2019 AIME I Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\ldots99}_{321 \text{ digits}}.$$

Find the sum of the digits of N.

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Solution:

Each summand is $10^k - 1$, so

$$N = \sum_{k=1}^{321} \left(10^k - 1\right) = \underbrace{11\ldots10}_{321} - 321.$$

The subtraction changes only the last four digits: $1110 - 321 = 789$, so those four digits become 0789. Thus N consists of 318 ones followed by 0789, and the digit sum is $318 + 0 + 7 + 8 + 9 = 342$.

2. Jenn randomly chooses a number J from $1, 2, 3, \ldots, 19, 20$. Bela then randomly chooses a number B from $1, 2, 3, \ldots, 19, 20$ distinct from J. The value of $B - J$ is at least 2 with a probability that can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

There are $20 \cdot 19 = 380$ equally likely ordered pairs (J, B) with $B \neq J$. The condition $B \geq J + 2$ allows $19 - J$ choices of B for each $J \leq 18$, so the number of favorable pairs is

$$\sum_{J=1}^{18} (19 - J) = 18 + 17 + \cdots + 1 = 171.$$

The probability is $\frac{171}{380} = \frac{9}{20}$, so $m + n = 9 + 20 = 29$.

3. In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.

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Solution:

Since $15^2 + 20^2 = 25^2$, the triangle is right-angled at R, and its area is $\frac{1}{2} \cdot 15 \cdot 20 = 150$. Also $\sin P = \frac{20}{25} = \frac{4}{5}$ and $\sin Q = \frac{15}{25} = \frac{3}{5}$.

The hexagon is the triangle minus three corner triangles, each with two sides of length $5:$ at P, area $\frac{1}{2} \cdot 5 \cdot 5 \cdot \frac{4}{5} = 10$; at Q, area $\frac{1}{2} \cdot 5 \cdot 5 \cdot \frac{3}{5} = \frac{15}{2}$; at R, area $\frac{1}{2} \cdot 5 \cdot 5 = \frac{25}{2}$.

Therefore the hexagon has area $150 - 10 - \frac{15}{2} - \frac{25}{2} = 120$.

4. A soccer team has 22 available players. A fixed set of 11 players starts the game, while the other 11 are available as substitutes. During the game, the coach may make as many as 3 substitutions, where any one of the 11 players in the game is replaced by one of the substitutes. No player removed from the game may reenter the game, although a substitute entering the game may be replaced later. No two substitutions can happen at the same time. The players involved and the order of the substitutions matter. Let n be the number of ways the coach can make substitutions during the game (including the possibility of making no substitutions). Find the remainder when n is divided by 1000.

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Solution:

At every moment there are 11 players in the game, any of whom may be removed, while the bench shrinks by one with each substitution. So the first substitution can be made in $11 \cdot 11$ ways, the second in $11 \cdot 10$ ways, and the third in $11 \cdot 9$ ways.

Summing over $0, 1, 2,$ or 3 substitutions,

$$n = 1 + 11 \cdot 11 + 11^2 \cdot 11 \cdot 10 + 11^3 \cdot 11 \cdot 10 \cdot 9 = 1 + 121 + 13310 + 1317690 = 1331122.$$

The remainder upon division by 1000 is $122.$

5. A moving particle starts at the point $(4,4)$ and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (a,b), it moves at random to one of the points $(a-1,b)$, $(a,b-1)$, or $(a-1,b-1)$, each with probability $\frac{1}{3}$, independently of its previous moves. The probability that it will hit the coordinate axes at $(0,0)$ is $\frac{m}{3^n}$, where m and n are positive integers, and m is not divisible by 3. Find $m+n$.

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Solution:

Coordinates never increase, so the first axis point reached is $(0,0)$ exactly when the particle reaches $(1,1)$ and then takes the diagonal step. Every path from $(4,4)$ to $(1,1)$ automatically stays off the axes, since its coordinates remain at least 1.

A path from $(4,4)$ to $(1,1)$ with d diagonal steps also has $3-d$ left steps and $3-d$ down steps, for $6-d$ steps in all, and there are $\frac{(6-d)!}{d!\,(3-d)!\,(3-d)!}$ orderings: $20, 30, 12, 1$ for $d=0,1,2,3$. Since a path with $6-d$ steps has probability $\left(\frac{1}{3}\right)^{6-d}$, the probability of reaching $(1,1)$ and then stepping to $(0,0)$ is

$$\frac{1}{3}\left(\frac{20}{3^6}+\frac{30}{3^5}+\frac{12}{3^4}+\frac{1}{3^3}\right)=\frac{1}{3}\cdot\frac{20+90+108+27}{3^6}=\frac{245}{3^7}.$$

Since $245=5\cdot 7^2$ is not divisible by 3, we get $m+n=245+7=252$.

