer is $45 + 12 \cdot 4 = 93.$

5. Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order.

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Solution:

First count arrangements from which some card's removal leaves the rest ascending. Any such arrangement arises by choosing the card to remove (6 ways) and inserting it into one of the 6 gaps of the other five cards written in increasing order, for 36 constructions. But the fully sorted row 123456 arises from all 6 card choices, and each of the 5 arrangements obtained by swapping two adjacent cards of the sorted row arises twice (move either card of the pair past the other). Every other construction gives a distinct arrangement.

So the ascending count is $1 + 5 + (36 - 6 - 10) = 26,$ and by symmetry there are 26 descending arrangements. No arrangement is counted in both totals: that would require an ascending and a descending subsequence of five cards, needing at least $5 + 5 - 1 = 9$ cards.

The total is $26 + 26 = 52.$

6. A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. The square of the radius of the spheres is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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Solution:

A sphere of radius r resting in a circular hole of radius a has its center on the axis of the hole; since the center is at distance r from every point of the hole's rim, it sits at distance $\sqrt{r^2-a^2}$ from the plane of the board. So the two centers lie at depths $\sqrt{r^2-1}$ and $\sqrt{r^2-4}$ on the same side of the board, with horizontal separation 7.

Tangency of the spheres means the centers are $2r$ apart:

$$49+\left(\sqrt{r^2-1}-\sqrt{r^2-4}\right)^2=4r^2.$$

Expanding gives $49+2r^2-5-2\sqrt{(r^2-1)(r^2-4)}=4r^2$, so $\sqrt{(r^2-1)(r^2-4)}=22-r^2$. Squaring, $r^4-5r^2+4=484-44r^2+r^4$, hence $39r^2=480$ and $r^2=\frac{160}{13}$.

Thus $m+n=160+13=173.$

7. A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

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Solution:

A committee with k men has $k+1$ women, so

$$N = \sum_{k=0}^{11} \binom{11}{k}\binom{12}{k+1} = \sum_{k=0}^{11} \binom{11}{k}\binom{12}{11-k} = \binom{23}{11}$$

by Vandermonde's identity (both sides count ways to choose 11 people from all 23).

Now factor $\binom{23}{11} = \frac{23!}{11!\,12!}$. The primes $13, 17, 19, 23$ each appear in the numerator but not the denominator. By Legendre's formula the exponent of 2 is $19-8-10=1$, of 3 is $9-4-5=0$, of 5 is $4-2-2=0$, of 7 is $3-1-1=1$, and of 11 is $2-1-1=0$. Hence $N = 2\cdot 7\cdot 13\cdot 17\cdot 19\cdot 23$.

The sum of the primes dividing N is $2+7+13+17+19+23=81$.

8. A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Work in the complex plane with O at the origin and east along the positive real axis. Each day's displacement is the previous one multiplied by $z = \frac{1}{2}e^{i\pi/3}$, so

$$P = 5\left(1 + z + z^2 + \cdots\right) = \frac{5}{1 - z}.$$

Since $z = \frac{1}{4} + \frac{\sqrt{3}}{4}i$, we get $1 - z = \frac{3}{4} - \frac{\sqrt{3}}{4}i$, whose squared magnitude is $\frac{9}{16} + \frac{3}{16} = \frac{3}{4}$. Therefore

$$OP^2 = \frac{25}{3/4} = \frac{100}{3},$$

and $m + n = 100 + 3 = 103.$

9. Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random from the set S and labeled $a_1, a_2,$ and a_3 in the order they are chosen. The probability that both a_1 divides a_2 and a_2 divides a_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

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Solution:

Write $20^9 = 2^{18} \cdot 5^9$, so each $a_i = 2^{x_i} 5^{y_i}$ with $0 \le x_i \le 18$ and $0 \le y_i \le 9$; there are $19 \cdot 10 = 190$ divisors, and the exponents of the two primes are chosen independently and uniformly. The chain $a_1 \mid a_2 \mid a_3$ holds exactly when $x_1 \le x_2 \le x_3$ and $y_1 \le y_2 \le y_3$.

Non-decreasing triples from a set of k values correspond to multisets of size 3, counted by $\binom{k+2}{3}$. So the probability is

$$\frac{\binom{21}{3}}{19^3} \cdot \frac{\binom{12}{3}}{10^3} = \frac{1330}{6859} \cdot \frac{220}{1000} = \frac{70}{361} \cdot \frac{11}{50} = \frac{77}{1805}.$$

Since $1805 = 5 \cdot 19^2$ shares no factor with $77 = 7 \cdot 11$, the fraction is in lowest terms and $m = 77$.