6. In convex quadrilateral $KLMN$, side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

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Solution:

Let F be the foot of the perpendicular from L to $\overline{KN}$, so O lies on segment LF. In right triangle KLN (right angle at L), the altitude LF to the hypotenuse gives the geometric mean relation $KF \cdot KN = KL^2 = 28^2 = 784.$

Triangles KFO and KMN share angle K, and $\angle KFO = 90^\circ = \angle KMN$, so they are similar. Hence $\frac{KF}{KM} = \frac{KO}{KN}$, that is, $KO \cdot KM = KF \cdot KN = 784$. With $KO = 8$ this gives $KM = 98$, so

$$MO = KM - KO = 98 - 8 = 90.$$

7. There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2\log_{10}(\gcd(x,y)) = 60$$

$$\log_{10} y + 2\log_{10}(\operatorname{lcm}(x,y)) = 570.$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$.

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Solution:

The equations say $x \cdot \gcd(x,y)^2 = 10^{60}$ and $y \cdot \operatorname{lcm}(x,y)^2 = 10^{570}$, so x and y are products of the primes 2 and 5 only. Fix one of these primes and let a and b be its exponents in x and y. Since the gcd takes the smaller exponent and the lcm the larger,

$$a + 2\min(a,b) = 60, \qquad b + 2\max(a,b) = 570.$$

If $a > b$, then $a + 2b = 60$ and $b + 2a = 570$; adding gives $a + b = 210$, and subtracting gives $a - b = 510$, forcing $b < 0$, impossible. So $a \le b$, and the equations become $3a = 60$ and $3b = 570$, giving $a = 20$ and $b = 190$ for both primes.

Thus $x = 2^{20}5^{20}$ and $y = 2^{190}5^{190}$, so $m = 40, n = 380$, and $3m + 2n = 120 + 760 = 880.$

8. Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Let $u = \sin^2 x$ and $v = \cos^2 x$, so $u + v = 1$, and set $p = uv$. Expanding $(u+v)^5$ gives $u^5 + v^5 = 1 - 5p(u+v)^3 + 5p^2(u+v) = 1 - 5p + 5p^2$, so the hypothesis reads

$$1 - 5p + 5p^2 = \frac{11}{36} \qquad \Longrightarrow \qquad 36p^2 - 36p + 5 = 0,$$

with roots $p = \frac{1}{6}$ and $p = \frac{5}{6}$. Since $p = \sin^2 x \cos^2 x = \frac{1}{4}\sin^2 2x \le \frac{1}{4}$, we must have $p = \frac{1}{6}$. Similarly $u^6 + v^6 = (u^2 + v^2)^3 - 3p^2(u^2 + v^2) = (1 - 2p)^3 - 3p^2(1 - 2p) = 1 - 6p + 9p^2 - 2p^3$. Substituting $p = \frac{1}{6}$,

$$u^6 + v^6 = 1 - 1 + \frac{1}{4} - \frac{1}{108} = \frac{26}{108} = \frac{13}{54},$$

so $m + n = 13 + 54 = 67$.

9. Let $\tau(n)$ denote the number of positive integer divisors of n. Find the sum of the six least positive integers n that are solutions to $\tau(n) + \tau(n+1) = 7$.

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Solution:

Since $7 = 2 + 5 = 3 + 4$, one of $\tau(n), \tau(n+1)$ equals 3 or 5, and $\tau = 3$ means a prime square p^2 while $\tau = 5$ means a prime fourth power p^4. So one of $n, n+1$ lies in $\{4, 9, 25, 49, 121, 169, 289, 361, \ldots\} \cup \{16, 81, 625, \ldots\}$, and its neighbor must have $\tau = 4$ (for a square) or be prime (for a fourth power).

Checking neighbors in increasing order: $n = 8$ works $(\tau(8) = 4, \tau(9) = 3)$; $n = 9$ works $(\tau(10) = 4)$; $n = 16$ works $(\tau(16) = 5, 17$ prime$)$; $n = 25$ works $(\tau(26) = 4)$. Then $49, 81, 169,$ and 289 all fail: $\tau(48) = 10, \tau(50) = 6, \tau(80) = 10, 82$ is not prime, $\tau(168) = 16, \tau(170) = 8,$ $\tau(288) = 18, \tau(290) = 8$. Next, $n = 121$ works $(\tau(122) = 4)$ and $n = 361$ works $(\tau(362) = 4)$.

The six least solutions are $8, 9, 16, 25, 121, 361,$ with sum 540.

10. For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x - z_1)^3 (x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \le j < k \le 673} z_j z_k \right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

The polynomial's 2019 roots are the numbers z_j, each repeated three times. By Vieta's formulas, the coefficient of x^{2018} is minus the sum of all roots: $3\sum_j z_j = -20$, so $\sum_j z_j = -\frac{20}{3}$.

The coefficient of x^{2017} is the sum over unordered pairs of roots. A pair may use two copies from one triple ($\binom{3}{2} = 3$ pairs for each j) or copies from two different triples ($3 \cdot 3 = 9$ pairs for each $j < k$). Writing $S = \sum_{j<k} z_j z_k$,

$$19 = 3\sum_j z_j^2 + 9S = 3\left[\left(-\tfrac{20}{3}\right)^2 - 2S\right] + 9S = \frac{400}{3} + 3S,$$

so $3S = 19 - \frac{400}{3} = -\frac{343}{3}$ and $S = -\frac{343}{9}$.

Hence $|S| = \frac{343}{9}$, which is in lowest terms, and $m + n = 343 + 9 = 352.$

11. In $\triangle ABC$, the sides have integer lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An *excircle* of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

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Solution:

Let $BC = a$ and $AB = AC = b$. Place $B = \left(-\frac{a}{2}, 0\right), C = \left(\frac{a}{2}, 0\right), A = (0, h)$ with $h = \sqrt{b^2 - \frac{a^2}{4}}$, so the semiperimeter is $s = b + \frac{a}{2}$ and the area is $K = \frac{ah}{2}$. The inradius and exradii are $r = \frac{K}{s} = \frac{ah}{a+2b}$, $r_A = \frac{K}{s-a} = \frac{ah}{2b-a}$, and $r_B = \frac{K}{s-b} = h$. The incenter is $I = (0, r)$ and the A-excircle has center $(0, -r_A)$. The B-excircle touches line BC at distance s from B, that is, at $x = b$, so its center is (b, h).

Internal tangency with the A-excircle: the center distance is $r + r_A$, so the radius ρ of ω satisfies $\rho - r_A = r + r_A$, i.e. $\rho = r + 2r_A$. External tangency with the B-excircle requires $b^2 + (h - r)^2 = (\rho + h)^2 = (h + r + 2r_A)^2$, which rearranges to $b^2 = 4(r + r_A)(h + r_A)$. Since

$$r + r_A = \frac{4abh}{4b^2 - a^2} \qquad \text{and} \qquad h + r_A = \frac{2bh}{2b - a},$$

and $h^2 = \frac{4b^2 - a^2}{4}$, the condition becomes $b^2 = \frac{8ab^2}{2b-a}$, that is, $2b - a = 8a$, so $2b = 9a$.

For integer sides, $a = 2t$ and $b = 9t$ for a positive integer t, giving perimeter $20t$. The minimum is 20, achieved by the triangle with sides $9, 9, 2$.

12. Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.

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Solution:

Since $f(w) - w = w(w - 20)$, the two legs at $f(z)$ are

$$f(z) - z = z(z - 20), \qquad f(f(z)) - f(z) = f(z)\big(f(z) - 20\big) = z(z - 19)(z - 20)(z + 1),$$

using $f(z) = z(z - 19)$ and $f(z) - 20 = (z - 20)(z + 1)$. They are perpendicular exactly when their quotient $(z - 19)(z + 1)$ is purely imaginary and nonzero.

Write $z = x + 11i$. The real part of $(z - 19)(z + 1) = z^2 - 18z - 19$ is $x^2 - 121 - 18x - 19$, so we need $x^2 - 18x - 140 = 0$, giving $x = 9 \pm \sqrt{221}$. The form $m + \sqrt{n} + 11i$ with m, n positive integers requires $x = 9 + \sqrt{221}$.

Hence $m + n = 9 + 221 = 230$.

13. Triangle ABC has side lengths $AB = 4, BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

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Solution:

Points D, E lie beyond B on ray AB, and F lies on the opposite side of line AB from C. Since $ACFD$ and $BCFE$ are cyclic, the inscribed angles give $\angle FDA = \angle FCA$ and $\angle FEB = \angle FCB$. Writing $\alpha = \angle FCA$ and $\beta = \angle FCB$, triangle DEF has angles $\angle FDE = 180^\circ - \alpha$ and $\angle FED = \beta$, so $\angle DFE = \alpha - \beta = \angle ACB$. From triangle ABC, $\cos \angle ACB = \frac{25+36-16}{2 \cdot 5 \cdot 6} = \frac{3}{4}$, so the law of cosines in triangle DFE gives

$$DE^2 = 2^2 + 7^2 - 2 \cdot 2 \cdot 7 \cdot \tfrac{3}{4} = 32, \qquad DE = 4\sqrt{2}.$$