10. Let m and n be positive integers satisfying the conditions

- $\gcd(m+n, 210) = 1,$
- m^m is a multiple of n^n, and
- m is not a multiple of $n.$

Find the least possible value of $m+n.$

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Solution:

If a prime p divides $n,$ then $p \mid n^n \mid m^m,$ so $p \mid m$ and hence $p \mid m+n.$ Since $\gcd(m+n, 210) = 1,$ no prime factor of n is $2, 3, 5,$ or $7:$ every prime factor of n is at least $11.$ Because m is not a multiple of $n,$ some prime p has $b = v_p(m) < a = v_p(n),$ where v_p denotes the exponent of $p.$ Comparing exponents of p in $n^n \mid m^m$ gives $bm \geq an,$ so $m \geq \frac{a}{b}n \geq 2n.$ In particular $a \geq 2,$ so $p^2 \mid n$ and $n \geq 11^2 = 121.$

Take $n = 121$ with $p = 11, a = 2, b = 1:$ then m is a multiple of 11 but not of $121,$ and $m \geq 2 \cdot 121 = 242.$ The candidates $m = 253, 264, 275$ give $m + n = 374 = 2 \cdot 11 \cdot 17, 385 = 5 \cdot 7 \cdot 11, 396 = 2^2 \cdot 3^2 \cdot 11,$ all sharing a factor with $210,$ while $m = 242 = 2 \cdot 11^2$ is a multiple of $121.$ But $m = 286 = 2 \cdot 11 \cdot 13$ works: $v_{11}(m^m) = 286 \geq 242 = v_{11}(n^n),$ so $n^n \mid m^m,$ and $m + n = 407 = 11 \cdot 37$ is coprime to $210.$

Any other admissible n is at least $13^2 = 169,$ forcing $m + n \geq 3n \geq 507.$ Hence the least possible value is $407.$

11. For integers $a, b, c,$ and d, let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.

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Solution:

The condition says the integers $f(2) = 4 + 2a + b$ and $f(4) = 16 + 4a + b$ are both roots of the monic quadratic g. These two values are equal exactly when $a = -6$.

If $a = -6$, then for any b and any c the choice $d = -f(2)^2 - c\,f(2)$ makes $f(2) = f(4)$ a root of g, giving $21 \cdot 21 = 441$ triples. If $a \neq -6$, the two distinct values must be the two roots of g, so Vieta forces $c = -(f(2) + f(4)) = -(20 + 6a + 2b)$, and then $d = f(2)f(4)$ is an integer. The requirement $|c| \leq 10$ becomes $-15 \leq 3a + b \leq -5$.

For each a, count integers $b \in [-10, 10]$ with $-15 - 3a \leq b \leq -5 - 3a$: the counts are $2, 5, 11, 11, 11, 11, 9, 6, 3$ for $a = -8, -7, -5, -4, -3, -2, -1, 0, 1$ respectively, and 0 for all other $a \neq -6$, totaling 69. The answer is $441 + 69 = 510$.

12. Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive divisors of n.

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Solution:

Work prime by prime. Since $149 - 2 = 147 = 3 \cdot 7^2$, the lifting-the-exponent lemma gives $v_3(149^n - 2^n) = v_3(147) + v_3(n) = 1 + v_3(n)$ and $v_7(149^n - 2^n) = 2 + v_7(n)$ for every positive integer n. Requiring at least 3 and 7 forces $3^2 \mid n$ and $7^5 \mid n$.

For 5 we first need $149^n \equiv 2^n \pmod{5}$, i.e. $4^n \equiv 2^n$, i.e. $2^n \equiv 1 \pmod{5}$, which requires $4 \mid n$. Write $n = 4k$. In $149^4 - 2^4 = (149 - 2)(149 + 2)(149^2 + 4)$, only the last factor is divisible by 5, and only once, since $149^2 + 4 = 22205 = 5 \cdot 4441$. Lifting the exponent from the base $149^4, 2^4$ gives $v_5(149^n - 2^n) = 1 + v_5(k)$, so $5^4 \mid k$, i.e. $4 \cdot 5^4 \mid n$.

The least valid n is $2^2 \cdot 3^2 \cdot 5^4 \cdot 7^5$, which has $(2+1)(2+1)(4+1)(5+1) = 270$ positive divisors.