In triangle DFE, $\cos \angle FDE = \frac{4+32-49}{2 \cdot 2 \cdot 4\sqrt{2}} = -\frac{13\sqrt{2}}{32}$, so α is acute with $\cos \alpha = \frac{13\sqrt{2}}{32}$ and $\sin \alpha = \sqrt{1 - \frac{338}{1024}} = \frac{7\sqrt{14}}{32}$. Let G be the intersection of line CF with line AB. In triangle ACG, $\angle GAC = \angle BAC$ has $\cos \angle BAC = \frac{16+36-25}{2 \cdot 4 \cdot 6} = \frac{9}{16}$, $\sin \angle BAC = \frac{5\sqrt{7}}{16}$, and $\angle ACG = \alpha$, so $\sin(\angle BAC + \alpha) = \frac{5\sqrt{7}}{16} \cdot \frac{13\sqrt{2}}{32} + \frac{9}{16} \cdot \frac{7\sqrt{14}}{32} = \frac{\sqrt{14}}{4}$ and

$$AG = \frac{AC \sin \alpha}{\sin(\angle BAC + \alpha)} = \frac{6 \cdot \frac{7\sqrt{14}}{32}}{\frac{\sqrt{14}}{4}} = \frac{21}{4}.$$

Line CF is the radical axis of the two circles, so $GA \cdot GD = GB \cdot GE$. With $x = BD$ and $BE = x + DE = x + 4\sqrt{2}$:

$$\frac{21}{4}\left(x - \frac{5}{4}\right) = \frac{5}{4}\left(x - \frac{5}{4} + 4\sqrt{2}\right),$$

since $GD = 4 + x - \frac{21}{4}$ and $GB = \frac{21}{4} - 4$. This gives $16\left(x - \frac{5}{4}\right) = 20\sqrt{2}$, so $x = \frac{5+5\sqrt{2}}{4}$ and $BE = \frac{5+21\sqrt{2}}{4}$. Therefore $a + b + c + d = 5 + 21 + 2 + 4 = 32$.

14. Find the least odd prime factor of $2019^8 + 1$.

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Solution:

Suppose an odd prime p divides $2019^8 + 1$. Then $2019^8 \equiv -1 \pmod{p}$, so $2019^{16} \equiv 1$ while $2019^8 \not\equiv 1:$ the multiplicative order of 2019 modulo p is exactly 16. Since the order divides $p - 1$, we need $p \equiv 1 \pmod{16}$, and the smallest such primes are 17 and 97.

Modulo $17: 2019 \equiv 13$, and $13^2 = 169 \equiv -1$, so $2019^8 \equiv (-1)^4 = 1$ and $2019^8 + 1 \equiv 2 \neq 0$.
Modulo $97: 2019 \equiv -18$, and squaring repeatedly,

$$2019^2 \equiv 324 \equiv 33, \qquad 2019^4 \equiv 33^2 = 1089 \equiv 22, \qquad 2019^8 \equiv 22^2 = 484 \equiv -1 \pmod{97}.$$

So 97 divides $2019^8 + 1$, and it is the least odd prime factor: 97.

15. Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5, PB = 3, XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Solution:

Since A lies on both ω and ω_1 and internally tangent circles meet only at their point of tangency, ω_1 is tangent to ω at A; likewise ω_2 is tangent at B. Let Z be the intersection of the tangent lines to ω at A and B. Each tangent line is also tangent to the corresponding inner circle, so the powers of Z with respect to ω_1 and ω_2 are ZA^2 and ZB^2, which are equal. Hence Z lies on the radical axis PQ, and along the line through Z, X, P, Q, Y :

$$ZP \cdot ZQ = ZA^2 = ZX \cdot ZY,$$

the last equality because ZA is tangent to ω.

Because $ZA = ZB$, the point Z lies on the perpendicular bisector of $\overline{AB}$; if M is the midpoint of $\overline{AB}$, then $ZA^2 - ZP^2 = MA^2 - MP^2 = 4^2 - 1^2 = 15$. Also the power of P in ω gives $XP \cdot PY = AP \cdot PB = 15$. Set $s = ZP$ and $u = ZX$, so $ZY = u + 11$. The relations become

$$u(u + 11) = s^2 + 15, \qquad (s - u)(u + 11 - s) = 15.$$

Expanding the second and substituting the first yields $u = s - \frac{11}{2} + \frac{15}{s}$, and substituting back gives $\left(s + \frac{15}{s}\right)^2 - \frac{121}{4} = s^2 + 15$, so $\frac{225}{s^2} = \frac{61}{4}$.

Finally $ZQ = \frac{ZA^2}{ZP} = s + \frac{15}{s}$, so $PQ = ZQ - ZP = \frac{15}{s}$ and $PQ^2 = \frac{225}{s^2} = \frac{61}{4}$. Therefore $m + n = 61 + 4 = 65$.

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