13. Point D lies on side $\overline{BC}$ of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4, BC = 5,$ and $CA = 6,$ the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

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Solution:

In triangle $ABD,$ the internal bisector of the angle at B meets the circumcircle of ABD again at the midpoint of arc AD not containing $B,$ and that arc midpoint lies on the perpendicular bisector of $\overline{AD}$ — so E is exactly that arc midpoint. The inscribed angles $\angle EAD$ and $\angle EBD$ subtend the same arc $ED,$ so $\angle EAD = \frac{B}{2}$. Similarly $\angle FAD = \frac{C}{2}$, and E, F lie on opposite sides of line $AD.$

Let M be the midpoint of $\overline{AD}$. In right triangles AME and $AMF,$ $ME = AM \tan \frac{B}{2}$ and $MF = AM \tan \frac{C}{2}$, so $EF = AM \left(\tan \frac{B}{2} + \tan \frac{C}{2}\right)$, while the distance from A to line EF is $AM.$ Hence $[AEF] = \frac{1}{2}AM^2 \left(\tan \frac{B}{2} + \tan \frac{C}{2}\right).$

Here $BD = 2$ and $DC = 3,$ so $AD^2 = AB \cdot AC - BD \cdot DC = 24 - 6 = 18$ and $AM^2 = \frac{9}{2}$. The law of cosines gives $\cos B = \frac{1}{8}$ and $\cos C = \frac{3}{4}$, so $\tan \frac{B}{2} = \sqrt{\frac{1-1/8}{1+1/8}} = \frac{\sqrt{7}}{3}$ and $\tan \frac{C}{2} = \frac{1}{\sqrt{7}}$, with sum $\frac{10\sqrt{7}}{21}$. The area is $\frac{1}{2} \cdot \frac{9}{2} \cdot \frac{10\sqrt{7}}{21} = \frac{15\sqrt{7}}{14}$, so $m + n + p = 15 + 7 + 14 = 36.$

14. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a+b)^2$.

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Solution:

Write $P(x) = x^2 + px + q$ with roots r_1 and r_2. The solutions of $P(P(x)) = 0$ split into the two solutions of $P(x) = r_1$ and the two of $P(x) = r_2$, and each pair sums to $-p$.

If 3 and 4 form one pair, then $a + b = -p = 3 + 4 = 7$, so $(a+b)^2 = 49$. This is achievable: $P(x) = (x-3)(x-4) + r_1$ with r_1 satisfying $r_1^2 - 6r_1 + 12 = 0$, which has (complex) solutions, and the four roots are distinct.

Otherwise 3 and 4 lie in different pairs: $3 + a = 4 + b = -p = s$, and $\{P(3), P(4)\} = \{r_1, r_2\}$. The root sum gives $P(3) + P(4) = 25 + 7p + 2q = s$, so with $p = -s$ we get $q = 4s - \frac{25}{2}$, and then $P(3) = s - \frac{7}{2}$ and $P(4) = \frac{7}{2}$. The root product gives $\frac{7}{2}\left(s - \frac{7}{2}\right) = q = 4s - \frac{25}{2}$, whose solution is $s = \frac{1}{2}$. Then $a + b = (s-3) + (s-4) = -6$, so $(a+b)^2 = 36$, with $a = -\frac{5}{2}, b = -\frac{7}{2}$ all distinct from 3 and 4. The sum of all possible values is $49 + 36 = 85$.

15. Let $\triangle ABC$ be an acute triangle with circumcircle ω and orthocenter H. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3, HX = 2$, and $HY = 6$. The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

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Solution:

Reflecting H over line BC lands on ω, so the circumcircle of HBC is the reflection of ω over BC. Take the circumcenter O as the origin, so that $H = A + B + C$ as vectors. If M is the midpoint of $\overline{BC}$, then $OM \perp BC$, so the reflected center is $2M - O = B + C = H - A$. Tangency at H means XY is perpendicular to the radius from $B + C$ to H, which is the vector A : the chord XY is perpendicular to OA.

Place $A = (0, R)$ so that XY is horizontal at height h, with $H = (x_0, h)$. The half-chord length is $\sqrt{R^2 - h^2}$, and $HX = 2, HY = 6$ give $\sqrt{R^2 - h^2} = 4$ with $|x_0| = 2$. From $HA = 3 : 4 + (R - h)^2 = 9$, so $R - h = \sqrt{5}$. Then

$$16 = R^2 - h^2 = (R - h)(R + h) = \sqrt{5}\left(2R - \sqrt{5}\right),$$

giving $R = \frac{21}{2\sqrt{5}}$.

Now $B + C = H - A = (\pm 2, -\sqrt{5})$, so $M = \left(\pm 1, -\frac{\sqrt{5}}{2}\right)$ and $OM = \frac{3}{2}$, whence $BC = 2\sqrt{R^2 - \frac{9}{4}} = 2\sqrt{\frac{99}{5}}$. The distance from A to line BC (through M, perpendicular to OM) is $\frac{|A \cdot M - OM^2|}{OM} = \frac{21/4 + 9/4}{3/2} = 5$, using $A \cdot M = -\frac{\sqrt{5}R}{2} = -\frac{21}{4}$. Hence

$$[ABC] = \frac{1}{2} \cdot 2\sqrt{\frac{99}{5}} \cdot 5 = \sqrt{495} = 3\sqrt{55},$$

and $m + n = 3 + 55 = 58$.

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